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On the k-restricted Intersection Graph

Mariane Eliz D. Pelagio[1]*, Kathlen C. Mendoza[1], Neil M. Mame[1]

[1] *College of Arts and Sciences, Batangas State University - The National Engineering University, Batangas City, Batangas, Philippines*

Abstract. The problem of intersection graphs was introduced by Szpilrajn-Marczewski in 1945. This study introduces a new variant of the intersection graph, called the k-restricted intersection graph. Let $\mathcal{S}_n$ be a nonempty n-element set, for some positive integer n, and let $S_{(n,k)}$ be the set of all the k-element subsets of $\mathcal{S}_n$ where $0 \leq k \leq n$. A k-restricted intersection graph, denoted by $G_{S_{(n,k)}}$, is a graph with vertex set $S_{(n,k)}$ such that two vertices $A, B \in S_{(n,k)}$ are adjacent whenever $A \cap B \neq \emptyset$ and $A \neq B$. Here, we determined the order and size of $G_{S_{(n,k)}}$. Moreover, some parameters such as independence number, domination number, and isolate domination number of the k-restricted intersection graph were established. Finally, necessary and sufficient conditions for a $G_{S_{(n,k)}}$ to be isomorphic to the cycle graph and complete graph were determined.

2020 Mathematics Subject Classifications: 05C62, 05C69

Key Words and Phrases: Intersection Graph, k-restricted Intersection Graph, k-element Subsets

1. Introduction

Graph theory has been linked to some areas of mathematics such as set theory. Specifically, the utilization of family sets as vertices of graphs was one of the examples of associating set theory with graph theory as stated by Golumbic, M. (1980) in his study "Algorithmic Graph Theory and Perfect Graphs" [5]. For instance, some of the special types of graphs like the Hamming graph by Richard Hamming and the Johnson graph by Selmer M. John were both derived from the system of sets and were being used in coding theory–which was also a field of mathematics [1].

Connecting the concept of a set to graph theory paved the way for the introduction of intersection graphs. An intersection graph contained a family of sets as its vertices and each vertices were connected by an edge whenever the sets had a nonempty intersection. This graph was introduced by Szpilrajn-Marczewski (1945) in their paper entitled "On Two Properties of Set Classes"[14], wherein they also asserted that all graphs may be

*Corresponding author.
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Email addresses: marianeeliz.pelagio@g.batstate-u.edu.ph (M.E. Pelagio), kathlen.mendoza@g.batstate-u.edu.ph (K. Mendoza), neil.mame@g.batstate-u.edu.ph (N. Mame)

represented as an intersection graph. This was proven and supported by Erdős, Goodman, and Pósa in 1966 in their study "The Representation of a Graph by Set Intersections"[10]. In this study, they provided a more efficient construction of an intersection graph and defined the total number of set elements which required a smaller number of vertices. Furthermore, variations on intersection graphs was also introduced as most of these were derived from sets on some kind of geometric configuration which was specified in [11]. Some of these were the circle graph (or the intersection graph from the chords of a circle), string graph (the intersection graph of curves on a plane), and circular arc graph (intersection graph derived from the arcs of the circle)[15], to name a few.

The assertion that all graphs can be represented as an intersection graph is an interesting concept. Since the majority of the intersection graphs that are discovered employ the usage of sets in the field of geometry, this gives the motivation to introduce and explore a related study adopting the area of set theory. Specifically, finding an intersection graph which is restricted to all the k-element subsets of an n-element set, where n is a positive integer and k is a nonnegative integer such that $k \leq n$. Hence, by using the concepts of k-element subsets and intersection graphs, this paper introduces a k-restricted intersection graph, which is an additional variation of an intersection graph. To determine this graph, its vertex set contains all the k-element subsets of an n-element set and two vertices are adjacent if they have a nonempty intersection. With these notions, it can be gleaned that the difference between an intersection graph and a k-restricted intersection graph is their vertex set wherein the first graph contains nonempty family of sets while the second one involves the collection of all k-element subsets of a set.

This study introduces a new variant of the intersection graph which is the k-restricted intersection graph. The formal definition of this graph is presented in Chapter 3. Also, this study provides the conditions when a k-restricted intersection graph is isomorphic to some special classes of graph. Lastly, some of the graph parameters are determined such as the order, size, independence number, domination number, and isolate domination number.

2. Preliminaries

Some necessary definitions of sets, combinations and subsets, and graph theory are presented in this section. Also, the discussion includes known theorems from combinatorics and graph theory and is presented without proof.

