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One Resolution of the Conjecture between the Projective Dimension of a Simple Module S of an Artinian Ring on Zero Radical's Cube and the First Bifunctor Extension on S

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Abstract. Through concepts from non-commutative algebra and homology, this paper resolves the conjecture that lets the first bifunctor extension to be zero when the projective dimension is finite, for a simple module S of an Artinian ring whose cube of its Jacobson radical is zero and under the condition that any simple module over this ring of finite projective dimension has a radical square zero of the cover projective of its first syzygy. For that, we use a property of the simple module which realizes the minimum of the finite projective dimensions of simple modules. Our main result is presented in the form of a corollary in the case of an Artinian ring with radical cubed zero such that the projective cover of its radical is of Loewy length two and its supremum being finite. In the part of discussions, we succeed to show the no loop conjecture through two examples. The first one is about its weak version by taking a quiver algebra A verifying $J^3 = 0$ and without considering that $rad^2(P(\Omega(S))) = 0$ for every simple module, while the second one shows that if the extension quiver has a loop in a simple module then its projective dimension is infinite for every nilpotence index of the Jacobson radical. More importantly, we finally provide a practical third example for our special case.

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1. Introduction

1.1. Background from non-commutative algebra and homology

Among the main branches of abstract algebra that deal with algebraic structures and operations are the commutative and non-commutative algebra. Both represent active research areas with many open problems and ongoing developments [1–3, 7, 8, 12, 16]. The

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primary distinction between them is that, in commutative algebra, the order of operations is irrelevant, and ideals on left and right are identical. As for non-commutative rings, they are distinguished by the need to consider ideals on the right and left separately. It is common for the study of non-commutative rings to impose a condition on one of these types of ideals without requiring that it has to be valid for its opposite side, whereas moving to the quotient of the non-commutative ring by an ideal imposes that it must be bilateral, a case which is verified by the Jacobson radical in an Artinian ring and which has several properties including that of its nilpotence. Thus, the study of simple, projective and injective modules of finite type on this ring is described by idempotent primitives [10]. In parallel, homological algebra has been categorically interested in the study of the properties of the special functor $\mathrm{Hom}(M, -)$ and its precision, the functor Hom is only exact on the left, while the tensor product functor is only exact on the right. To obtain an exact sequence from a functor of the form $\mathrm{Hom}(M, -)$ (with M a module), it is possible to use an approximation of M by projective modules. Such an approximation is called the projective resolution of M, and if we apply the functor Hom to a projective resolution of M, we therefore obtain a sequence which is generally not exact, but that is complex. The notion of homology corrects the lack of inexactness of this complex by associating it with a long exact sequence, called a homology sequence [10]. . This makes it possible to associate new functors with the functor Hom, called extension functors and denoted $\mathrm{Ext}^n(M, -)$ see [13].

Let A be an Artinian ring with J as its Jacobson radical, and let $mod(A)$ be the category of left A-modules of finite type. An invariant important of A is its global dimension denoted $gdim(A)$ and which is the supremum of projective dimensions of left A-modules of finite type, see Page 92-[13]. It is known that $gdim(A)$ is the supremum of the projective dimensions of simple A-modules and the number of non-isomorphic simple A-modules is finite and also J is nilpotent [10]. We define therafter the quiver (that is to say a directed graph) of extensions of A, whose set of vertices is a complete set of representatives of isomorphism classes of simple A-modules, and given two vertices S and T, there is an arrow from S to T if the extension group $\mathrm{Ext}^1_A(S, T)$ is not zero.

1.2. History of the conjecture until our contribution

The first version of the conjecture that we are dealing with here, was stating that if $gdim(A)$ is finite, then the quiver of extensions of A, is characterized by the $\mathrm{Ext}^1_A(S, S)$ null for every simple A-module S, then the second version of the conjecture, let us say the more localized and complicated, and that interests us more since the first one was resolved in the case of the supremum by Green et al. in 1985 [14]. In fact, this one was presented as the seventh conjecture in the book of representation theory of Artin Algebras [10], by the implication $\mathrm{Ext}^1_A(S, S) \neq 0 \Rightarrow \mathrm{pd}_A(S) = \infty$ which is the same as saying that if the projective dimension of each simple A-module S is finite, there is no arrow from S to S.

