EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1842-1854
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Comparative Analysis of Conformable, Non-conformable, Riemann-Liouville, and Caputo Fractional Derivatives

A. Ait Brahim[1,*], J. El Ghordaf[1], A. El Hajaji[2], K. Hilal[1], J. E.Nápoles Valdes[3]

[1] *AMSC Laboratory, University of Sciences and Technology , Beni Mellal, Morocco*
[2] *OEE Departement, ENCGJ, University of Chouaib Doukali, El jadida, Morocco*
[3] *FACENA Laboratory Universidad Nacional del Nordeste , (3400) Corrientes, Argentina*

Abstract. This study undertakes a comparative analysis of the non conformable and conformable fractional derivatives alongside the Riemann-Liouville and Caputo fractional derivatives. It examines their efficacy in solving fractional ordinary differential equations and explores their applications in physics through numerical simulations. The findings suggest that the conformable fractional derivative emerges as a promising substitute for the non conformable, Riemann-Liouville and Caputo fractional derivatives within the range of order α where $1/2 < \alpha < 1$.

2020 Mathematics Subject Classifications: 26A33, 34A08, 65R20, 35R11

Key Words and Phrases: Conformbale fractional derivative, non-conformable fractional derivative, Riemann-Liouville, Caputo fractional derivatives

1. Introduction

The principles of classical calculus, which form the bedrock of mathematical analysis, have been instrumental in understanding and modeling various phenomena across science and engineering. However, there exist numerous real-world phenomena that defy straightforward modeling using classical calculus due to their non-local behavior or long-term memory effects. Fractional calculus emerges as a powerful framework to address such complexities by extending the traditional notions of derivatives and integrals to non-integer orders [1–3, 5]. In recent years, fractional calculus has garnered significant attention across a multitude of scientific and engineering disciplines. This surge in interest is driven by its remarkable capacity to capture and describe phenomena that exhibit intricate temporal dependencies or spatial interactions. Whether it's the diffusion of particles in porous media, the dynamics of complex networks, or the behavior of viscoelastic

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5237

Email addresses: abdessamad11117@gmail.com (A. Ait Brahim),
a_elhajaji@yahou.fr (A. El Hajaji), hilalkhalid2005@yahou.fr (K. Hilal),
elg_jalila@yahou.fr (J. El Ghordaf), jnapoles@exa.unne.edu.ar (J.N. Valdes)

materials, fractional calculus provides a versatile toolkit for modeling and analyzing these systems with unprecedented accuracy and depth. The applications of fractional calculus span a wide spectrum, encompassing fields such as physics, biology, finance, signal processing, and control theory, among others. From characterizing anomalous diffusion processes in biological systems to optimizing the performance of control systems with fractional-order controllers, the influence of fractional calculus permeates numerous facets of modern science and technology. In this context, the significance of fractional calculus extends far beyond its theoretical foundations; it serves as a practical bridge between theoretical insights and real-world applications. As researchers continue to explore and refine the methodologies within this field, the potential for breakthroughs in understanding complex phenomena and developing innovative solutions grows ever greater. In this dynamic landscape, the exploration of fractional calculus promises to yield profound insights and transformative advancements across diverse domains, shaping the future of science and engineering in profound ways. During the late 1950s and early 1960s, local non-integer differential operators were introduced, defined in terms of incremental quotients. However, their utilization remained sporadic over the following five decades, hindering widespread dissemination and understanding of their potential. It wasn't until 2014 that the first comprehensive formalization of these local operators emerged, marking a significant milestone in the development of non-integer order calculus. This contribution addressed the limitations of global operators. R. Khalil et al. in their article "A new definition of the fractional derivative" (refer to [9]), introduced the concept of the "Conformable fractional derivative" with the following definition: let $\alpha \in]0,1]$ and $M :]0,+\infty[\longrightarrow \mathbb{R}$. Therefore, the conformable fractional derivative of order α at $t_0 \succ 0$ is defined by

$$M^{(\alpha)}(t_0) = \lim_{\varepsilon \longrightarrow 0} \frac{M\left(t + \varepsilon {t_0}^{1-\alpha}\right) - M(t_0)}{\varepsilon}, \tag{1}$$

if it exists.