2.1. Combination and Subsets

This section contains the concept of applying the method of combination in enumerating subsets. Also, the notions of sets and subsets discussed were employed. The references used are found in [3], [4], and [8].

A **set** S is a collection of distinct well-defined objects where an 'object' is a generic term that refers to elements (or members) of the set. The **cardinality of** S, denoted by $|S|$, refers to the number of elements of S. If x is an element of S, then we write $x \in S$, otherwise, $x \notin S$. Also, if S has no elements, then S is called an **empty set** and is written as $S = \emptyset$. Now, If S is an n-element set, then we can rewrite this as $\mathcal{S}_n$. In the succeeding discussions, sets with indicated cardinality shall be denoted as $\mathcal{S}_n$.

Definition 1. *Let S and T be sets. Then T is a **subset** of S, written $T \subseteq S$, if for all $x \in T$, then $x \in S$.*

A power set of S, denoted by $\mathcal{P}(S)$, is the set containing all the subsets of S. Note that a set is also a subset of itself. Now, let S and T be nonempty sets. The **difference** between S and T, written $S \setminus T$ and read as "S minus T", is the set containing all elements of S that are not in T. There is no defined cardinality for a generalized difference between two sets. However, if we get the difference between a set and its subset, then we can easily tell the cardinality of their difference.

Remark 1. *Let S and T be nonempty sets. If $T \subseteq S$, then $|S \setminus T| = |S| - |T|$.*

Consider the nonempty sets S and T. The **intersection** of S and T, denoted by $S \cap T$, is defined as the set containing all the elements that belong to both S and T. If two sets do not have any element(s) in common, then we call this as an **empty intersection**, denoted by $S \cap T = \emptyset$. Otherwise, we write as $S \cap T \neq \emptyset$ and refer this as a **nonempty intersection**.

We now introduce the notion of binomial coefficient. For the integers $n, k \geq 0$, and $0 \leq k \leq n$, the **binomial coefficient** is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Note that the number $\binom{n}{k}$, read as "n choose k"and also called as a **combination** or **combinatorial formula**, is the number of ways choosing k unordered outcomes from n possibilities. We will utilize this formula to count how many k-element subsets can be formed from $\mathcal{S}_n$.

Remark 2. *The binomial coefficient $\binom{0}{0}$ is equal to 1.*

Remark 2 discusses the trivial case for the binomial coefficient. Now, consider the 0-element set $\mathcal{S}_0 = \emptyset$. Observe that the only subset that every $\mathcal{S}_0$ has empty subset. By this, we will also consider the 0-element set $\mathcal{S}_0$ as a trivial case.

Remark 3. *The only subset of $\mathcal{S}_0$ is $\emptyset$.*

Given Remark 3, $\mathcal{S}_n$ with $n = 0$ will be regarded as a trivial case for the set $\mathcal{S}_n$. With this, the proceeding discussions shall now be focused on the n-element set $\mathcal{S}_n$ where n is a positive integer.

Theorem 1. *For any positive integer n, $\binom{n}{k} = 1$ if and only if $k = 0$ or $k = n$.*

Theorem 1 is the boundary values for the recursive formula of the binomial coefficient. The readers may refer to [2] for the proof of Theorem 1.

Definition 2. *Let $\mathcal{S}_n$ be an n-element set where n is a positive integer and let k be a nonnegative integer such that $k \leq n$. Then the* ***set containing all k-element subsets of*** *$\mathcal{S}_n$, written as $S_{(n,k)}$, is the set containing all the subsets of $\mathcal{S}_n$ having k-elements.*

Since this study does not consider multiset, it follows that $S_{(n,k)}$ contains the distinct k-subsets of $\mathcal{S}_n$. Moreover, by utilizing the binomial coefficient, we define the cardinality of the set $S_{(n,k)}$ be equal to the number $\binom{n}{k}$ where n is a positive integer and k is a nonnegative integer.

Given that $|S_{(n,k)}| = \binom{n}{k}$, it can be observed that we cannot form a k-element subset from $\mathcal{S}_n$ when $n < k$. Thus $S_{(n,k)} = \emptyset$ which implies that $|S_{(n,k)}| = \binom{n}{k} = 0$. Equivalently, we have Remark 4.

Remark 4. *If $n < k$, then $\binom{n}{k} = 0$.*

2.2. Graph Theory

This section discusses some basic concepts in graph theory. We adapt the definitions in [9] for the concepts used here. Also, the references [6], [7], [12], and [13] are utilized.