Historically, this conjecture was demonstrated in several cases long before it was formally stated. At the end of 1960, Helmut Lenzing proved this conjecture in the case where A is an algebra over an algebraically closed field [6] as it took back up the idea of Hattori-Stallings [4, 11] on the notion of (the trace of the endomorphism of a projective module). In chronological order, the next result in favor of the conjecture goes back to 1983 when Green et al. showed that conjecture is true when the global dimension of algebra is bounded by two [14]. Their proof consists of a recurrence of the number of isomorphic classes of simple modules. Subsequently, in 1986, Fuller and Zimmermann-Huisgen demonstrated in [17], the conjecture when $(\mathrm{rad}(A))^3 = 0$ and when A is a left serial algebra. Their approach uses the matrix Cartan filtered by radical algebra to prove that the determinant of the Cartan matrix is one.

1.1 Remark. We note that Fuller and Zimmermann-Huisgen have demonstrated that conjecture using a strong condition and which is about working with the global dimension. Our approach is interesting in the sense that we succeed to weaken that condition by proving that we can just take the projective dimension in every simple module.

Later, K. Igusa proved in 1990 [5] the conjecture in a case that all algebras of endomorphism of simple modules are separable. The author used concepts from the K-theory in his proof to point out his result included that of H. Lenzing since all fields are separable algebras. Then, the Lenzing trace function has been localized to endomorphisms of modules in $\mathrm{mod}(A)$ with the e-bounded projective resolution, where e is an idempotent in A and they have proved the conjecture for Artinian rings A with $J^2 = 0$ for finite dimensional algebras over an algebraically closed field.

In short, our research work aims to establish that last conjecture for Artinian rings A with $J^3 = 0$ in the particular case where each simple module having the finite projective dimension whose the projective cover of its first syzygy is canceled by J^2, and this is by taking inspiration from the algebra of endomorphisms of a projective A-module as well as the Jacobson radical of this one and the characterization of simple and projective $\mathrm{End}(P)$-modules with P is a projective A- module.

2. Main Theorem

Given a module M in $\mathrm{mod}(A)$, we denote by,

- $\Omega(M)$ the first syzygy,
- $\mathrm{pd}M$ the projective dimension of M,
- $\Omega^2(M) = \Omega(\Omega(M))$,
- $\mathrm{pd}(\Omega(M))$ the projective dimension of the projective cover of the first syzygy of M.

The following Theorem represents the main result of this paper, and in order to prove it, we will need the results that we have developed in Theorem 4.3. thereafter.

2.1 Theorem. *Let A be an Artinian ring with $J^3 = 0$. If $\mathrm{rad}^2(P(\Omega(S))) = 0$ for every simple module S, then $\mathrm{Ext}^1_A(S, S) = 0$.*

We also fix a complete set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents in A and let $S_i = Ae_i/Je_i$ the simple A-module associated with e_i.

For convenience, we quote the following well-known result.

2.2 Lemma. *Let A be an Artinian ring with a short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod}A$. *The following statement holds.*

(1) $\mathrm{pd}N \leq \max\{\mathrm{pd}M, \mathrm{pd}L + 1\}$*, and the equality occurs in case* $\mathrm{pd}M \neq \mathrm{pd}L$.

(2) $\mathrm{pd}L \leq \max\{\mathrm{pd}M, \mathrm{pd}N - 1\}$*, and the equality occurs in case* $\mathrm{pd}M \neq \mathrm{pd}N$.

(3) $\mathrm{pd}M \leq \max\{\mathrm{pd}L, \mathrm{pd}N\}$*, and the equality occurs in case* $\mathrm{pd}N \neq \mathrm{pd}L + 1$.

3. Minimal projective dimension

3.1 Lemma. *Let A be an Artinian ring with a radical cubed zero. Let M be a module in* $\operatorname{mod}A$ *of finite projective dimension with* $\mathrm{pd}M \leq \min\{\mathrm{pd}(S_1), \ldots, \mathrm{pd}(S_n)\}$ *and* $\mathrm{rad}^2(M) = 0$. *If $f : P \to M$ is a projective cover of M, then* $\mathrm{rad}(\Omega M) = \mathrm{rad}^2(P)$.