In fact, this novel form of derivation upholds all properties of conventional derivation except for semi-groups. R. Khalil et al., along with T. Abdeljawad, the "pioneer" of this innovative concept, laid the theoretical groundwork in their article "On Conformable fractional calculus" [3]. A. Abdelhakim [1], in a 2019 publication, demonstrated that the existence of limits is tantamount to differentiability in the classical sense, highlighting both the fractional nature of this approach and the broader significance of the theory initiated by R. Khalil et al. and T. Abdeljawad. This has sparked ongoing debates among proponents and critics of the methodology.

This paper examines the research by D. R. Anderson and D. J. Ulness published in 2015 under the title "Newly defined conformable derivatives" [4]. Building upon the provided definition, we presume ordinary differentiability. Furthermore, it reflects the viewpoints of R. Khalil et al. as referenced in [9]. Since differentiability aligns with the principles outlined in [4], we present the primary computations without proofs, establish the rules specified in [4], and introduce additional properties.

In response to these limitations, Khalil et al. introduced a new definition of fractional derivative that satisfies these properties, which are absent in Riemann-Liouville and Ca-

puto fractional derivatives. This new derivative is known as the conformable fractional derivative. Abdeljawad (2015) further refined the concept of the conformable fractional derivative, providing a more comprehensive understanding of its properties. Notably, the conformable fractional derivative is defined in a simpler manner compared to Riemann-Liouville and Caputo fractional derivatives. Among these definitions, two of the most prominent are:
If n is a positive integer and $\alpha \in [n-1, n]$ derivative of N is given by:

- Riemann-Liouville definition:

$$\mathcal{D}_a^{\alpha}(M)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{M(u)}{(t-u)^{\alpha-n+1}} du. \tag{2}$$

- Caputo definition:

If n is a positive integer and $\alpha \in [n-1, n), \alpha$ derivative of N is given by

$$\mathcal{D}_a^{\alpha}(M)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{M^{(n)}(u)}{(t-u)^{\alpha-n+1}} du. \tag{3}$$

In their work referenced as [9], R. Khalil et al. introduced a novel concept termed the "conformable fractional derivative." Unlike the traditional Riemann-Liouville and Caputo derivatives, which operate as nonlocal operators, the conformable fractional derivative functions locally, closely resembling the conventional derivative.

Furthermore, a new definition of "non-conformable" derivatives has been introduced in a separate work [11]. Although these definitions effectively apply in the case of $0 < \alpha < 1$, there arises a necessity for a general definition catering to conformable derivatives of any order, integer or otherwise, thereby extending the scope of conformable derivatives to higher orders [11]. Interested readers can also refer to additional sources such as [7], [16], and [17] for further insights. Moreover, the realm of fractional and generalized calculus has witnessed a surge in applications across various scientific and technological domains [6, 10, 13], showcasing advancements beyond the traditional integer orders. For a comprehensive understanding of historical and technical aspects, readers are encouraged to explore [14].

This study delves into the analysis and comparison of conformable, Riemann-Liouville, and Caputo fractional derivatives with order α, where $0 < \alpha < 1$. To enable a comprehensive comparison, we conduct numerical simulations to explore the solutions of fractional ordinary differential equations incorporating these three types of fractional derivatives.

2. Main results of conformable derivative

First, let's recall the definition of $N_1^{\alpha} M(t)$, which represents a non-conformable fractional derivative of a function at a point t, as defined in [13]. This definition forms the basis of our results, which closely resemble those found in classical calculus.