A **graph**, denoted by G, is an ordered pair $G = (V(G), E(G))$ where the vertex set $V(G)$ is a nonempty set of elements called **vertices** and the edge set $E(G)$ is a set of unordered pairs of vertices called **edges**. We write the edges of a graph G as $[x, y]$ for $x, y \in V(G)$. The edges of G are said to be unordered pairs so we also say that $[x, y] = [y, x]$. Moreover, the number $|V(G)|$ is called the **order** of G while the number $|E(G)|$ is referred to as the **size** of G.

A graph of order $n \geq 1$ having no edges is called an **empty graph**, denoted as $\overline{K}_n$. Furthermore, a graph with only one vertex is referred to as a **trivial graph**. It was taken into consideration that a graph with an empty edge set is still considered as a graph. However, note that if $V(G) = \emptyset$, then G here is undefined since we cannot form a graph with no vertices.

The **degree of vertex** $x \in V(G)$, denoted by $\deg(x)$, is the number of edges adjacent to the vertex x. If the vertex x has $\deg(x) = 0$, then x is called an **isolated vertex**. Meaning to say, an isolated vertex is a vertex that is not adjacent to any other vertex of G.

There are times when the degrees of every vertex help determine the size of a graph specifically if the degrees are the same in number. In this regard, we say that a graph with the same number of degrees for every vertex is a regular graph. A graph G is **regular** if every vertex has the same degree. Moreover, G is said to be regular of degree r (or r-**regular**) if $\deg(x) = r$ for all vertices x in G.

The size of an r-regular graph can be determined as follows:

$$\begin{aligned} \sum_{x \in V(G)} deg(x) &= 2m \\ nr &= 2m \\ m &= \frac{nr}{2} \end{aligned} \tag{1}$$

provided that nr is even.

Let $W : x_1, x_2, ..., x_k, x_{k+1}$ be a walk of length $k > 0$. This walk is closed if $x_1 = x_{k+1}$. Moreover, a closed walk is called a cycle if the vertices $x_1, x_2, ..., x_k$ are distinct. A graph G of order $n \geq 3$ is called a **cycle graph of order n**, denoted by C_n, if the vertices of G are labeled $x_1, x_2, ..., x_n$ so that the edges $[x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n], [x_n, x_1]$ form a cycle. It is noted that the order and size of a C_n is n.

A graph of order n is said to be a **complete graph of order** n, denoted by K_n, if every vertex is adjacent to every other vertex. Since every vertex of K_n is adjacent to every vertex of this graph it follows that the degrees of every vertex are equal to $n-1$, since we consider a simple graph K_n having no loops and multiple edges. This implies that K_n is an $(n-1)$-regular graph. Now, to determine the size of K_n, by utilizing Equation(1), we have

$$m = \frac{nr}{2}$$
$$m = \frac{n(n-1)}{2}.$$

Observe that m is always defined since the product of the consecutive integers n and $n-1$ is even.

Any cycle graph, in particular, C_3 whose order and size are equal to 3, is a 2-regular graph. Furthermore, a complete graph of order 3, which is a $(3-1) = 2$-regular graph, has size equal to 3. Now, it can be observed that C_3 and K_3 have the same number of order, size, and degree of every vertex. Through the notion of graph isomorphism, denoted by $\simeq$, it can be seen that there exists an isomorphism regarding the two graphs.

Remark 5. *Let C_3 and K_3 be a cycle and complete graphs of order 3, respectively. Then $C_3 \simeq K_3$.*

Definition 3. *Let $T = \{T_1, T_2, ..., T_m\}$ be a nonempty collection of sets. A graph G is called an* ***intersection graph*** *whose $V(G) = T$ and $[T_i, T_j] \in E(G)$ for $1 \leq i, j \leq m$, where $i \neq j$ is an edge in G if they have a nonempty intersection.*

For every intersection graph G, we have $|V(G)| = |T|$, where T is a nonempty family of sets. To date, there is no defined size for a generalized intersection graph.