Proof. Let $f : P \to M$ be a projective cover of M. Then, $\Omega M \subseteq \mathrm{rad}(P)$, and hence, $\mathrm{rad}(\Omega M) \subseteq \mathrm{rad}^2(P)$. Since $\mathrm{rad}^2(M) = 0$, $\mathrm{rad}^2(P) \subseteq \Omega M$.

Suppose that $\mathrm{rad}^2(P) \nsubseteq \mathrm{rad}(\Omega M)$. Then there exists a maximal submodule L of ΩM such that $\mathrm{rad}^2(P) \nsubseteq L$. Since $\mathrm{rad}^3(A) = 0$, $\mathrm{rad}^2(P)$ is semi-simple. Thus, $S \nsubseteq L$ where S is some simple submodule of $\mathrm{rad}^2(P)$, and consequently, $\Omega M = S \oplus L$. This yields that $\mathrm{pd}(S) \leq \mathrm{pd}(\Omega M) < \mathrm{pd}M \leq \mathrm{pd}(S)$, a contradiction. The proof is completed.

3.2 Lemma. *Let A be an Artinian ring with radical cubed zero, and let S be the simple module of minimal projective dimension among the simple modules in* $\operatorname{mod}A$. *If* $\mathrm{pd}S < \infty$ *and* $\mathrm{rad}^2(P_1) = 0$ *with P_1 is the projective cover of* $\Omega(S)$*, then* $\mathrm{pd}S \leq 1$.

Proof. Suppose that S admits a minimal projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0,$$

where $m > 1$ then, $\Omega^2(S) \neq 0$ and Since $\mathrm{pd}(\Omega(S)) < \mathrm{pd}(S)$ and like A has radical cubed zero we have, $\mathrm{rad}^2(\Omega(S)) = \mathrm{rad}^2(Je) = J^3e = 0$ where e is the idempotent associated with S, then by Lemma 1.1 and according to the hypothesis of our present Lemma, $\mathrm{rad}(\Omega^2(S)) = rad^2(P_1) = 0$, and therefore $\Omega^2(S)$ is a semi-simple module, then $\mathrm{pd}(S) \leq \mathrm{pd}(\Omega^2(S))$ which is absurd because $\mathrm{pd}(\Omega^2(S)) = \mathrm{pd}(S) - 2$.

Recall that if M and N are two modules, by choosing a projective resolution P_* of M, then $\mathrm{Ext}^n_A(M, N) = H^n(\mathrm{Hom}_A(P_*, N))$ is the n^{th} co-homology of the cochain complex of

K-modules $\mathrm{Hom}_A(P_*, N)$ which is

$$... \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(P_0, N) \longrightarrow \mathrm{Hom}_A(P_1, N) \longrightarrow \mathrm{Hom}_A(P_2, N) \longrightarrow ...$$

where $\mathrm{Hom}_A(P_n, N)$ is in degree n and P_* the complex deduced from the projective resolution of M, see [10] .

3.3 Theorem. *Let A be an Artinian ring and $S = Ae/Je$ be a simple A-module with e primitive idempotent such that* $\mathrm{pd}(S) \leq 1$*, then* $\mathrm{Ext}^1_A(S, S) = 0$*.*

Proof. if S is projective ie $\mathrm{pd}(S) = 0$, then the result holds .
If $\mathrm{pd}(S) = 1$ and $\mathrm{Ext}^1_A(S, S) \neq 0$, then $\mathrm{Hom}(Je, S) \neq 0$, therefore Je is projective then Ae is isomorphic to a summand of Je since Ae is a projective cover of S, therefore $l(Ae) \leqslant l(Je)$ and since $l(Je) \leqslant l(Ae)$, then $l(S) = 0$ and $S = 0$ contradiction.

