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An Efficient Numerical Approach Based On the Adomian Chebyshev Decomposition Method for Two-Point Boundary Value Problems

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Abstract. The current manuscript devises an efficient numerical method for solving two-point nonhomogeneous Boundary Value Problems (BVPs) with Dirichlet conditions. The method is based on the application of the celebrated Adomian Decomposition Method (ADM) and, the Chebyshev polynomials. This method which refers to "Adomian Chebyshev Decomposition Method" (ACDM) is further proved to be a robust numerical method as the associated nonhomogeneous terms are successfully reinstated with a reliable Chebyshev series. Lastly, a comparative study between the acquired numerical results and the existing exact solutions of the test problems has been established to demonstrate the salient features of the devised method.

2020 Mathematics Subject Classifications: 34A30,34A34,34B15

Key Words and Phrases: Boundary Value Problems (BVPs), Adomian Decomposition Method (ADM), Chebyshev polynomials, Adomian Chebyshev Decomposition Method (ACDM), Dirichlet conditions

1. Introduction

The celebrated Adomian decomposition method (ADM) is a well-known semi-analytical method that has been attested by many researchers to be an effective approach for solving wide classes of differential, integral and, integro-differential equations. The method reveals the solution as a rapidly convergent series that tends to accurately converge to the exact analytical solution [2, 3, 7, 8, 11, 24]. Additionally, the method is so flexible to be applied to the governing functional equations directly without the need of either

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restricting the method/solution to a particular domain, or by making use of certain simplifications like perturbation, discretization, or linearization [23] . Furthermore, like any other mathematical method in the literature, various researchers have equally proposed different extensions and modifications to the ADM, thereby improving the convergence rate of the method and, also, extending its applicability, [6, 10, 12, 13, 20]. In addition, the utilization of Green's function approach on the BVPs for treating ordinary differential equations (ODEs) has been overcome with the invention of the ADM[5, 21]; as the acquisition of approximate analytic solutions are attained and greatly facilitated. There are numerous efficient methods based on the ADM for solving BVPs of ODEs such as the double decomposition method [3, 4] and, the Duan-Rach modified decomposition method [16] to mention a few.

However, in the current manuscript, we shall devise an efficient numerical method based on the application of the ADM and, Chebyshev polynomials to solve a class of two-point nonhomogeneous BVPs with Dirichlet boundary conditions. This method that called ACDM makes use of the traditional ADM together with the Chebyshev orthogonal polynomials to solve BVPs. More, the method first appeared in [19] to solve only certain initial value problems and; thus, the method is extended in the present study in order to be able to cover a wide range of applications. Additionally, a comparative study between the acquired numerical results by the devised method and the existing exact solutions of the test problems will be established to demonstrate certain salient features of the devised method.

Lastly, we arrange the manuscript in the following way: in sections 2 and 3, we present the methods based on the traditional ADM and ACDM for solving BVPs with Dirichlet boundary conditions, respectively. Section 4 gives the numerical application of the devised method in Section 3; while Section 5 gives some concluding notes.

2. Adomian Decomposition Method (ADM) for BVPs

Let us consider the generalized second-order nonlinear version of the nonhomogeneous ODE expressed in operator form as follows

$$Lu(x) + Ru(x) + Nu(x) = g(x), \tag{1}$$

together with the following Dirichlet boundary conditions

$$u(\alpha_1) = \beta_1 \text{ and } u(\alpha_2) = \beta_2, \tag{2}$$

where β_1 and β_2 are real constants, L is the highest linear operator and, R is an operator with degree less than of $L; N$ is the nonlinear operator, while the function $g(x)$ on the other side of the above equation is a prescribed source term, which is a given continuous function.

What's more, if the operator L is assumed to be invertible and, further apply its corresponding inverse operator L^{-1} to all the parts of Eq. (1), we get after solving for $Lu(x)$ the following expression for $u(x)$

$$u(x) = \Phi(x) + L^{-1}(g(x)) - L^{-1}(Ru(x)) - L^{-1}(Nu(x)), \tag{3}$$

where $\Phi(x)$ satisfies $L\Phi(x) = 0$. In particular, since we are considering a generalized second-order nonlinear nonhomogeneous ODE, let us consider the inverse operator of L to be a two-fold definite integration of the form

$$L^{-1}(.) = \int_{\alpha_1}^{x}\int_{\alpha_1}^{x}(.)dxdx. \tag{4}$$

Appling L^{-1} defined above to all the parts of Eq. (1) and using the boundary condition $u(\alpha_1) = \beta_1$ yields

$$u(x) = u(\alpha_1) + z(x - \alpha_1) + L^{-1}g(x) - L^{-1}Ru - L^{-1}Nu, \tag{5}$$

where $z = u'(\alpha_1)$. Next, by decomposing the solution u and the nonlinear term Nu to series forms, say $u = \sum_{n=0}^{\infty} u_n$ and, $Nu = \sum_{n=0}^{\infty} A_n$, where the components $u_n(x), n \geq 0$, are to be determined in a recursive manner, and A_n 's are the Adomian polynomials that take care of the nonlinear terms and, recurrently determined via the following formula [2, 3, 7, 8, 11, 24]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{6}$$

and thereafter substitute them into the above equation yields the following

$$\sum_{n=0}^{\infty} u_n = u(\alpha_1) + z(x - \alpha_1) + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{7}$$

More so, the ADM identifies the zeroth component $u_0(x)$ with the terms emanating from the nonhomogeneous term and boundary conditions, and the rest follows recurrently as given in the following recurrence relation

$$\begin{cases} u_0(x) = u(\alpha_1) + z(x - \alpha_1) + L^{-1}g(x), \\ u_{n+1}(x) = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \end{cases} \tag{8}$$

The recurrence relation or procedure could be carried out to calculate as many components as we like. Furthermore, using the second boundary condition $u(\alpha_2) = \beta_2$ in the obtained series solution and comparing coefficients of like powers of x leads to the determination of z.