Example 1. *Consider the intersection graph G over the set*

$$T = \{\{1, 2, 3\}, \{2, 4\}, \{1, 3\}\}.$$

It can be observed that the order of G is 3. Also, since $\{1, 2, 3\} \cap \{2, 4\} = \{2\}$ and $\{1, 2, 3\} \cap \{1, 3\} = \{1, 3\}$ both contain nonempty intersection it follows that $[\{1, 2, 3\}, \{2, 4\}]$ and $[\{1, 2, 3\}, \{1, 3\}]$ are edges of G. Moreover, $\{2, 4\}$ and $\{1, 3\}$ has no intersection, so $[\{2, 4\}, \{1, 3\}]$ is not in $E(G)$. By these, then

$$E(G) = \big\{[\{1, 2, 3\}, \{2, 4\}], [\{1, 2, 3\}, \{1, 3\}]\big\}.$$

Shown in Figure 1 is a pictorial illustration of the intersection graph over the set $T = \{\{1, 2, 3\}, \{2, 4\}, \{1, 3\}\}$.

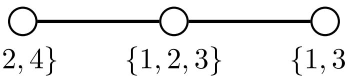


Figure 1: An Illustration of the Intersection Graph of the Set T.

Consider a graph G. The complement of G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and such that for every pair $x, y \in V(G)$, $[x, y]$ is an edge of $\overline{G}$ if and only if $[x, y]$ is not an edge of G. Recall that the degree of every vertex in a regular graph is equal in number. This can infer that the vertices in a complement graph of a regular graph yield also a degree that is equal in number. In sum, the complement graph of a regular graph is considered a regular graph.

Definition 4. *Let G be a graph. The nonempty set $T \subseteq V(G)$ is called an **independent set** in a graph G if for every $x, y \in T$, then $[x, y] \notin E(G)$. The **independence number** of a graph, denoted by $\alpha(G)$, is the cardinality of the largest independent set of G.*

In a graph G, if there exists an independent set $T \subseteq V(G)$, it follows that $\alpha(G) \geq |T|$. The next theorem determines the independence number of a complete graph of order n.

Definition 5. *Let $G = (V(G), E(G))$ be a graph. A nonempty subset T of $V(G)$ is called the **dominating set** of G if every element of $V(G) \setminus T$ is adjacent to some element of T. Moreover, the **domination number**, written as $\gamma(G)$, of a graph G is the minimum cardinality among all the dominating sets of G.*

The vertex set $V(G)$ of a graph G is a dominating set since $V(G) \setminus V(G) = \emptyset$ which implies that there are no other vertices that are needed to be considered.

A graph H is called a subgraph of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, a subgraph H of a graph G is called an induced-subgraph (also called a "vertex-induced subgraph"), written as $< H >$, if whenever $x, y \in H$ and $[x, y] \in E(G)$, then $[x, y]$ is an edge of $< H >$.

Definition 6. *A dominating set T is called an **isolate dominating set** if the subgraph induced by T on graph G has at least one isolated vertex. An **isolate domination number**, denoted by γ_0, has the minimal cardinality among all the isolate dominating sets of G.*

Note that the domination number must be the smallest among the cardinality of the domination sets of $V(G)$. By this, the isolate domination number is always equal to or greater than the domination number. Also, note that if there exists an isolate dominating set in G, say $T \subseteq V(G)$, then $\gamma(G) \leq \gamma_0(G) \leq |T|$.

3. The k-restricted Intersection Graph

Recall that $S_{(n,k)}$ is the collection of all k-element subsets of $\mathcal{S}_n$. From Remark 3, it was regarded that $\mathcal{S}_0$ is a trivial case such that $S_{(0,0)} = \{\emptyset\}$. Then, $S_{(0,0)}$ is considered a trivial case for a k-restricted intersection graph. As a result, the 0-restricted intersection graph of $\mathcal{S}_0$ is a trivial graph containing the vertex $\emptyset$.

Throughout the succeeding discussion, we shall consider n as a positive integer for $\mathcal{S}_n$ and a nonnegative integer k such that $k \leq n$.

Definition 7. *Let $\mathcal{S}_n$ be an n-element set where n is a positive integer, let $k \leq n$ be a nonnegative integer, and let $S_{(n,k)}$ be the collection of all k-element subsets of $\mathcal{S}_n$. A **k-restricted intersection graph**, denoted by $G_{S_{(n,k)}}$, is the graph whose vertex set is $S_{(n,k)}$ and two vertices A and B are adjacent whenever $A \cap B \neq \emptyset$ and $A \neq B$.*

The elements of $V(G_{S_{(n,k)}})$ are the k-subsets of $\mathcal{S}_n$ so for every $A \in V(G_{S_{(n,k)}})$, $|A| = k$. Moreover, an unordered pair of vertex $[A, B] \in E(G_{S_{(n,k)}})$ if A and B have a nonempty intersection and A and B are distinct. Although for every nonempty subset $A \in V(G_{S_{(n,k)}})$, $A \cap A = A$, it can be observed that every edge in $G_{S_{(n,k)}}$ should contain distinct vertices so $G_{S_{(n,k)}}$ does not contain any loops. Now, given in Example 2 is an example for $G_{S_{(n,k)}}$ of a 4-element set $\mathcal{S}_4$ where $k = 2$.