4. Jacobson radical of end(p) with p an A-projective module.

Let P be an A-projective module in mod$A$, we consider the algebra $E = \mathrm{End}_A(P)$ and we denote $J(E)$ the Jacobson radical of the algebra $\mathrm{End}_A(P)$ or otherwise $J(E) = \mathrm{rad}_{\mathrm{End}_A(P)}(\mathrm{End}_A(P))$ and according to Lemma 1 in [15], we have,
$J(E) = \{\Phi \in E/\Phi(P) \textit{ is a small submodule of } P\}$

4.1 Proposition. *For $n \geq 0$ and P an A-projective module in* mod$A$ *we have:*

(1) $J^n(E) \subseteq \mathrm{Hom}(P, \mathrm{rad}^n_A(P))$

(2) $\mathrm{rad}^n_{\mathrm{End}(P)}(\mathrm{Hom}(P, M)) \subseteq \mathrm{Hom}(P, \mathrm{rad}^n_A(M))$

Proof. For the first assertion, if $n = 0$ is obvious, we have equality.
If $\Phi \in J(E)$, we still have by Lemma 1-[15], $\Phi(P) \subseteq \mathrm{rad}_A(P)$, because otherwise, there exists a maximal submodule m of P such that $\Phi(P) + m = P$ but $m \neq P$ and this contradicts that $\Phi(P)$ is small in P, and then consequently $\Phi \in \mathrm{Hom}(P, \mathrm{rad}_A(P))$ and the inclusion is true for $n = 1$.
Similarly, we have, $J^2(E) = \{\sum g \circ f / f \in J(E) \textit{ et } g \in J(E)\}$,
then $f(P) \subseteq \mathrm{rad}_A(P)$, and $g(f(P)) \subseteq g(\mathrm{rad}_A(P))$, that is to say $g \circ f(P) \subseteq g(\mathrm{rad}_A(A)P)$, and thus $g \circ f(P) \subseteq \mathrm{rad}_A(A)g(P)$ and $g \circ f(P) \subseteq \mathrm{rad}^2_A(A)P$ because $g(P) \subseteq \mathrm{rad}_A(P)$, hence $J^2(E) \subseteq \mathrm{Hom}(P, \mathrm{rad}^2_A(P))$, so the Proposition is true by induction, while the second inclusion comes from the fact that $\mathrm{Hom}(P, M)$ is $\mathrm{End}(P)$-module on the right and $\mathrm{rad}_{\mathrm{End}(P)}(\mathrm{Hom}(P, M)) = \mathrm{Hom}(P, M).J(E)$

5. Conjecture for Artinian ring with $J^3 = 0$

Let A be an Artinian ring whose $J^3 = 0$ and let $\{P_i = Ae_i\}_{1 \leq i \leq n}$ the complete set of non-isomorphic indecomposable projective A-modules, for any simple A-module S we assume that $\mathrm{rad}^2(P(\Omega(S))) = 0$ with $P(\Omega(S))$ being the projective cover of the first syzygy

$\Omega(S)$, let I be the set such that any simple module S_i for $i \in I$ has a finite projective dimension, let $I_1 \subseteq I$ such that $top(P_j) \simeq S_{i_1}$ for $j \in I_1$ where S_{i_1} is the simple A-module of minimal projective dimension among the simple modules in $\mathrm{mod}A$ for all $i \in I_1$.

We consider the algebra $\Gamma_1 = \mathrm{End}(\underset{i\in I-I_1}{\oplus} P_i)^{op}$ and we denote $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S)$ by $\tilde{S}$ for every simple A-module S, then we have the following Proposition.

5.1 Proposition. *The ring* Γ_1 *is Artinian with:*

(1) $\mathrm{rad}^3_{\Gamma_1}(\Gamma_1) = 0$.

(2) $\mathrm{pd}(\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S)) < \infty$ *for every simple A-module S such that* $\mathrm{pd}(S) < \infty$.

(3) $\mathrm{rad}^2 P(\Omega(\tilde{S})) = 0$ *with* $P(\Omega(\tilde{S}))$ *the projective cover of the first syzygy* $\Omega(\tilde{S})$.

