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Separation Axioms via F-open Sets

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Abstract. In this paper, we introduce some types of separation axioms via F-open sets, namely FT_i $(i = 0, 1, 2, 3, 4)$, F-regular and F-normal spaces, and investigate their properties, relationships and characterizations. We show that every FT_i space is a T_i space for $i = 0, 1, 2, 3, 4$. However, the converse is true whenever X is finite.

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1. Introduction

In 2012, Alias et al.[3] introduced some types of separation axioms via $\omega-$open sets, namely $\omega-$regular, completely $\omega-$regular and $\omega-$normal space and investigated their fundamental properties. After that, in 2024, Alqahtani and Abd El-latif [1] interduced some kinds of separation axioms via $\aleph-$poen sets, namley $\aleph - T_0-$space, $\aleph - T_1-$space and $\aleph - T_2$-space. Quite recently, Alqahtani [2] has introduced the notion of F-open sets in a topological space and obtained the fundamental properties of F-open sets. Furthermore, several notions such as F-continuous functions, F-compact spaces and related properties are defined and investigated.

Motivated by these works, and to simplify the path for many future articles on this topic, in this paper, we introduce some types of separation axioms via F-open sets, namely, FT_i $(i = 0, 1, 2, 3, 4)$, F-regular and F-normal spaces, and their relationships and characterizations are obtained. Moreover, it is shown that 1) an F-compact FT_2 space is FT_4, 2) F-normal spaces are preserved under F-closed preserving and continuous surjections.

First, we recall some notions defind by Alqahtani[2]. Let (X, τ) be a topological space and A be a subset of X. The closure of A and interior of A are denoted by $cl(A)$ and

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$int(A)$, respectively.

Definition 1. *[2] An open subset A of a topological space (X, τ) is called an $F-$open set if $cl(A)\setminus A$ is a finite set.*

Definition 2. *[2] A closed subset A of a topological space (X, τ) is called an $F-$closed set if $A\setminus int(A)$ is a finite set.*

Definition 3. *[2] Let A be a subset of a topological space (X, τ). The $F-$closure of A is defined as the intersection of all $F-$closed sets containing A, and is denoted by $cl^F(A)$.*

Definition 4. *[2] Let (X, τ) be a topological space. Then (X, τ) is called an $F-$compact space if any open cover of X has a finite subcover of $F-$open sets.*

2. Separation axioms via F-open sets

For any topological space (X, τ), by τ_F we denote the collection of all $F-$ open subsets of X.

Example 1.

1) *For any discrete topology τ on X, $\tau_F = \tau$ since every $A \subseteq X$ is clopen, and therefore $cl(A)\setminus A = \emptyset$ is finite.*

2) *For any indiscrete topology τ on X, we have $\tau_F = \tau$. The only open sets in X are X and $\emptyset$. Since both are clopen, so $cl(X)\setminus X = \emptyset$ and $cl(\emptyset)\setminus\emptyset = \emptyset$.*

3) *The odd-even topology on $\mathbb{N}$ induced by its bases*
   $P = \{\{2k-1, 2k\}, k\in \mathbb{N}\} = \{\{1,2\}, \{3,4\}, \{5,6\}, \cdots\}$ *satisfies $\tau = \tau_F$.*

4) *Also, the deleted integer topology defined by letting $X = \bigcup_{n\in\mathbb{N}} (n-1, n) \subset \mathbb{R}$ and $P = \{(0,1), (1,2), (2,3), \ldots\}$ satisfies $\tau = \tau_F$ since every open set is clopen.*

5) *For any metrizable space, the topology τ induced by the metric also satisfies $\tau = \tau_F$. This is because every singleton is closed and therefore every open set is clopen.*

Remark 1. *The converse of 5 of Example 1 is not true by the following counter examples.*

Example 2. *The indiscrete topology is not metrizable but yet satisfies $\tau = \tau_F$.*

Example 3. *The cofinite topology on $\mathbb{R}$ which is defined for a nonempty set X by*

$$\tau_{cof} = \{U \subseteq X : X\setminus U \; is finite\} \cup \{\emptyset\}$$

i.e., $U \subset \mathbb{R}$ is open whenever $\mathbb{R} \setminus U$ is finite is not metrizable since $\mathbb{R}$ is infinite.