Example 2. *Let $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ and let $k = 2$. Then $G_{S_{(4,2)}}$ has the vertex set*

$$V(G_{S_{(4,2)}}) = \big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\big\}.$$

Now, observe that $\{x_2, x_3\} \cap \{x_2, x_4\} = \{x_2\}$, so $[\{x_2, x_3\}, \{x_2, x_4\}] \in E(G_{S_{(4,2)}})$. Also, $\{x_1, x_2\} \cap \{x_1, x_3\} = \{x_1\}$ which implies that $[\{x_1, x_2\}, \{x_1, x_3\}]$ is also an edge of $G_{S_{(4,2)}}$. However, the vertices $\{x_1, x_2\}$ and $\{x_3, x_4\}$ bear an empty intersection so they are not adjacent to each other. To enumerate the edges of $G_{S_{(4,2)}}$, we have:

$$\begin{aligned} E(G_{S_{(4,2)}}) = \big\{ & [\{x_1, x_2\}, \{x_1, x_3\}], [\{x_1, x_2\}, \{x_1, x_4\}], [\{x_1, x_2\}, \{x_2, x_3\}], [\{x_1, x_2\}, \{x_2, x_4\}], \\ & [\{x_1, x_3\}, \{x_1, x_4\}], [\{x_1, x_3\}, \{x_2, x_3\}], [\{x_1, x_3\}, \{x_3, x_4\}], [\{x_1, x_4\}, \{x_2, x_4\}], \\ & [\{x_1, x_4\}, \{x_3, x_4\}], [\{x_2, x_3\}, \{x_2, x_4\}], [\{x_2, x_3\}, \{x_3, x_4\}], [\{x_2, x_4\}, \{x_3, x_4\}]\big\}. \end{aligned}$$

It can be observed that the order of $G_{S_{(4,2)}}$ is 6 and its size is 12. The graph illustrated in Figure 2 is a pictorial representation of $G_{S_{(4,2)}}$.

Since $[A, A] \notin E(G_{S_{(n,k)}})$, it follows that a k-restricted intersection graph does not contain any loop. Also, note that $V(G_{S_{(n,k)}})$ is the set containing the distinct k-subsets of $\mathcal{S}_n$, this means that $E(G_{S_{(n,k)}})$ is not a multiset. Hence, $G_{S_{(n,k)}}$ has no multiple edges.

Remark 6. *A k-restricted intersection graph $G_{S_{(n,k)}}$ is a simple graph.*

Theorem 2 determines the order of a $G_{S_{(n,k)}}$. Recall that the order of a graph refers to the cardinality of its vertex set.

Theorem 2. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. Then the order of $G_{S_{(n,k)}}$ is $\binom{n}{k}$.*

Proof. Assume $G_{S_{(n,k)}}$ is a k-restricted intersection graph. By Definition 7, $V(G_{S_{(n,k)}})$ is equal to $S_{(n,k)}$. Since $|S_{(n,k)}| = \binom{n}{k}$, it follows that $|V(G_{S_{(n,k)}})| = \binom{n}{k}$.

The number $\binom{n}{k}$ provided that n a positive integer and k is nonnegative, has a positive integer value. Meaning to say, $V(G_{S_{(n,k)}})$ is nonempty for all n and k. Therefore, $G_{S_{(n,k)}}$ is defined for all the given values of n and k.

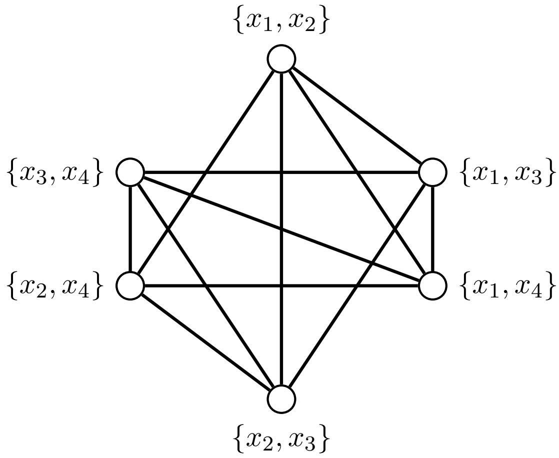


Figure 2: A 2-Restricted Intersection Graph $G_{S_{(4,2)}}$.