Proof. It is easy to verify that Γ_1 is an Artinian ring, we have by Proposition 2.1, $J^3(\Gamma_1) \subseteq \mathrm{Hom}(\Gamma_1, \mathrm{rad}^3_A(\underset{i\in I-I_1}{\oplus} P_i))$, as $J^3 = 0$, then $\mathrm{rad}^3_{\Gamma_1}(\Gamma_1) = 0$, and since there is an equivalence between the two categories $\mathrm{mod}(A)$ and $\mathrm{mod}(\Gamma_1)$ see Proposition 2.5 in [10], then the projective Γ_1-modules are of the form $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, Q)$ denoted by $\tilde{Q}$ with Q is a projective A-module and the simple Γ_1-modules are of the form $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S)$ with S a simple A-module, the same if $\mathrm{pd}(S) = m$ for a simple A-module S, then by application of the functor $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, -)$ in the projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0$$

we will have,

$$0 \longrightarrow \tilde{P_m} \longrightarrow \cdots \longrightarrow \tilde{P_2} \longrightarrow \tilde{P_1} \longrightarrow \tilde{P_0} \longrightarrow \tilde{S} \longrightarrow 0$$

so $\mathrm{pd}_{\Gamma_1}(\tilde{S}) \leq m$ because $\mathrm{rad}^2_A(P(\Omega(S))) = 0$ and $P(\Omega(\tilde{S})) = P(\tilde{\Omega(S)}) = P(\Omega(S))$ and according to Proposition 2.1, $\mathrm{rad}^2_{\Gamma_1}(\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, P(\Omega(S)))) \subseteq \mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, \mathrm{rad}^2_A(P(\Omega(S))))$ and the third assertion is verified.

5.2 Remark. The Simple A-modules $(S_i)_{i\in I-I_1}$ have finite projective dimensions so the Γ_1-simple modules are also simple, and by the third assertion in Proposition 3.1 and by Lemma 1.2, the Γ_1-simple module $\tilde{S_{i_2}}$ of minimal projective dimension among the simple modules in $\mathrm{mod}\Gamma_1$ checks $\mathrm{pd}_{\Gamma_1}(\tilde{S_{i_2}}) \leq 1$ and $\mathrm{Ext}^1_{\Gamma_1}(\tilde{S_{i_2}}, \tilde{S_{i_2}}) = 0$ by Theorem 1.3.

5.3 Theorem. *If* $\tilde{S_{i_2}}$ *is the* Γ_1*-simple module of minimal projective dimension among the simple modules of* $\mathrm{mod}\Gamma_1$*, then* $\mathrm{Ext}^1_A(S_{i_2}, S_{i_2}) = 0$

Proof. $\tilde{S_{i_2}}$ *is a simple* Γ_1*-module and* $\tilde{S_{i_2}} = \mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2})$ *with* $S_{i_2} = Ae_{i_2}/Je_{i_2}$ *where* e_{i_2} *is a primitive idempotent a simple A -module not isomorphic to* S_{i_1}*, then* $\mathrm{Ext}^1_A(\tilde{S_{i_2}}, \tilde{S_{i_2}}) = 0$ *with* $\mathrm{pd}_{\Gamma_1}(\tilde{S_{i_2}}) = 0$ *or* 1.

Thus,
- *If* $\tilde{S_{i_2}}$ *is projective, then* S_{i_2} *is projective and* $\mathrm{Ext}^1_A(S_{i_2}, S_{i_2}) = 0$.
- *If* $\tilde{S_{i_2}}$ *is not projective, then* $\mathrm{Ext}^1_A(S_{i_2}, S_{i_2}) \neq 0$*, therefore* $\mathrm{Hom}(Je_{i_2}, S_{i_2}) \neq 0$ *and the exact sequence,*

$0 \longrightarrow Je_{i_2} \longrightarrow P_{i_2} \longrightarrow S_{i_2} \longrightarrow 0$ *will not be split,*

by the same $\mathrm{pd}_A(S_{i_2}) \geq 2$ *because* $i_2 \notin I_1$ *and by application of the functor* $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, -)$*, we will have the exact sequence,*

$$0 \longrightarrow \mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, Je_{i_2}) \longrightarrow \mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, P_{i_2}) \longrightarrow \mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2}) \longrightarrow 0$$

and since $\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, Je_{i_2})$ *is not projective, then* $\mathrm{pd}_{\Gamma_1}(\mathrm{Hom}(\underset{i\in I-I_1}{\oplus} P_i, S_{i_2}) > 1$ *i.e.* $\mathrm{pd}_{\Gamma_1}(\tilde{S_{i_2}}) > 1$ *which is absurd, thus* $\mathrm{Ext}^1_A(S_{i_2}, S_{i_2}) = 0$