Moreover, in many BVPs of exact solutions, the obtained series provides the solution in a closed-form.

3. Adomian Chebyshev Decomposition Method (ACDM) for BVPs

Here, we present the methodology as the mixture of the ADM and, the Chebyshev polynomials. In numerical analysis, it is often important to find the polynomials P_n, of specified maximum degree n, such that, for a function $f(x)$ on the interval $[a,b], |f(x) - P_n(x)|$ has as small maximum value over $[a,b]$ as possible. Usually, this polynomials P_n is rather difficult to find, but a very good approximation to P_n can be found with relative easily. This approximation takes the form of a linear combination of a special group of polynomials that are named after the prolific mathematician Pafnuty Lvovich Chebyshev (1821-1894).

Definition 1. [14]
(1). The Chebyshev polynomials of the first kind are defined as

$$T_n(x) = \cos n\theta, \quad x = \cos\theta, \quad -1 \leq x \leq 1, \quad n = 0, 1, 2, \dots \tag{9}$$

(2). The Chebyshev polynomials of the second kind are defined as

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}, x = \cos\theta, \quad -1 \leq x \leq 1, n = 0, 1, 2, \dots \tag{10}$$

It may be instantly obvious from the above definitions that Chebyshev polynomials are indeed polynomials. Therefore, note that the trigonometric identity,

$$\cos[(n+1)\theta] + \cos[(n-1)\theta] = 2\cos\theta\cos n\theta, \tag{11}$$

supplies a recurrence relation for the functions as,

$$T_n(x) = \cos n\theta, \tag{12}$$

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \tag{13}$$

or equally expressed as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{14}$$

Since $T_0(x) = \cos 0 = 1$ and $T_1(x) = \cos\theta = x$, Eq. (14) can be used to generate any number of the $T_n(x)$ in their specific polynomial forms as

$$\begin{aligned} &T_2(x) = 2x^2 - 1, \\ &T_3(x) = 4x^3 - 3x, \\ &T_4(x) = 8x^4 - 8x^2 + 1, \\ &T_5(x) = 16x^5 - 20x^3 + 5x, \\ &\vdots \end{aligned} \tag{15}$$

Besides, many mathematical techniques have been devised based on the Chebyshev polynomials to treat different types of functional equations; more especially, with regards to its discrete orthogonality relations which were found to have considerable advantages. For instance, Hoseini [19] presented a promising modification of the ADM through the application of the Chebyshev polynomials to solve initial value problems. Moreover, we shall expand the work done in [19] by performing approximations using Chebyshev polynomials with fractional function as in the Padé method [9], but except by replacing each term of

x in Padé approximation with the Chebyshev polynomials of degree k.
Furthermore, to derive the ACDM based on the ADM and Chebyshev polynomials, we first rewrite the function $g(x)$ in Eq. (8) in terms of the Chebyshev series as follows

$$g(x) = \sum_{i=0}^{m} a_i T_i(x), \tag{16}$$

where $T_i(x)$ are the first kind of orthogonal Chebyshev polynomials, and the coefficients a_k are determined from the orthogonal polynomials as follows

$$\begin{cases} a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_0}{\sqrt{1-x^2}}dx, \\ a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}dx, k \geq 1. \end{cases} \tag{17}$$

Therefore, from Eq. (8), we, thus obtain the following ACDM recurrent relation as follows

$$\begin{cases} u_0(x) = u(\alpha_1) + z(x-\alpha_1) + L^{-1}\left(a_0T_0(x) + a_1T_1(x) + \cdots + a_mT_m(x)\right), \\ u_{n+1}(x) = -L^{-1}Ru_n - L^{-1}A_n, n \geq 0. \end{cases} \tag{18}$$

Finally, in what follows, we will be demonstrating the devised ACDM scheme given in Eq. (18) to obtain an approximate solution via $\varphi_n = \sum_{n=0}^{n-1} u_n$.

Remark 1. **[14]** *The technique for using the Chebyshev polynomial is extended to a general closed interval $[a, b]$ through the change of the following variables*

$$t = \frac{1}{2}[(b-a)x + a + b]. \tag{19}$$

4. Numerical applications

In this section, certain BVPs have been are considered as test problems to demonstrate the efficiency of the devised ACDM methodology. It is worthy to mention that all computations are performed by using Maple 18 version.

Example 1. *Consider the linear nonhomogeneous BVP [22]*

$$u'' = u + 2e^x, \quad u(0) = 0, \quad u(1) = e, \tag{20}$$

that admits the following exact analytical solution $u(x) = xe^x$.

Thus, according to the ADM, the BVP admits the following

$$\begin{cases} u_0(x) = u(0) + zx + L^{-1}g(x), \\ u_{n+1}(x) = L^{-1}u_n, \quad n \geq 0. \end{cases}$$

where $L(.) = \frac{d^2}{dx^2}(.), L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$, and, $g(x) = 2e^x$. We now make use of the Chebyshev expansion for the function $g(x)$ from Eq. (16) say, $g(x) \cong \sum_{i=0}^{10} a_i T_i(2x-1)$, where the coefficients a_0 and a_i are computed from Eq. (17) as

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(0.5+0.5x)T_0(x)}{\sqrt{1-x^2}}dx,$$
$$a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g(0.5+0.5x)T_i(x)}{\sqrt{1-x^2}}dx, i = 1, 2, \ldots 10.$$

Thus, $g(x)$ becomes

$$g(x) \cong 2.0000000000000387 + 1.99999999999064x + 1.0000000003722x^2 \\ +0.3333333275x^3 + 0.0833333792x^4 + 0.01666645453x^5 + \cdots$$