However, for any open set, $\text{cl}(U)\backslash U\subseteq \mathbb{R}\backslash U$, so it is finite and therefore U is $F-$open. Therefore, $\tau_F = \tau$.

In general, if every open set is clopen, then $\tau_F = \tau$. We can also see that if X is finite, then $\tau_F = \tau$.

Note that τ_F is not necessarily another topology on X, since countable union of $F-$open set is not necessarily an $F-$open set as shown by the following Example.

Example 4. *[2] Let $A_n = (n,\ n+1)$ be a subset of $(\mathbb{R},\mathcal{U})$ for all $n \in N$, then, $A_n = (n,\ n+1) \in \mathcal{U}$ and $cl(n,\ n+1) \setminus (n,\ n+1) = \{n,\ n+1\}$ is finite for all $n \in N$. Hence, $A_n = (n,\ n+1)$ is $F-$open for all $n \in N$. Now, we have $\bigcup_{n\in\mathbb{N}}(n,\ n+1) = [1,\infty)\setminus\mathbb{N}$ is an open set. However, $cl(\bigcup_{n\in\mathbb{N}}(n,\ n+1))\backslash\bigcup_{n\in\mathbb{N}}(n,\ n+1) = cl([1,\infty)\backslash\mathbb{N})\backslash([1,\infty)\setminus\mathbb{N}) = [1,\infty)\backslash([1,\infty)\setminus\mathbb{N}) = \mathbb{N}$ is not a finite set. Therefore, $\bigcup_{n\in\mathbb{N}}(n,\ n+1)$ is not $F-$open.*

Definition 5. *A topological space (X,τ) is called an FT_0- space if for any two distinct points $x,y\in X$ there is an $F-$open set U in X containing x but not y or an $F-$ open set U containing y but not x.*

Example 5. *Let $(\mathbb{R},\mathcal{U})$ be a topological space, where $\mathcal{U}$ is the usual topology. Let (a,b) be any open interval in $(\mathbb{R},\mathcal{U})$, since $cl(a,b)\backslash(a,b) = [a,b]\backslash(a,b) = \{a,b\}$ is finte, then any open interval in $(\mathbb{R},\mathcal{U})$ is F-open. So, for any two distinct points $x,y\in\mathbb{R}$ there is an open interval (a,b)containing x but not y or an open interval (a,b) containing y but not x.Therefore, $(\mathbb{R},\mathcal{U})$ is an FT_0- space.*

Definition 6. *A topological space (X,τ) is called an FT_1- space if for any two distinct points $x,y\in X$ there is an $F-$open set U in X containing x but not y and an $F-$ open set V in X containing y but not x.*

Example 6. *Let $(\mathbb{R},\mathcal{U})$ be the usual topology for $\mathbb{R}$. Let $x,y\in\mathbb{R}$, $y>x$, and $y-x=c$. Then $U=\left(x-\frac{c}{3},\ x+\frac{c}{3}\right)$ and $V=\left(y-\frac{c}{3},\ y+\frac{c}{3}\right)$ are two $F-$open sets where $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$.Therefore, $(\mathbb{R},\mathcal{U})$ is an FT_1- space.*

Theorem 1. *An FT_1- space is an FT_0- space, but the converse is not true.*

Proof. Let (X,τ) be an $FT_1-space$. Then for any distinct points $x,y\in X$, there exist $F-$open sets $U,V\in\tau_F$ such that:

$$x\in U,\ y\notin U\ and\ y\in V,\ x\notin V$$

Hence, there exists an $F-$open set $U\in\tau_F$ such that:

$$x\in U \text{ and } y\notin U\ or\ \ y\in U \text{ and } \ x\notin U$$

Therefore,(X,τ) is an FT_0-space.

For the converse, let us consider the Sierpinski space, where $S = \{0,1\}$ and $\tau=\{\emptyset,\ \{1\},\{0,1\}\}$. It is clear that the Sierpinski space is FT_o but not FT_1.