Now, to deduce the proof of our main contribution, namely Theorem 1.1. In fact, by induction and in the same way, we consider the Artin algebras,
$\Gamma_k = \mathrm{End}(\underset{i\in I-I_1\cup I_2\cup I_3...\cup I_k}{\oplus} P_i)^{op}$ where $I_k \subseteq I$ such as $top(P_i) \simeq S_{i_{k+1}}$ for all index $i \in I - I_1 \cup I_2 \cup I_3 ... \cup I_k$ and $S_{i_{k+1}}$ be a simple module of minimal projective dimension among the simple modules in modΓ_k for all $i \in I - I_1 \cup I_2 \cup I_3 ... \cup I_k$, and $\mathrm{Ext}^1_A(S_{i_k}, S_{i_k}) = 0$, then we get the result.

6. Theoretical application as a corollary

6.1 Corollary. *Let* A *be an Artinian ring with radical cubed zero such that the projective cover of* $\mathrm{rad}(A)$ *is of Loewy length two. If* $gdim(A)$ *is finite, then* $\mathrm{Ext}^1_A(S, S) = 0$ *for every simple module* S.

Proof. If $\mathrm{rad}^2(P(\mathrm{rad}(A)) = 0$, then $\mathrm{rad}^2(P(\Omega(S))) = 0$ for all simple A-modules indeed $A/\mathrm{rad}(A) = \underset{i\in J}{\oplus} S_i^{a_i}$ where the $(S_i)_{i\in J}$ are all the A non-isomorphic simple modules and as we have the exact sequence,

$$0 \longrightarrow \mathrm{rad}(A) \longrightarrow A \longrightarrow A/\mathrm{rad}(A) \longrightarrow 0$$

then,

$\Omega(A/\mathrm{rad}(A)) = \mathrm{rad}(A)$ and if $\mathrm{rad}^2(P(\mathrm{rad}(A))) = 0$, then,
$\mathrm{rad}^2(P(\Omega(A/\mathrm{rad}(A)))) = 0$, then we have $\mathrm{rad}^2(P(\Omega(S_i))) = 0$ and according to the Theorem 3.4, we obtain $\mathrm{Ext}^1_A(S, S) = 0$.

7. Discussions by practical examples

7.1. Couterexample to Theorem 2.3

The condition $\mathrm{pd}(S) \leq 1$ is not necessary but its consideration let us be sure to get $\mathrm{Ext}^1_A(S,S) = 0$ which is the more practical result as it leads to the existence of the arrows in the extension quiver. This is just to say that we should not deny that there may be some particular cases where we may have $\mathrm{Ext}^1_A(S,S) = 0$ even if $\mathrm{pd}(S) > 1$ as we can illustrate in the following example.

7.1 Example. Consider the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

and let $A = KQ/I$ be its quiver algebra bounded by $I =< abc, cd >$.

We have the list of all projective and indecomposable modules,
$P(1) : K \longrightarrow K \longrightarrow K \longrightarrow 0 \longrightarrow 0$
$P(2) : 0 \longrightarrow K \longrightarrow K \longrightarrow K \longrightarrow 0$
$P(3) : 0 \longrightarrow 0 \longrightarrow K \longrightarrow K \longrightarrow 0$
$P(4) : 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow K \longrightarrow K$
$P(5) : 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow K$

and also, $S(1) : K \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ is the simple module corresponding to vertices 1 which is injective and coincides by $I(1)$.
Then, $\mathrm{Ext}^1_A(S(1), S(1)) = 0$ because $S(1)$ is an injective module and we have,

$$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

which implies that $\mathrm{pd}(S) = 3$

7.2. Weak no loop conjecture

In what follows, we provide an example for the resolution of the weak no loop conjecture, by taking a quiver algebra A verifying $J^3 = 0$ and without considering that $rad^2(P(\Omega(S))) = 0$ for every simple module and that is to say that we may have some of these modules checking this condition. This is introduced just to help the reader to get used and understand the steps followed in our main final example and which the strong no loop conjecture.