Illustration 1. *Let $G_{S_{(4,2)}}$ be a 2-restricted intersection graph over the 4-element set $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ whose graph is pictorially represented in Figure 2. We see that $|V(G_{S_{(4,2)}})| = 6$. To verify this by utilizing Theorem 2, since $n = 4$ and $k = 2$, it follows that $|V(G_{S_{(4,2)}})| = \binom{4}{2} = 6$.*

There are instances that a $G_{S_{(n,k)}}$ is a trivial graph. Meaning to say, $G_{S_{(n,k)}}$ has only one vertex. Note that from Theorem 1, $\binom{n}{k} = |V(G_{S_{(n,k)}})| = 1$ if and only if $k = 0$ or $k = n$. Theorem 3 discusses $G_{S_{(n,k)}}$ when k is 0 or n.

Theorem 3. *Let $\mathcal{S}_n$ be an n-element set. Then $G_{S_{(n,k)}}$ is a trivial graph if and only if $k = 0$ or $k = n$.*

Proof. Let $G_{S_{(n,k)}}$ be a trivial graph. Hence, we have $|V(G_{S_{(n,k)}})| = 1$. Since the order of $G_{S_{(n,k)}}$ is equal to $\binom{n}{k}$, hence we have $\binom{n}{k} = 1$. By Theorem 1, k is either 0 or n. Conversely, assume that $k = 0$ or $k = n$. If $k = 0$, then $S_{(n,0)} = \{\emptyset\}$ which implies that $V(G_{S_{(n,0)}}) = \{\emptyset\}$. An empty subset has no element to consider so it does not intersect to itself. It follows that $E(G_{S_{(n,0)}})$ is empty. Thus, $G_{S_{(n,0)}}$ is a trivial graph. Moreover, if $k = n$, then $S_{(n,n)} = \{\mathcal{S}_n\}$ meaning, $V(G_{S_{(n,n)}}) = \{\mathcal{S}_n\}$. This implies that $G_{S_{(n,n)}}$ has only one vertex. By Remark 6, $[\mathcal{S}_n, \mathcal{S}_n] \notin E(G_{S_{(n,n)}})$ so $E(G_{S_{(n,n)}}) = \emptyset$. Therefore, $G_{S_{(n,n)}}$ is also a trivial graph.

Illustration 2. *Let $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$. If $k = 0$, then $V(G_{S_{(4,0)}}) = \{\emptyset\}$. Furthermore, if $k = 4$, then $V(G_{S_{(4,4)}}) = \big\{\{x_1, x_2, x_3, x_4\}\big\}$. Pictorial representations of $G_{S_{(4,0)}}$ and $G_{S_{(4,4)}}$ are presented in Figure 3.*

For every $G_{S_{(n,k)}}$, if $k = 1$, then $V(G_{S_{(n,1)}})$ contains the distinct 1-element subsets of $\mathcal{S}_n$. Discussed in Theorem 4 is a $G_{S_{(n,k)}}$ when $k = 1$.

Theorem 4. *Let $\mathcal{S}_n = \{x_1, x_2, ..., x_n\}$ be an n-element set. If $k = 1$, then $G_{S_{(n,1)}}$ is an empty graph of order n.*

Figure 3: Illustrating $G_{S_{(4,0)}}$ (Left) and $G_{S_{(4,4)}}$ (Right).

Proof. If $k = 1$, then $V(G_{S_{(n,1)}}) = \{\{x_1\}, \{x_2\}, ..., \{x_n\}\}$. It can be observed that for all $1 \leq i, j \leq n$ and $i \neq j$, $\{x_i\} \cap \{x_j\} = \emptyset$. This implies that every vertex in $G_{S_{(n,1)}}$ is not adjacent to each other. Hence, $G_{S_{(n,1)}}$ is an empty graph. Note that $|V(G_{S_{(n,1)}})| = \binom{n}{1} = n$, which implies that $G_{S_{(n,1)}}$ is an empty graph of order n.

Note that all trivial graphs are empty graphs while not every empty graphs are trivial graphs. Hence, $G_{S_{(n,n)}}$ and $G_{S_{(n,0)}}$ are also empty graphs.