Theorem 2. *A topological space (X, τ) is an FT_1 space if and only if for each $x \in X$, the singleton set $\{x\}$ is $F-$closed.*

Proof. Let (X, τ) be an FT_1-space and let $x \in X$ be any point. We want to show that $\{x\}$ is $F-$closed, i.e., $\{x\}$ is closed and $\{x\}\backslash int(\{x\})$ is a finite set. Since X is an FT_1 space, then it is a T_1-space and every singleton $\{x\}$ is closed. Now

$$\{x\} \backslash int(\{x\}) = \{x\} \backslash \emptyset = \{x\}$$

is finite. Therefore, $\{x\}$ is $F-$closed.
Conversely, let every singleton $\{x\}$ be $F-$closed, and let $x \neq y$ be two points in X, then since $\{x\}$ is $F-$closed, $X\backslash\{x\}$ is $F-$ open and contains y but not x. Similarly, $X\backslash\{y\}$ is $F-$ open and contains x but not y. Therefore, (X, τ) is an FT_1- space.

Definition 7. *A topological space (X, τ) is called an $FT_2 -$ space if for any two distinct points $x, y \in X$ there exist $F-$open sets U and V in X, such that $x \in U$ and $y \in V$, and $U \cap V = \emptyset$.*

Example 7. *Let us consider the usual topology for $\mathbb{R}$ as in Example 5.Then $U = \left(x - \frac{c}{3},\ x + \frac{c}{3}\right)$ and $V = \left(y - \frac{c}{3},\ y + \frac{c}{3}\right)$ are two $F-$open sets where $x \in U$, $y \in V$, x and $U \cap V = \emptyset$. Therefore, $(\mathbb{R}, \mathcal{U})$ is an $FT_2 -$ space.*

Theorem 3. *An $FT_2 -$ space is an $FT_1 -$ space, but the converse is not true.*

Proof. Let (X, τ) be an $FT_2 - space$. Then $\forall x, y \in X : x \neq y$, there exist $F-$open sets U, V such that:

$$x \in U,\ y \in V \text{ and } U \cap V = \emptyset$$

Therefore, we have

$$U \in \tau_F : x \in U,\ y \notin U$$

and

$$V \in \tau_F : y \in V,\ x \notin V$$

This shows that (X, τ) is an $FT_1 -$ space.

For the converse, let us consider the cofinite topology τ_{cof} on an infinite set X. Then (X, τ_{cof}) is an FT_1-space but is not FT_2. Let $x \in X$ be any point, since $\{x\} \backslash int(\{x\}) = \{x\} \backslash \emptyset = \{x\}$ is finite, then the singleton $\{x\}$ is $F-$closed and hence, by Theorem 2, X is an $FT_1 -$ space. To show that X is not an FT_2-space, let U and V be any two open subsets of X such that $U \cap V = \emptyset$. Since, U is open and $\text{cl}(U)\backslash U \subseteq \mathbb{R}\backslash U$, then $\text{cl}(U)\backslash U$ is finite and then U is $F-$open. Similarly, V is also an $F-$open set. Now, since $U \cap V = \emptyset$, then

$$(X\backslash U) \cup (X\backslash V) = X$$

This is a contradiction since X is infinite. Hence, there is no two distinct points in X belong to two disjoint $F-$open sets. Therefore, X is not an FT_2-space.

Definition 8. *A topological space (X, τ) is called an $F-$regular space (briefly $Fr-$ space) if for each closed subset $W \subset X$ and each $x \notin W$, there exist $F-$open sets U and V in X, such that $x \in U$, $W \subseteq V$, and $U \cap V = \emptyset$. An $F-$ regular FT_1-space is called an FT_3-space.*

Example 8. *Let us consider the usual topology for $\mathbb{R}$ as in Example 5. It's easy to show that $(\mathbb{R},\mathcal{U})$ is FT_3-space.*

Theorem 4. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is F-regular;

(2) For any $x \in X$ and any open set U containing x, there exists an F-open set V such that $x \in V \subseteq cl^F(V) \subseteq U$;

(3) For any $x \in X$ and any closed set B such that $x \notin B$, there exists an F-open set U such that $x \in U$ and $B \cap cl^F(U) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let X be an F-regular space and U be any open set containing x. Let $K = X \setminus U$, then K is a closed set not containing x. Since (X, τ) is F-regular, then there exist F-open sets V, W of X such that $x \in V, K \subset W$ and $V \cap W = \emptyset$. Hence $cl^F(V) \cap W = \emptyset$. Therefore, we have $x \in V \subseteq cl^F(V) \subseteq U$.