Therefore, having determined $g(x)$, the ACDM solution is recurrently given from Eq. (18) as

$$u_0 = zx + 1.0000000000000193x^2 + 0.33333333333177x^3 + 0.083333333643x^4 \\ + 0.016666666377x^5 + \cdots,$$
$$u_1 = 0.166666667zx^3 + 0.0833333333x^4 + 0.0166666666x^5 + 0.0027777777x^6 + \\ 0.0003968253899x^7 + \cdots,$$
$$\vdots$$

In the same way, we compute u_2, u_3 and u_4, then we have $\varphi_5 = \sum_{i=0}^{4} u_i$. Now, to determine z, we substitute the boundary condition at $x = 1$ in φ_5 to get

$$\varphi_5 = 1.543080630 + 1.175201168z.$$

More, φ_5 requires the boundary condition $\varphi_5(1) = e$, such after solving this relation gives $z = 1.0000000253$. Thus, since the constant z is determined, the solution in a series form follows instantly. The matching of φ_5 with the exact solution is accurate as demonstrated in Table 1. In Figure 1, we depict the curves of φ_5 and the exact analytical solution $u(x) = xe^x$ in the approximate region $0 \leq x \leq 1$.

Table 1: Absolute error between the exact and ACDM solutions in Example 1.

x	Exact solution	ACDM solution	Absolute error
0.0	0	0	0
0.1	$1.10517092 \times 10^{-1}$	$1.10517094 \times 10^{-1}$	$2.53649074 \times 10^{-9}$
0.2	$2.44280552 \times 10^{-1}$	$2.44280557 \times 10^{-1}$	$5.09836706 \times 10^{-9}$
0.3	$4.04957642 \times 10^{-1}$	$4.04957650 \times 10^{-1}$	$7.71122380 \times 10^{-9}$
0.4	$5.96729879 \times 10^{-1}$	$5.96729889 \times 10^{-1}$	$1.04002264 \times 10^{-8}$
0.5	$8.24360635 \times 10^{-1}$	$8.24360649 \times 10^{-1}$	$1.31822171 \times 10^{-8}$
0.6	1.09327128	1.09327130	$1.60211319 \times 10^{-8}$
0.7	1.40962690	1.4096269	$1.86512214 \times 10^{-8}$
0.8	1.78043274	1.78043276	$2.00216909 \times 10^{-8}$
0.9	2.21364280	2.21364282	$1.68194966 \times 10^{-8}$
1.0	2.71828183	2.71828183	$1. \times 10^{-29}$

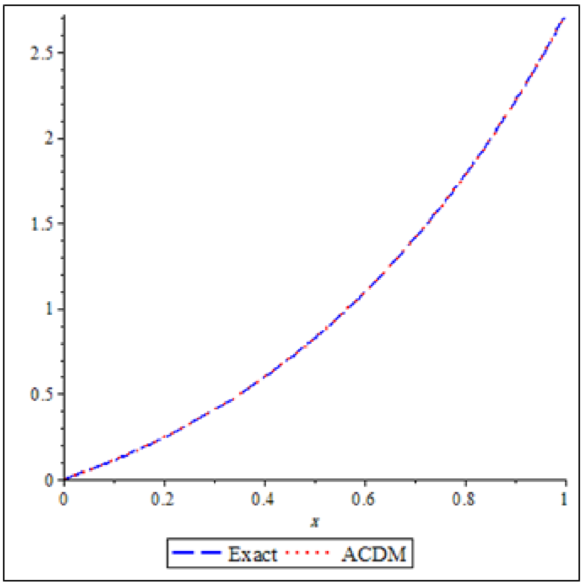

Figure 1: Comparison between the exact and ACDM solutions in Example 1.

Example 2. *Consider the nonlinear nonhomogeneous BVP [17]*

$$u'' = u^2 + 2\pi^2 \cos 2\pi x - \sin^4 \pi x, u(0) = u(1) = 0, \tag{21}$$

having the following exact analytical solution $u(x) = \sin^2 \pi x$.

Again, based on the ADM, the BVP have the following

$$\begin{cases} u_0(x) = u(0) + zx + L^{-1}g(x), \\ u_{n+1}(x) = L^{-1}A_n, \quad n \geq 0. \end{cases}$$

where Adomian polynomials A_n of a nonlinear term u^2 are determined from Eq. (6) as follows:

$$A_0 = u_0^2,$$
$$A_1 = 2u_0u_1,$$
$$A_2 = 2u_0u_2 + u_1^2,$$
$$A_3 = 2u_0u_3 + 2u_1u_2,$$
$$\vdots$$

In similar manner, $g(x)$ becomes

$$g(x) \cong 19.739432398 - 0.063110987x - 386.709633571x^2 - 52.590635088x^3 + 1664.947764846x^4 - 2526.987198109x^5 + \cdots$$

Therefore, having determined $g(x)$ above, the ACDM solution is recurrently given from Eq. (18), thus obtain $\varphi_5 = \sum_{i=0}^{4} u_i$, then we find $z = 4.213845191 \times 10^{-6}$.

Therefore, since the constant value of z is determined, the series solution of the problem follows instantly. The matching of φ_5 with the exact solution is accurate as demonstrated in Table 2. In Figure 2, we depict the curves of φ_5 and the exact analytical solution $u(x) = \sin^2 \pi x$ in the approximate region $0 \leq x \leq 1$.