7.2 Example. Consider the quiver defined by:

$$\begin{tikzcd} & & & 4 \\ 1 \arrow[r, "a"] & 2 \arrow[r, shift left, "b"] \arrow[r, shift right, "c"'] & 3 \arrow[ur, "d"] \arrow[dr, "e"'] & \\ & & & 5 \end{tikzcd}$$

and let $A = KQ/I$ its quiver algebra bounded by $I =< ab, ac >$ It's clear that any path of length 3 is null therefore $\text{rad}^3(A) = 0$ and we listed the way how to find the resolution projectives of any simple A-module in our example:
we have

$$\begin{tikzcd} & & & 0 \\ P(1) : K \arrow[r] & K \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\ & & & 0 \end{tikzcd}$$

$$\begin{tikzcd} & & & K^2 \\ P(2) : 0 \arrow[r] & K \arrow[r, shift left] \arrow[r, shift right] & K^2 \arrow[ur] \arrow[dr] & \\ & & & K^2 \end{tikzcd}$$

$$\begin{tikzcd} & & & K \\ P(3) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & K \arrow[ur] \arrow[dr] & \\ & & & K \end{tikzcd}$$

$$\begin{tikzcd} & & & K \\ P(4) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\ & & & 0 \end{tikzcd}$$

$$\begin{tikzcd} & & & 0 \\ P(5) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\ & & & K \end{tikzcd}$$

and we also have,

$0 \longrightarrow \Omega(S(1)) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$

with,

$$\begin{tikzcd}
 & & & 0 \\
\Omega(S(1)) = S(2) : 0 \arrow[r] & K \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\
 & & & 0
\end{tikzcd}$$

and we can check that $\mathrm{rad}^2(P(\Omega(S(1))) \neq 0$.

Then, we have the exact sequence,

$0 \longrightarrow \Omega(S(2)) \longrightarrow P(2) \longrightarrow \Omega(S(1)) \longrightarrow 0$

with,

$$\begin{tikzcd}
 & & & K^2 \\
\Omega(S(2)) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & K^2 \arrow[ur] \arrow[dr] & \\
 & & & K^2
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & K^2 \\
\mathrm{rad}(\Omega(S(2))) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\
 & & & K^2
\end{tikzcd}$$

then, $\Omega(S(2))/rad(\Omega(S(2))) = S(3)^2$, and we have the exact sequence:
$0 \longrightarrow \Omega^2(S(2)) \longrightarrow P(3)^2 \longrightarrow (S(3))^2 \longrightarrow 0$ with

$$\begin{tikzcd}
 & & & K^2 \\
\Omega^2(S(2)) : 0 \arrow[r] & 0 \arrow[r, shift left] \arrow[r, shift right] & 0 \arrow[ur] \arrow[dr] & \\
 & & & K^2
\end{tikzcd}$$

then we have,

$$0 \longrightarrow P(4)^2 \bigoplus P(5)^2 \longrightarrow \Omega^2(S(2)) \longrightarrow 0$$

and finally the projective resolution of $S(1)$ is,

$$0 \longrightarrow P(4)^2 \bigoplus P(5)^2 \longrightarrow (P(3))^2 \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

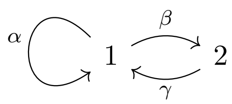
In the same way, we get,
$0 \longrightarrow P(4)^2 \bigoplus P(5)^2 \longrightarrow (P(3))^2 \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0$
$0 \longrightarrow P(4) \bigoplus P(5) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$
$0 \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow S(5) \longrightarrow 0$

By using the formula,
$\mathrm{Hom}(P(i), S(j)) = \mathrm{Hom}(Ae_i, S(j)) \simeq e_i S(j) = \begin{cases} K & if \quad j = i \\ 0 & if \quad j \neq i \end{cases}$
and by application of the functor $\mathrm{Hom}(-, S(1))$ to the resolution $P_*(S(1))$, i.e. the complex obtained by eliminating of $S(1)$ in the resolution projective of $S(1)$ we find the complex: $0 \longrightarrow K \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ and $\mathrm{Ext}^1_A(S(1), S(1)) = 0$.
Thus, by the same way, $\mathrm{Ext}^1_A(S(i), S(i)) = 0$ for every $1 \leq i \leq 5$.