Definition 1. *(See[13])*

Given a function $M : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of M of order α is defined by

$$N_1^{\alpha} M(t) = \lim_{\varepsilon \to 0} \frac{M\left(t + \varepsilon e^{t^{-\alpha}}\right) - M(t)}{\varepsilon}, \tag{4}$$

for all $t > 0$, $\alpha \in (0, 1)$. If M is α-differentiable in some $(0, a)$, and $\lim_{t \to 0^+} N_1^{(\alpha)} M(t)$ exists, then define

$$N_1^{(\alpha)} M(0) = \lim_{t \to 0^+} N_1^{(\alpha)} M(t). \tag{5}$$

By following an analogous procedure to that of ordinary calculus, we can establish the following result.

Definition 2. *(See[11]) Be the function $M : [0, +\infty) \to \mathbb{R}$. The N-derivative of function M of order α is defined by*

$$N_F^{\alpha} M(t) = \lim_{\varepsilon \to 0} \frac{M(t + \varepsilon F(t, \alpha)) - M(t)}{\varepsilon}, \tag{6}$$

for all $t > 0, \alpha \in (0, 1)$ being $F(\alpha, t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}()$.

Theorem 1. *(See[11]) Let M and G be N-differentiable at a point $t > 0$ and $\alpha \in (0, 1]$:*
a) $N_F^{\alpha}(aM + bG)(t) = aN^{\alpha}{}_F(M)(t) + bN^{\alpha}{}_F(G)(t)$.
b) $N_F^{a}\left(t^P\right) = e^{t-\alpha} p t^{t-1}, p \in \mathbb{R}$.
c) If, in addition, M is differentiable then $N_F(M) = F(t, \alpha) M'(t)$.

Definition 3. *The conformable fractional derivative of M in order α is represented by*

$$\left(\mathcal{D}^{\alpha} M\right)(t) = \lim_{v \to 0} \frac{M\left(t + v e^{(\alpha-1)t}\right) - M(t)}{v}, \tag{7}$$

with , $\alpha \in [0, 1]$ and $t > 0$.
On the other hand if M is differentiable of order α in $[0, a], a > 0$, and $\lim_{t \to 0^+} \left(\mathcal{D}^{\alpha} M\right)(t)$ exists, we gets

$$\mathcal{D}^{\alpha} M(0) = \lim_{t \longrightarrow 0^+} (\mathcal{D}^{\alpha} M)(t). \tag{8}$$

Theorem 2. *(See [8])*

If M differentiable of order α at $t_0 > 0$ and $M : [0, +\infty) \longrightarrow \mathbb{R}$ then M is continuous at t_0.

Theorem 3. *(See[8]) If M be α differentiable at a point $t > 0$. We have*

(i) $\mathcal{D}^{\alpha}(aM + bM) = a\left(\mathcal{D}^{\alpha} M\right) + b\left(\mathcal{D}^{\alpha} M\right)$, for all $a, b \in \mathbf{R}$

(ii) $\left(\mathcal{D}^{\alpha}(M/H)\right) = \left(M\left(\mathcal{D}^{\alpha} H\right) + H\left(\mathcal{D}^{\alpha} M\right)\right)/H^2$.

(iii) If M is differentiable, then $\left(\mathcal{D}^{\alpha} M\right)(t) = e^{(\alpha-1)t} M'(t)$.

3. Fractional ordinary differential equations :

3.1. Fractional ordinary differential equation with non-conformable fractional derivative

The initial value problem of fractional ordinary differential equations with generalized derivative, for $0 < \alpha < 1$

$$N_F^{\alpha} y(t) = \lambda y(t), \tag{9}$$

with the initial condition $y(t_0) = y_0$ has the solution

$$y(t) = y_0 e^{\lambda(t-t_0)}. \tag{10}$$

In general form, if instead of (9) we have :

$$N_F^{\alpha} y(t) = f(t), \tag{11}$$

with the initial condition $y(t_0) = y_0$, the solution is

$$y(t) = y_0 e^{\int_{t_0}^{t} f(s) d_F^{\alpha} s} = y_0 e^{N_F J_{t_0}^{\alpha} f(t)}. \tag{12}$$

3.2. Fractional ordinary differential equation with Riemann Liouville, Caputo, and conformable fractional Derivatives

This section discusses solutions to fractional Ordinary Differential Equations using Riemann Liouville, Caputo, and conformable fractional Derivatives and then their numerical simulations. For more detail concerning these, one can refer to Abdeljawad (2015), Khalil et al. (2014), Podlubny (1999), and Kilbas (2006).