Illustration 3. *Let $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ and consider $k = 1$. Then, $V(G_{S_{(4,1)}}) = \big\{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\big\}$. It can be observed that the 1-element subsets have no element in common. Thus, $E(G_{S_{(4,1)}}) = \emptyset$ which means that $G_{S_{(4,1)}}$ is an empty graph of order 4. Shown in Figure 4 is a pictorial representation of $G_{S_{(4,1)}}$ of $\mathcal{S}_4$.*

Figure 4: Pictorial Illustration of $G_{S_{(4,1)}}$.

The degree of every vertex in $G_{S_{(n,k)}}$ depends on the value of the nonnegative integer k. Lemma 1 determines the degree of every vertex in a $G_{S_{(n,k)}}$ when $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Lemma 1. *Let $\mathcal{S}_n$ be an n-element set and let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. Then for all $A \in V(G_{S_{(n,k)}})$, $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$ if and only if $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Assume $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$ for all A in $V(G_{S_{(n,k)}})$ and suppose that $k = 0$ or $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$. If $k = 0$, $\binom{n}{0} - \left[\binom{n-0}{0} + 1\right] = 1 - (1+1) = -1$. This is a contradiction since the degree of a vertex cannot be a negative integer. If $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$, then $n - k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Since $n - k \leq \left\lfloor \frac{n}{2} \right\rfloor < k$ which implies that $n - k < k$, by Remark 4, it follows that $\binom{n-k}{k} = 0$. Hence, $deg(A) = \binom{n}{k} - (0+1) = \binom{n}{k} - 1$. This is a contradiction to the assumption that $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$. Therefore, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Conversely, assume that $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and let A be an arbitrary vertex of $V(G_{S_{(n,k)}})$. Since $A \subseteq \mathcal{S}_n$, then by Remark 1, it follows that $|\mathcal{S}_n \setminus A| = n - k$. Now, let $S_{(n-k,k)}$ be a set containing all the k-element subsets of $\mathcal{S}_n \setminus A$. Observe that $S_{(n-k,k)} \subseteq V(G_{S_{(n,k)}})$. If $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, then $\left\lfloor \frac{n}{2} \right\rfloor \leq n - k$. This implies that $k \leq n - k$ so $|S_{(n-k,k)}| = \binom{n-k}{k}$. Note that $A \cap (\mathcal{S}_n \setminus A) = \emptyset$. Hence, for every $B \in S_{(n-k,k)}$, $A \cap B = \emptyset$. This means that A is not adjacent to $\binom{n-k}{k}$ elements of $V(G_{S_{(n,k)}})$. By Theorem 2, since $|V(G_{S_{(n,k)}})| = \binom{n}{k}$, it follows that $deg(A) = \binom{n}{k} - \binom{n-k}{k}$. Additionally, by Remark 6, $[A, A] \notin E(G_{S_{(n,k)}})$. This implies that $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$. Since A was arbitrarily chosen, it follows that every vertex of $G_{S_{(n,k)}}$, with $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, has degree equal to $\binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$.

Illustration 4. *Let $\mathcal{S}_5 = \{x_1, x_2, x_3, x_4, x_5\}$ and let $k = 2$. The vertex set of $G_{S_{(5,2)}}$ is given by*

$$\begin{aligned} V(G_{S_{(5,2)}}) =& \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \\ & \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}\} \end{aligned}$$

A pictorial representation of the graph $G_{S_{(5,2)}}$ is shown in Figure 5.

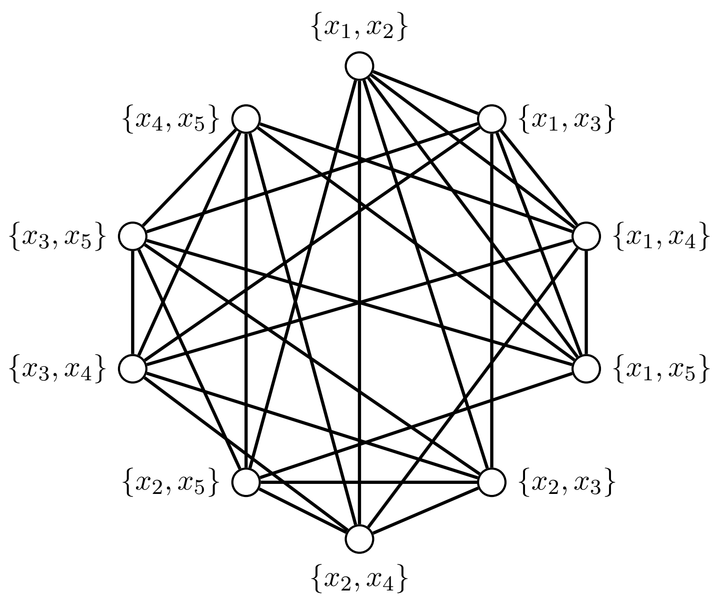


Figure 5: A Pictorial Representation of $G_{S_{(5,2)}}$.