(2) $\Rightarrow$ (3): Let x be any point of X and B be any closed set in X not containing x. Then $X \setminus B$ is an open set containing x. By (2), there exists an F-open set U such that $x \in U \subseteq cl^F(U) \subseteq (X \setminus B)$. Therefore, there exists an F-open set U such that $x \in U$ and $B \cap cl^F(U) = \emptyset$.

(3) $\Rightarrow$ (1): Let K be a closed set not containing x. Then $x \in X \setminus K$ and $X \setminus K$ is an open set. Hence there exists an F-open set U such that $x \in U$ and $K \cap cl^F(U) = \emptyset$. Let $V = X \setminus cl^F(U)$. Then V is F-open, $K \subseteq V$ and $U \cap V = U \cap (X \setminus cl\ F(U)) = \emptyset$. This shows that (X, τ) is F-regular.

Definition 9. *A topological space (X, τ) is called an $F-$normal space (briefly $Fn-$space) if for each pair of disjoint closed subsets W_1 and W_2 of X, there exist $F-$open sets U and V in X, such that $W_1 \subseteq U$, $W_2 \subseteq V$, and $U \cap V = \emptyset$. An $F-$ normal FT_1-space is called a FT_4-space.*

Example 9. *The usual topology for $\mathbb{R}$ as in Example 5 is an Fn and FT_1-space and hence it is an FT_4-space.*

Remark 2. *Every $F-$normal space in not necessarily $F-$regular.*

Example 10. *Let us consider the Sierpinski space, where* $S = \{ 0 , 1 \}$ *and* $\tau = \{\emptyset, \{1\}, \{0,1\}\}$ *. It is clear that the Sierpinski space is* $F-$*normal since the only closed subsets of* S *are* $\{0\}$ *and* $\emptyset$*, but* S *is not F-regular, because 1 and* $\{0\}$ *which is closed set cannot be separated by two disjoint F-open sets, i.e.,* $1 \notin \{0\}$ *which is closed sets,* $1 \in \{1\}$*,* $\{0\} \subseteq \{0,1\}$ *where* $\{1\}$ *,* $\{0,1\}$ *are* $F-$*open sets and* $\{1\} \cap \{0,1\} = \{1\} \neq \emptyset$

Theorem 5. *Every* $FT_4 -$ *space is an* $FT_3 -$ *space.*

Proof. Let (X, τ) be an $FT_4 - space$. Then (X, τ) is an FT_1 space, hence we need only show that X is an $Fr-$ space. For that, let $K \subseteq X$ be an $F-closed$ set such that $x \notin K$. Since X is an FT_1 space, then $\{x\}$ is an F-closed set such that $\{x\} \cap K = \emptyset$. Since X is an $Fn-$space, there exist $F-$open sets U and V in X, such that $\{x\} \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$. Hence, for each closed subset $K \subset X$ and each $x \notin K$, there exist $F-$open sets U and V in X such that $x \in U$, $K \subseteq V$, and $U \cap V = \emptyset$. Therefore, X is $Fr-$ regular.

Theorem 6. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is F-normal;*

(2) For any closed set F *and any open set* U *containing* F*, there exists an* F*-open set* V *such that* $F \subseteq V \subseteq cl^F(V) \subseteq U$*;*

(3) For any disjoint closed sets A, B*, there exists an* F*-open set* U *such that* $A \subseteq U$ *and* $B \cap cl^F(U) = \emptyset$*.*

Proof. The proof is similar with Theorem 4.