Solutions The initial value problem of fractional ordinary differential equations with Riemann-Liouville fractional derivative, for $0 < \alpha < 1$,

$${}^{R}D_t^{\alpha} w(t) = \lambda w(t), \quad t > 0, \tag{13}$$

with the initial condition

$$I_t^{1-\alpha} w(t)\big|_{t=0} = w_0, \tag{14}$$

has the solution

$$w(t) = w_0 t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \tag{15}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler Function defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{16}$$

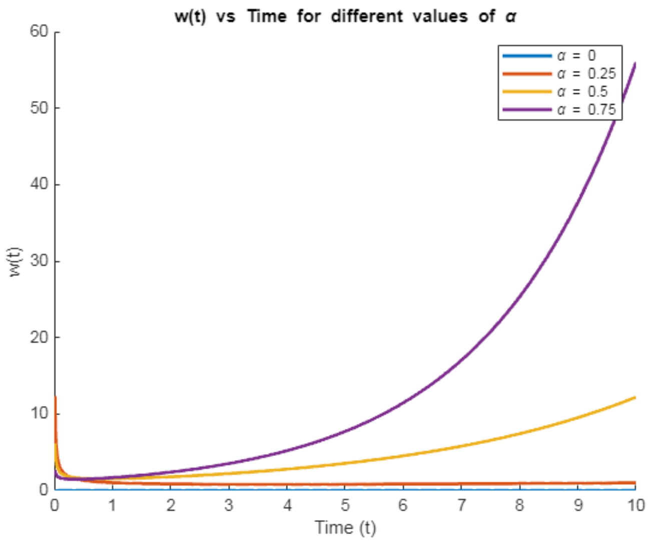

Figure 1 : Comparative solutions for serval values of α, with Riemann-Liouville fractional derivative.

The initial value problem of fractional ordinary differential equations with Caputo fractional derivative, for $0 < \alpha < 1$,

$$ {}^{C}D_t^{\alpha} w(t) = \lambda w(t), \quad t > 0, \tag{17} $$

with the initial condition

$$ w(t)|_{t=0} = w_0, \tag{18} $$

has the solution

$$ w(t) = w_0 E_{\alpha,1}\left(\lambda t^{\alpha}\right). \tag{19} $$

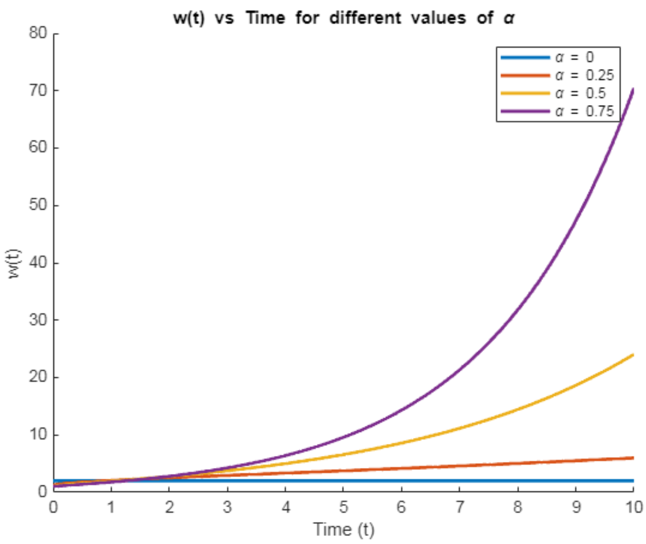

Figure 2 : Comparative solutions for serval values of α, with Caputo fractional derivative.