Table 2: Absolute error between the exact and ACDM solutions in Example 2

x	Exact solution	ACDM solution	Absolute error
0.0	0	0	0
0.1	$9.54915028 \times 10^{-2}$	$9.54918613 \times 10^{-2}$	$3.58489171 \times 10^{-7}$
0.2	$3.45491503 \times 10^{-1}$	$3.45492103 \times 10^{-1}$	$6.00678252 \times 10^{-7}$
0.3	$6.54508497 \times 10^{-1}$	$6.54509467 \times 10^{-1}$	$9.70293389 \times 10^{-7}$
0.4	$9.04508497 \times 10^{-1}$	$9.04510189 \times 10^{-1}$	$1.69139683 \times 10^{-6}$
0.5	1.00000000	1.00000133	$1.33401846 \times 10^{-6}$
0.6	$9.04508497 \times 10^{-1}$	$9.04510994 \times 10^{-1}$	$2.49639628 \times 10^{-6}$
0.7	$6.54508497 \times 10^{-1}$	$6.54511059 \times 10^{-1}$	$2.56152588 \times 10^{-6}$
0.8	$3.45491503 \times 10^{-1}$	$3.45494307 \times 10^{-1}$	$2.80459804 \times 10^{-6}$
0.9	$9.54915028 \times 10^{-2}$	$9.54939626 \times 10^{-2}$	$2.45981055 \times 10^{-6}$
1.0	$-1.30000000 \times 10^{-26}$	$2.47124121 \times 10^{-61}$	$1.30000000 \times 10^{-26}$

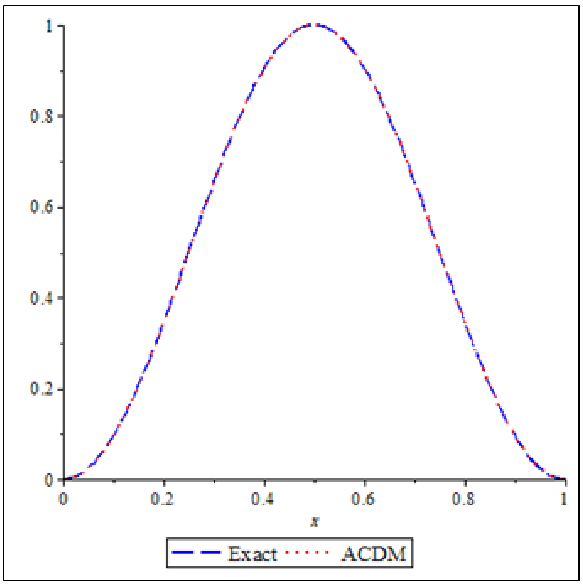

Figure 2: Comparison between the exact and ACDM solutions in Example 2

Example 3.
Consider the following nonhomogeneous BVP [18]

$$u'''(x) = u(x) - 3e^x, u(0) = 1, u(1) = u'(0) = 0, \tag{22}$$

that admits the following exact analytical solution $u(x) = (1 - x)e^x$.

Therefore, the ADM of the BVP is recurrently given as follows

$$\begin{cases} u_0(x) = u(0) + xu'(0) + \dfrac{tx^2}{2} + L^{-1}g(x), \\ u_{n+1}(x) = L^{-1}u_n, \quad n \geq 0, \end{cases}$$

where $L(.) = \frac{d^3(.)}{dx^x}, g(x) = -3e^x$ and, $L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)dxdxdx$. As proceeding before, we determine $g(x)$ as follows

$$g(x) \cong -3.0000000000000581 - 2.99999999998x - 1.500000000558x^2 - \\ 0.49999999133x^3 + \cdots$$

In similar manner, we get $t = -0.9999999999985$. Therefore, since the constant value of t is determined, the series solution of the problem follows instantly. The matching of φ_5 with the exact solution is remarkable as reported in Table 3. In Figure 3, we depict the curves of φ_5 and the exact analytical solution $u(x) = (1 - x)e^x$ in the approximate region $0 \leq x \leq 1$.

Table 3: Absolute error between the exact and ACDM solutions in Example 3

x	Exact solution	ACDM solution	Absolute error
0.0	1	1	0
0.1	$9.94653826 \times 10^{-1}$	$9.94653826 \times 10^{-1}$	$7.49228630 \times 10^{-15}$
0.2	$9.77122207 \times 10^{-1}$	$9.77122207 \times 10^{-1}$	$2.99634613 \times 10^{-14}$
0.3	$9.44901165 \times 10^{-1}$	$9.44901165 \times 10^{-1}$	$6.74469668 \times 10^{-14}$
0.4	$8.95094819 \times 10^{-1}$	$8.95094819 \times 10^{-1}$	$1.19977986 \times 10^{-13}$
0.5	$8.24360635 \times 10^{-1}$	$8.24360635 \times 10^{-1}$	$1.87622933 \times 10^{-13}$
0.6	$7.28847520 \times 10^{-1}$	7.2887520×10^{-1}	$2.70272115 \times 10^{-13}$
0.7	$6.04125812 \times 10^{-1}$	$6.04125812 \times 10^{-1}$	$3.65511599 \times 10^{-13}$
0.8	$4.45108186 \times 10^{-1}$	$4.45108186 \times 10^{-1}$	$4.56646544 \times 10^{-13}$
0.9	$2.45960311 \times 10^{-1}$	$2.45960311 \times 10^{-1}$	$4.57186783 \times 10^{-13}$
1.0	0	1×10^{-30}	1×10^{-30}

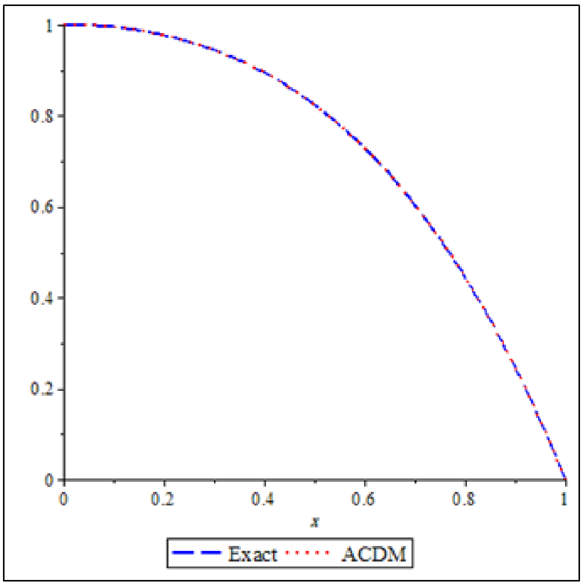

Figure 3: Comparison between the exact and ACDM solutions in Example 3

Example 4.
Consider the following nonlinear nonhomogeneous BVP [1]

$$u^{(iv)}(x) = u^2(x) - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \\ u(0) = u'(0) = 0, \quad u(1) = u'(1) = 1, \tag{23}$$

having the following exact analytical solution $u(x) = x^5 - 2x^4 + 2x^2$.