7.3. Strong no loop conjecture without necessary specification of the nilpotence index of the Jacobson radical

The following example clearly shows that if the extension quiver has a loop in a simple module then its projective dimension, is infinite although it is the nilpotence index of the Jacobson radical

7.3 Example. Let Q the quiver defined by



and let $A = KQ/I$ which is algebra bounded by the relations $\alpha^2 - \beta\gamma, \gamma\alpha\beta$ and $\gamma\beta$. By definition, in the quiver algebra, we get $\mathrm{Ext}^1_A(S(1), S(1)) = 0$.

Then, by further calculations, we can find the infinite resolution of $S(1)$ as following,

$$... \longrightarrow P(2) \longrightarrow P(1) \longrightarrow P(1) \bigoplus P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

with,

$$P(1): \begin{tikzcd} K^4 \arrow[loop left] \arrow[r, bend left] & K^2 \arrow[l, bend left] \end{tikzcd}$$

and,

$$P(2): \begin{tikzcd} K^2 \arrow[loop left] \arrow[r, bend left] & K \arrow[l, bend left] \end{tikzcd}$$

7.4. Strong no loop conjecture finally resolved in $J^3 = 0$

In this paper, we have succeed to resolve the strong no loop conjecture posed in [14] but here in the special case of $J^3 = 0$ under the condition that the radical's square of the cover projective of the first syzygy is zero, while using more tractable background focusing on non-commutative algebra and homology instead of some combination with K-theory as used in the past authors in [9] to solve this conjecture at least in the case of $J^2 = 0$, however, the problem has remained open since that time for the case of $J^3 = 0$ and which is of course very important to resolve for getting some inspiration and try to resolve further cases like $J^4 = 0$ or more generally $J^n = 0$, either by introducing one condition like we have done here by taking a zero radical's square of the cover projective of the first syzygy, or maybe some would think about more additional conditions in the future.

In what follows we will present an example of an Artin algebra with Jacobson cube radical 0 and that also satisfies the condition of the First lemma.

7.4 Example. Consider the algebra $A = KQ/I$ given over a field K by a quiver Q and an admissible ideal $I =< bc, ad >$ as following:

$$\begin{tikzcd} & & 3 \arrow[dr, "c"] & \\ 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[ur, "b"] \arrow[dr, "a"'] & & 5 \\ & & 4 \arrow[ur, "d"'] & \end{tikzcd}$$

Clearly, any path in A of length 3 is zero, and therefore $\text{rad}^3(KQ/I) = 0$.

Here are the calculations of the projective resolutions of simple modules $(S(i))_{1 \leq i \leq 5}$,
$0 \longrightarrow P(3) \oplus P(4) \longrightarrow P(2)^2 \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$

$0 \longrightarrow P(3) \oplus P(4) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0$
$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow S(5) \longrightarrow 0$

Then, we have $inf(\mathrm{pd}(S_i))_{1\leq i\leq 5} = \mathrm{pd}(S_2) = \mathrm{pd}(S_3) = \mathrm{pd}(S_4)$ and $\mathrm{rad}^2(P(\Omega(S))) = 0$, where $P(\Omega(S))$ the projective cover for $\Omega(S)$ for every simple module S.
Thus, $\mathrm{Ext}^1_A(S(i), S(i)) = 0$ for $i = 2; i = 3$ and $i = 4$.

8. Conclusion

Our work ultimately resolved the conjecture for Artinian rings by relying on our main results and that could be summarized on the statement of Theorem 2.1 and which we have succeeded to prove along this paper, as well as the statement of Corollary 6.1. which generalizes it, and finally without forgetting our other contribution point stated in Remark 1.1.

We may follow similar process in the future in the hope to demonstrate the conjecture in the case of zero radical cube without a condition on the syzygies but also extending this research by attacking the conjecture in the case of $J^4 = 0$ under constraints on the syzygies.

The whole approach may lead to a process of proving the conjecture in general.

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