The initial value of fractional ordinary differential equations employing the conformable fractional derivative, with $0 < \alpha < 1$,

$$T_t^{\alpha} w(t) = \lambda w(t), \quad t > 0, \tag{20}$$

with the initial condition

$$w(t)|_{t=0} = w_0, \tag{21}$$

has the solution

$$w(t) = w_0 e^{\lambda \frac{t^{\alpha}}{\alpha}}. \tag{22}$$

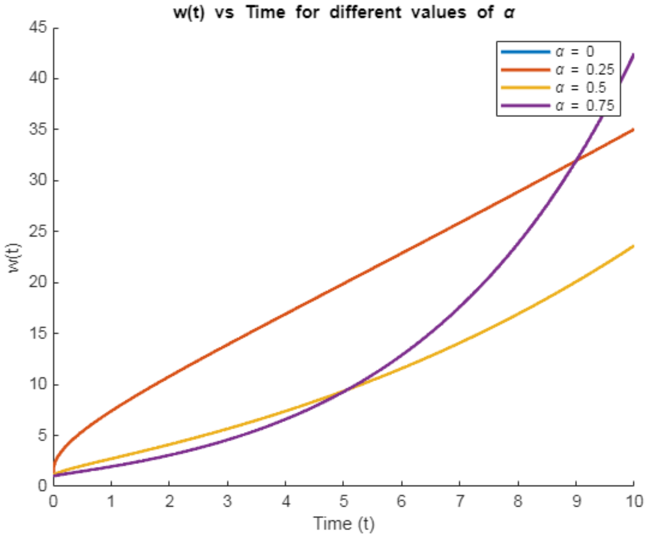

Figure 3 :Comparative solutions for serval values of α, with the conformable fractional derivative.

This study Figure 1, Figure 2 and Figure 3 delves into the analysis and comparison of conformable, Riemann-Liouville, and Caputo fractional derivatives of order α, where $0 < \alpha < 1$. To facilitate a comprehensive comparison, we conduct numerical simulations to examine the solutions of fractional ordinary differential equations that incorporate these three types of fractional derivatives.

4. Application to physics

4.1. Newton's Cooling Problem:

Newton's Cooling Problem is a fundamental concept in physics that studies how an object loses heat when exposed to a colder environment. This phenomenon, named after the famous physicist Sir Isaac Newton, is crucial in many fields, from engineering to meteorology to biology. We define the problem of Newton's cooling law states that the cooling by equation

$$\frac{dT}{dt} = -\kappa \left(T(t) - T_c\right), T(0) = T_0. \tag{23}$$

In $t = 0$ we have the initial temperature T_0, and we gets the solution:

$$T(t) = T_c + (T_0 - T_c)\, e^{-\kappa t}. \tag{24}$$

First of all, we must ensure that in the pass of equation (23) to its corresponding fractional equation, the physical parameters in it retain their physical units. Following [12], it is necessary to take into account "the distortion" suffered by the derivative when going from the ordinary case to the non-integer. We can use the physical parameter $[k] = s^{-1}$ to pass from the ordinary differential operator to the fractional one [11], as follows

$$\frac{d}{dt} \to F\left(k^{-1}, \alpha\right) N_F^{\alpha}. \tag{25}$$

In this way, the differential operator retains its dimensionality $[d/dt] = s^{-1}$, and α is the order of derivative. Susbstituting (25) in (23), we obtain the fractional model as

$$\begin{aligned} N_F^{\alpha} T(t) + F^{-1}\left(k^{-1}, \alpha\right) \kappa T(t) &= F^{-1}\left(k^{-1}, \alpha\right) \kappa \epsilon(t) \\ F\left(t, \alpha\right) T'(t) + F^{-1}\left(k^{-1}, \alpha\right) \kappa T(t) &= F^{-1}\left(k^{-1}, \alpha\right) \kappa \epsilon(t) \\ T'(t) + \frac{F^{-1}\left(k^{-1}, \alpha\right)}{F\left(t, \alpha\right)} \kappa T(t) &= \frac{F^{-1}\left(k^{-1}, \alpha\right)}{F\left(t, \alpha\right)} \kappa \epsilon(t). \end{aligned}$$