Accordingly, the ADM gives the following recurrent relation to the BVP

$$\begin{cases} u_0(x) = u(0) + xu'(0) + \dfrac{tx^2}{2!} + \dfrac{kx^3}{3!} + L^{-1}g(x), \\ u_{n+1}(x) = L^{-1}A_n, \quad n \geq 0. \end{cases}$$

where $L(.) = \frac{d^4(.)}{dx^4}$, $g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$ and $L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.) dxdxdxdx$.
In similar manner, $g(x)$ becomes

$$g(x) \cong -48 + 120x - 1 \times 10^{-29}x^2 - 4x^4 + 8x^6 - 4x^7 + \cdots .$$

Now, we determine $t = 3.9999999999$, and $k = 3.319466361 \times 10^{-17}$. Therefore, since the constant values of t and k are determined, the series solution of the problem follows instantly. The matching of φ_5 with the exact solution is amazing as reported in Table 4. In Figure 4, we depict the curves of φ_5 and that of the exact analytical solution $u(x) = x^5 - 2x^4 + 2x^2$ in the approximate region $0 \leq x \leq 1$.

Table 4: Absolute error between the exact and ACDM solutions in Example 1

x	Exact solution	ACDM solution	Absolute error
0.0	0	0	0
0.1	$1.98100000 \times 10^{-2}$	1.9810000×10^{-2}	$4.77923286 \times 10^{-20}$
0.2	$7.71200000 \times 10^{-2}$	$7.71200000 \times 10^{-2}$	$1.69039567 \times 10^{-19}$
0.3	$1.66230000 \times 10^{-1}$	$1.66230000 \times 10^{-1}$	$3.30547636 \times 10^{-19}$
0.4	$2.79040000 \times 10^{-1}$	$2.79040000 \times 10^{-1}$	$4.99126063 \times 10^{-19}$
0.5	$4.06250000 \times 10^{-1}$	$4.06250000 \times 10^{-1}$	$6.41596782 \times 10^{-19}$
0.6	$5.38560000 \times 10^{-1}$	$5.38560000 \times 10^{-1}$	$7.24810853 \times 10^{-19}$
0.7	$6.67870000 \times 10^{-1}$	$6.67870000 \times 10^{-1}$	$7.15675166 \times 10^{-19}$
0.8	$7.88480000 \times 10^{-1}$	$7.88480000 \times 10^{-1}$	$5.81377176 \times 10^{-19}$
0.9	$8.98290000 \times 10^{-1}$	$8.98290000 \times 10^{-1}$	$2.96356361 \times 10^{-19}$
1.0	1	1	$2.00000000 \times 10^{-29}$

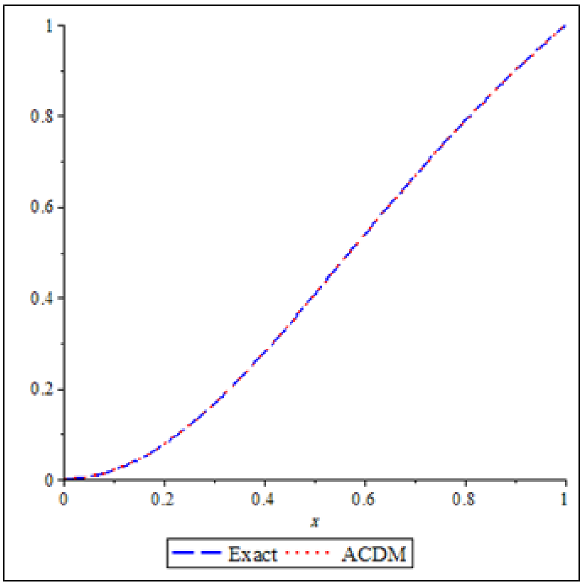

Figure 4: Comparison between the exact and ACDM solutions in Example 4

Example 5.
Consider the linear nonhomogeneous system of second-order BVPs [15]

$$\begin{cases} u''(x) + (2x-1)u'(x) + \cos(\pi x)v'(x) = g_1(x), \quad u(0) = u(1) = 0, \quad 0 \le x \le 1, \\ v''(x) + xu(x) = g_2(x), \quad v(0) = v(1) = 0, \quad 0 \le x \le 1, \end{cases} \tag{24}$$

where $g_1(x)$ and $g_2(x)$ are given by $g_1(x) = -\pi^2 \sin(\pi x) + (2x-1)\pi\cos(\pi x) + (2x-1)\cos(\pi x)$, and $g_2(x) = 2 + x\sin(\pi x)$. In addition, the following exact analytical solution set satisfies the system $u(x) = \sin(\pi x)$ and, $v(x) = x^2 - x$.