Lemma 1. *Let* (X, τ) *be a topological space and* Y *be a subset of* X*. If* U *is an* F*-open set of* X*, then* $U \cap Y$ *is an* F*-open set of the subspace* Y*.*

Proof. Since U is open in X, $U \cap Y$ is open in Y.
$cl_Y(U \cap Y) \setminus (U \cap Y) = cl_X(U \cap Y) \cap Y \setminus (U \cap Y) \subseteq cl_X(U) \cap Y \setminus (U \cap Y) \subset cl_X(U) \setminus U$. Since U is an F-open set of X, $cl_X(U) \setminus U$ is a finite set. Therefore, $U \cap Y$ is F-open in the subspace Y.

Theorem 7. *If* X *is an* $FT_i -$ *space, then any subspace* Y *of* X *is an* $FT_i -$ *space for* $i = 0, 1, 2, 3, 4$*.*

Proof. We consider only the cases $i = 1, 3$ and the other cases can be proved by the same argument.

1) Let $i = 1$. Let x, y be any distinct points of a subspace Y of an FT_1-space X. There exists an F-open set U of X such that $x \in U$ and $y \notin U$. By Lemma 1, $U \cap Y$ is an F-open set of Y such that $x \in U \cap Y$ and $y \notin U \cap Y$. Similarly, we can find an F-open set V in Y containing y but not x. Therefore, the subspace Y is an $FT_1 - space$.

2) Let $i = 3$. Let K be any closed set in Y and $y \in Y \setminus K$. Then there exists a closed set K_X in X such that $K = K_X \cap Y$, where $y \notin K_X$. Since X is FT_3, there exist disjoint F-open sets U_X, V_X in X such that $y \in V_X$ and $K_X \subset U_X$. Now, let $U = U_X \cap Y, V = V_X \cap Y$, then by Lemma 1, U, V are F-open sets in Y and $K = K_X \cap Y \subset U_X \cap Y = U$ and $y \in V_X \cap Y = V$. Moreover, $U \cap V \subset U_X \cap V_X = \emptyset$. Therefore, Y is FT_3.

Lemma 2. *[2] Let (X, τ) be a topological space. For F-open sets of X, the following properties hold:*

(1) The finite union of F-open sets is F-open,

(2) The finite intersection of F-open sets is F-open.

Theorem 8. *Let (X, τ) be a topological space. Then every FT_i space is a T_i space for $i = 0, 1, 2, 3, 4$. However, the converse is true whenever X is finite.*

Proof. Easy, since $F-$open sets are open sets. For the converse, let X be a finite set, then every open set is a $F-$open set and then every T_i space is a FT_i space for $i = 0, 1, 2, 3, 4$.

Theorem 9. *If (X, τ) is F-compact and FT_2, then (X, τ) is FT_4.*

Proof. Let K, L be any disjoint closed sets. Then they are F-compact. Let k be any point of K. For any point$x \in L$, $x \notin K$ and $x \neq k$. Since (X, τ) is FT_2, there exist disjoint F-open sets $U_x(k)$ and $V_k(x)$ such that $k \in U_x(k)$ and $x \in V_k(x)$. Now, fix x, then $\cup\{U_x(k) : k \in K\}$ is F-open cover of K and there exists a finite subset K_0 of K such that $K \subset \cup\{U_x(k) : k \in K_0\}$. Now, put $U(x) = \cup\{U_x(k) : k \in K_0\}$ and $V_{K_0}(x) = \cap\{V_k(x) : k \in K_0\}$. Then $U(x) \cap V_{K_0}(x) = \emptyset$ for each $x \in L$. Since $\cup\{V_{K_0}(x) : x \in L\}$ is an F-open cover of L and L is F-compact, there exists a finite subset L_0 of L such that $L \subset \cup\{V_{K_0}(x) : x \in L_0\}$. Now, put $V_L =\subset \cup\{V_{K_0}(x) : x \in L_0\}$ and $U_K = \cap\{U(x) : x \in L_0\}$. Then, by Lemma 2, U_K, V_L are disjoint F-open sets such that $L \subset V_L, K \subset U_k$. Therefore, (X, τ) is FT_4.