Since

$$P\left(t\right) = \frac{\kappa F^{-1}\left(k^{-1}, \alpha\right)}{F\left(t, \alpha\right)},$$

then

$$exp\left(\int \frac{\kappa F^{-1}\left(k^{-1}, \alpha\right)}{F\left(t, \alpha\right)} dt\right) = exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right).$$

$$\begin{aligned} & \frac{d}{dt}\left[exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) T\left(t\right)\right] \\ = \; & exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) \frac{F^{-1}\left(k^{-1}, \alpha\right)}{F\left(t, \alpha\right)} \kappa \epsilon(t) exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) T\left(t\right) \\ = \; & \kappa F^{-1}\left(k^{-1}, \alpha\right) \int exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) \frac{\epsilon\left(t\right)}{F\left(t, \alpha\right)} dt. \end{aligned}$$

Therefore,

$$T\left(t, \alpha\right) = exp\left(-\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) \kappa F^{-1}\left(k^{-1}, \alpha\right) \int exp\left(\kappa F^{-1}\left(k^{-1}, \alpha\right) \int \frac{dt}{F\left(t, \alpha\right)}\right) \frac{\epsilon\left(t\right)}{F\left(t, \alpha\right)} dt. \tag{26}$$

The representation of this solution of the new fractional derivative for the values of $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$ is compared with results obtained using the factor $T(t, \alpha) = e^{(\alpha-1)t}$ directly, as well as non-conformable fractional derivatives using the factor $T(t, \alpha) = t^{-\alpha}$ (see [5, 15]). These comparisons illustrate the differences and similarities in behavior between the new fractional derivative approach and the established methods. Specifically, the comparison with $T(t,\alpha) = e^{(\alpha-1)t}$ showcases how the exponential factor influences the solutions for various values of α. Meanwhile, the comparison with non-conformable fractional derivatives, which utilize the algebraic factor $T(t,\alpha) = t^{-\alpha}$, highlights the impact of a different functional form on the solution's characteristics. These graphical comparisons are essential for understanding the nuances and potential advantages of the new fractional derivative method over traditional approaches, providing deeper insights into its applicability and performance in various scenarios.

4.2. Falling body problem:

The problem of falling bodies, also known as the free fall problem, is a classic physics scenario that examines the motion of objects under the influence of gravity alone, disregarding other forces such as air resistance. It serves as a fundamental concept in physics, providing insights into motion, acceleration, and gravitational effects. When an object is in free fall, it experiences a constant acceleration due to gravity, which on the surface of the Earth is approximately 9.8 $m.s^{-1}$ directed towards the center of the Earth. This acceleration is represented by the symbol g.

The motion of falling bodies can be described using equations derived from Newton's laws of motion and kinematic equations. These equations allow us to predict various aspects of the motion, such as the time taken to fall, the velocity at any given time, and the distance traveled during the fall.

This problem considers the fall of a body of mass m, starting from rest, under the action of gravity. Suppose that the chosen reference system has as its origin the starting point (rest of the body) at a height A from the floor at the moment the fall begins, that is, at $t = 0$. The downward movement will be chosen as positive. At any point P on its trajectory, the distance traveled will be the function y dependent on time t, consequently, using ordinary derivatives, the instantaneous velocity and acceleration snapshot, respectively, are given by

$$v(t) = \frac{dy(t)}{dt}, a = \frac{dv(t)}{dt} = \frac{d^2y(t)}{dt^2}. \tag{27}$$

According to Newton's Law, we have $F = mg$, $\frac{dv(t)}{dt} = g$. Then this problem is modeled by the following differential equation with initial condition as follows

$$\frac{dv(t)}{dt} = g, v(0) = 0, y(0) = A. \tag{28}$$

Now consider the problem with the new conformable fractional derivative defined in Definition 1.