Consequently, the ADM recurrent relation for the system is given by

$$\begin{cases} u_0(x) = u(0) + z_1 x + L^{-1} g_1(x), \\ v_{0(x)} = v(0) + z_2 x + L^{-1} g_2(x), \\ u_{n+1}(x) = -L^{-1}\left((2x-1)u'_n\right) - L^{-1}\left(\cos(\pi x)v'_n\right), \quad n \ge 0, \\ v_{n+1}(x) = -L^{-1}\left(xu_n\right), n \ge 0. \end{cases}$$

we use the Chebyshev expansion to determine the series forms of $g_1(x)$ and $g_2(x)$ from Eq. (16) say, $g_1(x) \cong \sum_{i=0}^{10} a_i T_i(2x-1)$, and $g_2(x) \cong \sum_{i=0}^{10} b_i T_i(2x-1)$, where the coefficients a_0, b_0, a_i and b_i are computed from Eq. (17) as

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g_1(0.5+0.5x)T_0(x)}{\sqrt{1-x^2}}dx, \quad b_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g_2(0.5+0.5x)T_0(x)}{\sqrt{1-x^2}}dx$$
$$a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g_1(0.5+0.5x)T_i(x)}{\sqrt{1-x^2}}dx, \quad b_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g_2(0.5+0.5x)T_i(x)}{\sqrt{1-x^2}}dx \quad i = 1, 2, \dots 10.$$

Additionally, $g_1(x)$ and $g_2(x)$ are explicitly found from Eq. (16) as

$$g_1(x) \cong -4.141592658 - 22.723090021x + 20.437876252x^2 + 10.12859625x^3 + \cdots$$
$$g_2(x) \cong 1.999999988 + 0.000002814x + 3.141480108x^2 + 0.001753695x^3 + \cdots$$

Now, we determine $z_1 = 3.141584852$, and $z_2 = -1.0000214289$. Therefore, since the constant values of z_1 and z_2 are determined above, the series solution of the problem follows instantly. The matching of $\psi_5 = \sum_{i=0}^{4} v_i$ with the exact solution is commendable as reported in Table 5. In Figure 5, we depict the curves of ψ_5 and the exact analytical solution $v(x) = x^2 - x$ in the approximate region $0 \leq x \leq 1$.

Table 5: Absolute error between the exact and ACDM solutions $v(x)$ in Example 5

x	ACDM solution $\boldsymbol{\psi_5(x)}$	Exact solution $\boldsymbol{v(x)}$	Absolute error
0.0	0	0	0
0.1	$-9.00021428 \times 10^{-2}$	$-9.00000000 \times 10^{-2}$	$2.14282294 \times 10^{-6}$
0.2	$-1.60004284 \times 10^{-1}$	$-1.60000000 \times 10^{-1}$	$4.28395716 \times 10^{-6}$
0.3	$-2.10006407 \times 10^{-1}$	$-2.10000000 \times 10^{-1}$	$6.40680363 \times 10^{-6}$
0.4	$-2.40008430 \times 10^{-1}$	$-2.40000000 \times 10^{-1}$	$8.43004588 \times 10^{-6}$
0.5	$-2.50010128 \times 10^{-1}$	$-2.50000000 \times 10^{-1}$	$1.01279366 \times 10^{-5}$
0.6	$-2.40011187 \times 10^{-1}$	$-2.4000000 \times 10^{-1}$	$1.10872057 \times 10^{-5}$
0.7	$-2.10010776 \times 10^{-1}$	$-2.10000000 \times 10^{-1}$	$1.07759673 \times 10^{-5}$
0.8	$-1.60008755 \times 10^{-1}$	$-1.60000000 \times 10^{-1}$	$8.75509042 \times 10^{-6}$
0.9	$-9.00049736 \times 10^{-2}$	$-9.00000000 \times 10^{-2}$	$4.97362094 \times 10^{-6}$
1.0	$-3.45366895 \times 10^{-28}$	0	$3.45366895 \times 10^{-28}$

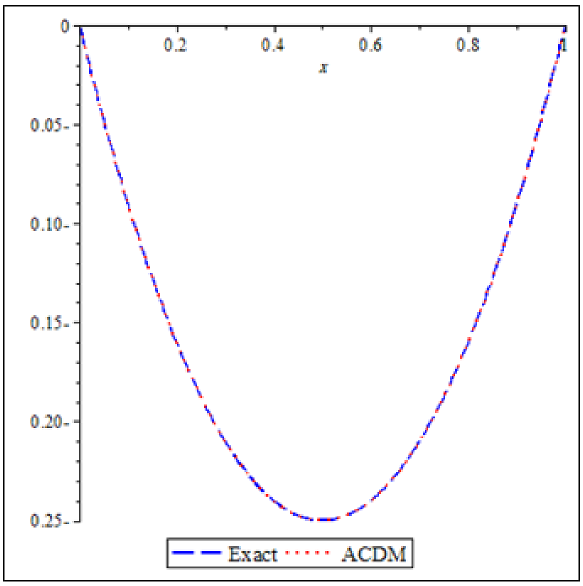

Figure 5: Comparison between the exact and ACDM solutions in Example 5 for $v(x)$

The matching of $\varphi_5 = \sum_{i=1}^{4} u_i$ with the exact solution is commendable as reported in Table 6. In Figure 6, we depict the curves of φ_5 and the exact analytical solution $u(x) = \sin(\pi x)$ in the approximate region $0 \le x \le 1$.