Definition 10. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be F-closed preserving if for any F-closed set K of X, $f(K)$ is F-closed in Y.*

Theorem 10. *A surjective function $f : (X, \tau) \to (Y, \sigma)$ is F-closed preserving if and only if for any subset S of Y and any F-open set U such that $f^{-1}(S) \subseteq U$, there exists an F-open set V in Y such that $S \subset V$ and $f^{-1}(V) \subseteq U$.*

Proof. ($\Rightarrow$) Let S be any subset of Y and U be any F-open set such that $f^{-1}(S) \subseteq U$. Since $X \setminus U$ is F-closed, $f(X \setminus U)$ is F-closed in Y. Since $f^{-1}(S) \subseteq U$, $X \setminus U \subseteq X \setminus f^{-1}(S)$. Let $V = Y \setminus f(X \setminus U)$. Then we have $S \subseteq Y \setminus f(X \setminus U) = V$ and $f^{-1}(V) \subseteq U$.

($\Leftarrow$) Let K be any F-closed in X. We show that $f(K)$ is F-closed in Y. Let $S = Y \setminus f(K)$. Then $f^{-1}(S) = f^{-1}(Y \setminus f(K)) \subseteq X \setminus K$. Since $X \setminus K$ is F-open, there exists an F-open set V such that $Y \setminus f(K) \subset V$ and $f^{-1}(V) \subset X \setminus K$. Hence $K \subset X \setminus f^{-1}(V) = f^{-1}(Y \setminus V)$ and $f(K) \subseteq Y \setminus V$. On the other hand, we have $Y \setminus V \subseteq f(K)$. Therefore, $Y \setminus V = f(K)$ and $f(K)$ is F-closed in Y.

Theorem 11. *Let $f : (X, \tau) \to (Y, \sigma)$ be an F-closed preserving continuous surjection. If (X, τ) is F-normal, then (Y, σ) is F-normal.*

Proof. Let (X, τ) be F-normal. For any disjoint closed sets K_1 and K_2 in Y, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets in X since f is cotinuous. Since (X, τ) is F-normal, there exist F-open sets U_1, U_2 in X such that $f^{-1}(K_i) \subseteq U_i$ $(i = 1, 2)$ and $U_1 \cap U_2 = \emptyset$. By Theorem 10, there exist F-open sets V_i such that $K_i \subset V_i, f^{-1}V_i) \subseteq U_i$ $(i = 1, 2)$. Since $U_1 \cap U_2 = \emptyset$ and f is surjective, $V_1 \cap V_2 = \emptyset$. Therefore, (Y, σ) is F-normal.

Corollary 1. *Let $f : (X, \tau) \to (Y, \sigma)$ be an F-closed preserving continuous surjection. If (X, τ) is FT_4, then (Y, σ) is FT_4.*

Proof. It is necessary to show that if (X, τ) is FT_1, then (Y, σ) is FT_1. For any point $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Since X is FT_1, by Theorem **??**, $\{x\}$ is F-closed in X and $f(\{x\}) = \{y\}$ is F-closed in Y.

Conclusion. By using F-open sets in a topological space, we defined separation axioms $FT_i (i = 0, 1, 2, 3, 4)$ and obtained their properties, characterization and relationships. Furthermore, it was shown that (1) F-compact FT_2-spaces are FT_4, (2) F-normality is preserved under F-closed preserving continuous surjections, (3) for $FT_i (i = 0, 1, 2, 3, 4)$ the following DIAGRAM holds:

$$\begin{array}{ccccccccc} FT_4 & \Rightarrow & FT_3 & \Rightarrow & FT_2 & \Rightarrow & FT_1 & \Rightarrow & FT_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ T_4 & \Rightarrow & T_3 & \Rightarrow & T_2 & \Rightarrow & T_1 & \Rightarrow & T_0 \end{array}$$

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References

[1] M. H. Alqahtani and A.M. Abd El-latif. Separation axioms via novel operators in the frame of topological spaces and applications. *AIMS Mathematicsl*, 9(6):14213–14227, 2024.

[2] M.H. Alqahtani. F-open and f-closed sets in topological spaces. *Eur. J. Pure Appl. Math.*, 6(2):819–832, 2023.

[3] A.B. Khalaf, H.M. Darwesh, and K. Kannan. Some Types of Separation Axioms in Topological Spaces. *Tamsui Oxford Journal of Information and Mathematical Sciences*, 28(3):303–326, 2012.