$$\mathcal{D}^{\alpha}v(t) = g, v(0) = 0, y(0) = A. \tag{29}$$

Using Theorem 2.4, part 6) we get:

$$e^{(\alpha-1)t}v'(t) = g. \tag{30}$$

Integrating, we obtain :

$$v(t) = g(\frac{1}{1-\alpha}e^{(1-\alpha)t}) + c. \tag{31}$$

Using the initial condition, namely $v(0) = 0$, we find that the constant is zero. Similary, we obtain :

$$y'(t) = ge^{(1-\alpha)t}(\frac{1}{1-\alpha}e^{(1-\alpha)t}). \tag{32}$$

Then, we have :

$$y(t) = \frac{g}{2(1-\alpha)^2}e^{2(1-\alpha)t} + M. \tag{33}$$

Using the condition initial, we find that :

$$y(t) = \frac{g}{2(1-\alpha)^2}e^{2(1-\alpha)t} + A. \tag{34}$$

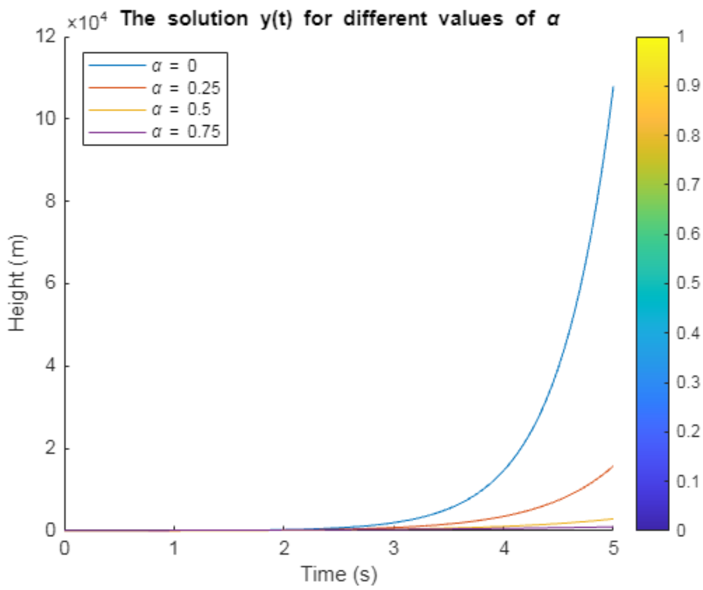

Figure 4: Comparative solutions for serval values of α, with the new conformable fractional derivative.

In this example, we present the solution for various values of α: $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$, utilizing the new conformable fractional calculus (see [1, 12]). We then compare these solutions with those obtained using other methods, such as Khalil's conformable fractional derivative (see [13]), as well as ordinary derivatives. This comparison highlights the differences in the behavior of the solutions across these methods, illustrating the potential advantages and limitations of the new conformable fractional calculus in contrast to more traditional approaches.

5. Conclusion

Numerical simulations conducted on the solutions to initial value problems of fractional ordinary differential equations reveal that the graphs of solutions using Riemann-Liouville and conformable fractional derivatives exhibit significant proximity for $0 < \alpha < 1$. Furthermore, the solutions' graphs employing all three fractional derivatives show substantial similarity for $1/2 < \alpha < 1$. Hence, conformable fractional derivatives offer a viable alternative to Riemann-Liouville and Caputo derivatives within the range of $1/2 < \alpha < 1$.

The simplicity of the conformable fractional derivative's definition stands out as a notable advantage over Riemann-Liouville and Caputo derivatives. Unlike these traditional fractional derivatives, the conformable fractional derivative is more intuitive and straightforward to implement, reducing the complexity often associated with fractional calculus. Additionally, the conformable fractional derivative aligns with several fundamental properties of the usual derivative, such as the product rule, quotient rule, and chain rule, which are not entirely satisfied by Riemann-Liouville and Caputo derivatives. This compatibility with classical derivative properties further enhances its appeal and broadens its applicability in various fields.