Table 6: Absolute error between the exact and ACDM solutions $u(x)$ in Example 5

x	ACDM solution $\boldsymbol{\varphi_5(x)}$	Exact solution $\boldsymbol{u(x)}$	Absolute error
0.0	$1.00000000 \times 10^{-27}$	0	$1.00000000 \times 10^{-27}$
0.1	$3.09016283 \times 10^{-1}$	$3.09016994 \times 10^{-1}$	$7.11807212 \times 10^{-7}$
0.2	$5.87783841 \times 10^{-1}$	$5.87785252 \times 10^{-1}$	$1.41152139 \times 10^{-6}$
0.3	$8.09014318 \times 10^{-1}$	$8.09016994 \times 10^{-1}$	$2.67611588 \times 10^{-6}$
0.4	$9.51051322 \times 10^{-1}$	$9.51056516 \times 10^{-1}$	$5.19434595 \times 10^{-6}$
0.5	$9.99991110 \times 10^{-1}$	1	$8.89040303 \times 10^{-6}$
0.6	$9.51043993 \times 10^{-1}$	$9.51056516 \times 10^{-1}$	1.2523000×10^{-5}
0.7	$8.09002815 \times 10^{-1}$	$8.09016994 \times 10^{-1}$	$1.41795077 \times 10^{-5}$
0.8	$5.87772869 \times 10^{-1}$	$5.87785252 \times 10^{-1}$	$1.23837067 \times 10^{-5}$
0.9	$3.09009889 \times 10^{-1}$	$3.09016994 \times 10^{-1}$	$7.10572450 \times 10^{-6}$
1.0	$-3.03006339 \times 10^{-27}$	$-4.97115803 \times 10^{-31}$	$3.02956627 \times 10^{-27}$

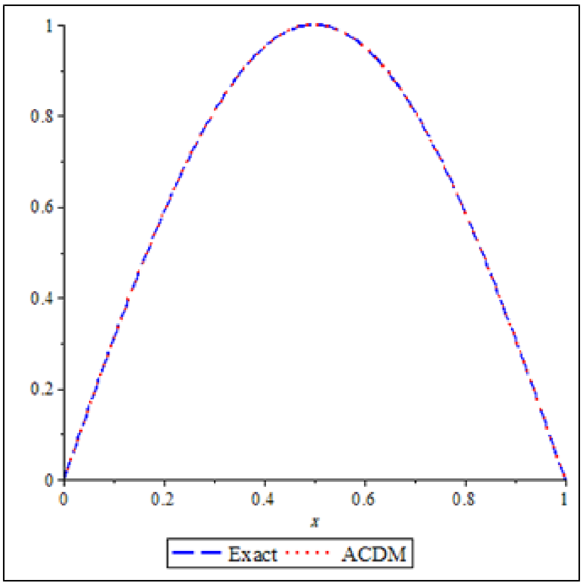

Figure 6: Comparison between the exact and ACDM solutions in Example 5 for $u(x)$

Example 6.
Consider the nonlinear nonhomogeneous system of second-order BVPs [15]

$$\begin{cases} u''(x) + xu'(x) + \cos(\pi x)v'(x) = g_1(x), & u(0) = u(1) = 0, \quad 0 \le x \le 1, \\ v''(x) + xu'(x) + x\left(v'(x)\right)^2 = g_2(x), & v(0) = v(1) = 0, \quad 0 \le x \le 1, \end{cases} \tag{25}$$

where $g_1(x)$ and $g_2(x)$ are given by $g_1(x) = \sin(x) + \left(x^2 - x + 2\right)\cos(x) + (1 - 2x)\cos(\pi x)$, $g_2(x) = -2 + x\sin(\pi x) + \left(x^2 - x\right)\cos(x) + x(1 - 2x)^2$. *Additionally, the above system admits the following exact analytical solution* $u(x) = (x - 1)\sin(x)$ *and* $v(x) = x - x^2$.

As preceded, the recurrent relation via the ADM takes the following form

$$\begin{cases} u_0(x) = u(0) + z_1 x + L^{-1} g_1(x), \\ v_0(x) = v(0) + z_2 x + L^{-1} g_2(x), \\ u_{n+1}(x) = -L^{-1} x u_n' - L^{-1}\cos(\pi x) v_n', & n \geq 0, \\ v_{n+1}(x) = -L^{-1} x u_n' - L^{-1} x A_n, & n \geq 0, \end{cases}$$

From Eq. (6), the nonlinear term $Nv = (v')^2$ is also expressed through the following Adomian polynomials A_n are given as

$$\begin{aligned} A_0 &= (v_0')^2, \\ A_1 &= 2(v_0')(v_1'), \\ A_2 &= 2(v_0')(v_2') + (v_1')^2, \\ A_3 &= 2(v_0')(v_3') + 2(v_1')(v_2'), \\ &\vdots \end{aligned}$$

In similar manner, using the relation given in Eq. (16), we equally obtain

$$\begin{aligned} g_1(x) &\cong 3.000000001 - 2.000000476x - 4.934779553x^2 + 10.20251612x^3 + \cdots, \\ g_2(x) &\cong -2.0 - 1.5896928 \times 10^{-10} x - 2.0x^2 + 4.500000096x^3 + \cdots \end{aligned}$$

So, we find $z_1 = -1.000001152, z_2 = 0.999830865$. Therefore, since the constant values of z_1 and z_2 are determined, the series solution of the problem follows instantly. The matching of $\psi_5 = \sum_{i=0}^{4} v_i$ with the exact solution is demonstrated in Table 7. In Figure 7 we depict the curves of ψ_5 and the exact analytical solution $v(x) = x - x^2$ in the approximate region $0 \leq x \leq 1$.

Table 7: Absolute error between the exact and ACDM solutions $v(x)$ in Example 6.

x	ACDM solution $\boldsymbol{\psi}_5(\boldsymbol{x})$	Exact solution $\boldsymbol{v}(\boldsymbol{x})$	Absolute error
0.0	$1.24003340 \times 10^{-22}$	0	$1.24003340 \times 10^{-22}$
0.1	$8.99831359 \times 10^{-2}$	$9.00000000 \times 10^{-2}$	$1.68640966 \times 10^{-5}$
0.2	$1.59966511 \times 10^{-1}$	1.6000000×10^{-1}	$3.34886510 \times 10^{-5}$
0.3	$2.09950234 \times 10^{-1}$	2.1000000×10^{-1}	$4.97663491 \times 10^{-5}$
0.4	$2.39934409 \times 10^{-1}$	2.4000000×10^{-1}	$6.55905833 \times 10^{-5}$
0.5	$2.49919498 \times 10^{-1}$	2.5000000×10^{-1}	$8.05024359 \times 10^{-5}$
0.6	$2.39906904 \times 10^{-1}$	2.4000000×10^{-1}	$9.30960974 \times 10^{-5}$
0.7	$2.09900091 \times 10^{-1}$	2.1000000×10^{-1}	$9.99091708 \times 10^{-5}$
0.8	$1.59906373 \times 10^{-1}$	1.6000000×10^{-1}	$9.36273128 \times 10^{-5}$
0.9	$8.99368847 \times 10^{-2}$	9.000000×10^{-2}	$6.31153078 \times 10^{-5}$
1.0	$1.29287648 \times 10^{-21}$	0	$1.29287648 \times 10^{-21}$