Furthermore, the conformable fractional derivative offers a unique flexibility by allowing for the consideration of both conformable and non-conformable kernels, as well as the ability to vary the order of α. This dual capability provides a significant advantage in modeling and analysis, enabling researchers and practitioners to tailor their approach to the specific characteristics of the problem at hand. The ability to switch between conformable and non-conformable kernels and to adjust the fractional order introduces a level of versatility that is not available in traditional ordinary or even other conformable models.

This adaptability makes the conformable fractional derivative particularly useful in applications where the behavior of the system or process can be better captured through a variable-order derivative, or where the inclusion of different kernel types can offer a more accurate representation. Consequently, the conformable fractional derivative stands out as a powerful tool in the realm of fractional calculus, providing both simplicity in definition and extensive flexibility in application.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. A. Abdelhakim. The flaw in the conformable calculus: It is conformable because it is not fractional. *Fractional Calculus and Applied Analysis*, 22(2):242–254, 2019.

[2] T. Abdeljawad. Dualities in fractional difference calculus within riemann. *Advances in Difference Equations*, 2013.

[3] T. Abdeljawad. On conformable fractional calculus. *Journal of Computational and Applied Mathematics*, 279:57–66, 2015.

[4] D.R. Anderson and D.J. Ulness. Newly defined conformable derivatives. *Advances in Dynamical Systems and Applications*, 10(02):109–137, 2015.

[5] A. Ait Brahim, A. El Hajaji, K. Hilal, and J. El Ghordaf. On a novel fractional calculus and its applications to well-known problems. *European Journal of Pure and Applied Mathematics*, April 2024.

[6] A. Fleitas, J. A. Méndez, J. E. Nápoles, and J. M. Sigarreta. On fractional liénard-type systems. *Revista Mexicana de Física*, 65:618–625, 2019.

[7] A. Fleitas, J. E. Nápoles, J. M. Rodríguez, and J. M. Sigarreta. Note on the generalized conformable derivative. *Revista de la UMA*, 62(2):443–457, 2021.

[8] Ahmed Kajouni, Ahmed Chafiki, Khalid Hilal, and Mohamed Oukessou. A new conformable fractional derivative and applications. *Hindawi*, 2022.

[9] R. Khalil, M. Al-Horani, A. Youcef, and M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264:65–70, 2014.

[10] F. Martínez and J. E. Nápoles Valdés. Towards a non-conformable fractional calculus of n-variables. *Journal of Mathematics and Applications*, (43):87–98, 2020.

[11] J. E. Nápoles, P. M. Guzmán, L. M. Lugo, and A. Kashuri. The local non conformable derivative and mittag leffler function. *Sigma Journal of Engineering and Natural Sciences*, 38(2):1007–1017, 2020.

[12] J.J. Rosales, J.F. Gómez, M. Guía, and V.I. Tkach. Lfnm. In *International Conference on Laser and Fiber-Optical Networks Modelling*, 2011.

[13] J. E. Nápoles Valdés, P. M. Guzmán, and L. M. Lugo. Some new results on non-conformable fractional calculus. *Advances in Dynamical Systems and Applications*, 13(2):167–175, 2018.

[14] Juan E. Nápoles Valdés. The non-integer local order calculus. *Physics Astronomy International Journal*, 7(3):163–168, 2023.

[15] M. Vivas-Cortez, A. Fleitas, P.M. Guzmán, J.E. Nápoles, and J.J. Rosales. Newton's law of cooling with generalized conformable derivatives. *Symmetry*, 13, 2021.

[16] Miguel Vivas-Cortez, Juan E. Nápoles Valdés, Shilpi Jain, and Praveen Agarwal. On a generalized fractional integral and related methodological remarks. *Applied Mathematics & Information Sciences*, 17(4):615–623, 2023.

[17] D. Zhao and M. Luo. General conformable fractional derivative and its physical interpretation. *Calcolo*, 54:903–917, 2017.