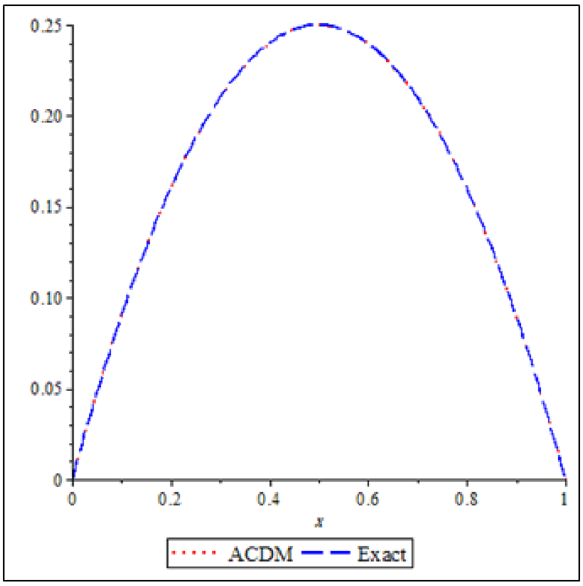

Figure 7: Comparison between the exact and ACDM solutions in Example 6 for $v(x)$

The matching of φ_5 with the exact solution is commendable as reported in Table 8. In Figure 8, we depict the curves of $\varphi_5 = \sum_{i=0}^{4} u_i$ and the exact analytical solution $u(x) = (x-1)\sin(x)$ in the approximate region $0 \leq x \leq 1$.

Table 8: Absolute error between the exact and ACDM solutions $u(x)$ in Example 6.

x	ACDM solution $\boldsymbol{\varphi_5(x)}$	Exact solution u$(\boldsymbol{x})$	Absolute error
0.0	$-2.15883226 \times 10^{-15}$	0	$2.15883226 \times 10^{-15}$
0.1	$-8.98493539 \times 10^{-2}$	$-8.98500750 \times 10^{-2}$	$7.21035532 \times 10^{-7}$
0.2	$-1.58932458 \times 10^{-1}$	$-1.558935465 \times 10^{-1}$	$3.00633437 \times 10^{-6}$
0.3	$-2.06857574 \times 10^{-1}$	$-2.06864145 \times 10^{-1}$	$6.57114337 \times 10^{-6}$
0.4	$-2.33639930 \times 10^{-1}$	$-2.3365100 \times 10^{-1}$	$1.10754214 \times 10^{-5}$
0.5	$-2.39696515 \times 10^{-1}$	$-2.39712769 \times 10^{-1}$	$1.62538649 \times 10^{-5}$
0.6	$-2.25835101 \times 10^{-1}$	$-2.25856989 \times 10^{-1}$	$2.18878914 \times 10^{-5}$
0.7	$-1.93237792 \times 10^{-1}$	$-1.93265306 \times 10^{-1}$	2.7514182910^{-5}
0.8	$-1.43439660 \times 10^{-1}$	$-1.43471218 \times 10^{-1}$	$3.15577978 \times 10^{-5}$
0.9	$-7.83043755 \times 10^{-2}$	$-7.83326910 \times 10^{-2}$	$2.83154484 \times 10^{-5}$
1.0	$-1.48468813 \times 10^{-14}$	0	$1.48468813 \times 10^{-14}$

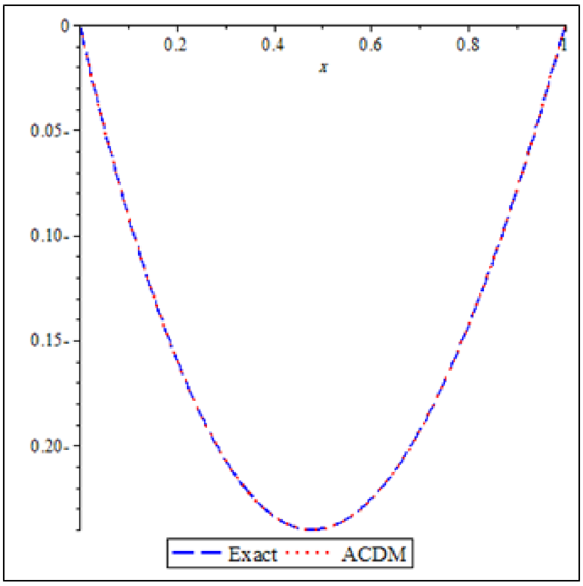

Figure 8: Comparison between the exact and ACDM solutions in Example 6 for $u(x)$.

5. Conclusion

In conclusion, the current manuscript devised an efficient numerical method based on the application of the celebrated ADM and, Chebyshev polynomials for solving two-point nonhomogeneous BVPs with Dirichlet boundary conditions. The devised method that is called the ACDM was further successfully applied to some test problems comprising of both linear and nonlinear problems, including coupled nonlinear systems. Additionally, a comparative study between the acquired numerical results and the existing exact solutions of the test problems was established to demonstrate certain salient features of the devised method. Lastly, the devised method is proved to be a robust numerical method for solving various classes of functional equations and, can be broadened to tackle other problems with different types of conditions.

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