It can be observed that the degree of every vertex in $G_{S_{(5,2)}}$ is 6. Note that $k = 2$ which implies that $k \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$. By using Lemma 1 with $n = 5$ and $k = 2$, the degree of each $A \in V(G_{S_{(5,2)}})$ is given by

$$\begin{aligned} deg(A) &= \binom{n}{k} - \left[\binom{n-k}{k} + 1\right] \\ &= \binom{5}{2} - \left[\binom{5-2}{2} + 1\right] \\ &= \binom{5}{2} - \left[\binom{3}{2} + 1\right] \\ &= 10 - (3 + 1) \\ &= 6. \end{aligned}$$

Furthermore, Lemma 2 shows the degree of every vertex in $G_{S_{(n,k)}}$ if and only if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.

Lemma 2. *Let $\mathcal{S}_n$ be an n-element set and let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. Then $deg(A) = \binom{n}{k} - 1$ for all $A \in V(G_{S_{(n,k)}})$ if and only if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.*

Proof. Assume that $deg(A) = \binom{n}{k} - 1$ for all A in $V(G_{S_{(n,k)}})$ and suppose that $1 < k \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 1, since $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$, it follows that $\binom{n}{k} - \left[\binom{n-k}{k} + 1\right] = \binom{n}{k} - 1$. This implies that $-\binom{n-k}{k} = 0$ which can be equated to $\binom{n-k}{k} = 0$. The only time that $\binom{n-k}{k} = 0$ is when $n - k < k$. Note that if $1 < k \leq \lfloor \frac{n}{2} \rfloor$, then $\lfloor \frac{n}{2} \rfloor \leq n - k$. So $k \leq n - k$. This is a contradiction to the fact that $n - k < k$. Therefore, $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. Conversely, assume that $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. If k is 0 or n, by Theorem 3, $G_{S_{(n,k)}}$ is a trivial graph. Since $|V(G_{S_{(n,k)}})| = \binom{n}{k} = 1$ and the degree of the vertex of a trivial graph is 0, it follows that $deg(A) = 0 = 1 - 1 = \binom{n}{k} - 1$ where $A \in V(G_{S_{(n,k)}})$. Now, let $\lfloor \frac{n}{2} \rfloor < k < n$ and let A be an arbitrary vertex of $G_{S_{(n,k)}}$. Since $A \subseteq \mathcal{S}_n$, then by Remark 1, it follows that $|\mathcal{S}_n \setminus A| = n - k$. Now, consider the set of all k-element subsets of $\mathcal{S}_n \setminus A$, denoted by $S_{(n-k,k)}$. Observe that $S_{(n-k,k)} \subseteq V(G_{S_{(n,k)}})$. If $\lfloor \frac{n}{2} \rfloor < k < n$, then $n - k \leq \lfloor \frac{n}{2} \rfloor$ which means that $n - k < k$. Thus by Remark 4, $|S_{(n-k,k)}| = \binom{n-k}{k} = 0$. Note that $A \cap (\mathcal{S}_n \setminus A) = \emptyset$, so for every $B \in S_{(n-k,k)}$, $A \cap B = \emptyset$. But $S_{(n-k,k)} = \emptyset$, so given that $|V(G_{S_{(n,k)}})| = \binom{n}{k}$, $deg(A) = \binom{n}{k} - 0$. Also, since $[A, A] \notin E(G_{S_{(n,k)}})$ it follows that $deg(A) = \binom{n}{k} - 1$. Since A was chosen arbitrarily, it follows that every vertex of $G_{S_{(n,k)}}$, with $\lfloor \frac{n}{2} \rfloor < k < n$, has a degree equal to $\binom{n}{k} - 1$.

Illustration 5 provides an example for the degree of every vertex of a $G_{S_{(n,k)}}$ when $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$ with $n = 4$ and k is 0, 4 and 3.

Illustration 5. *Consider $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$. To show an illustration for the case $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, we have the following:*

Let $k = 0$ and consider the pictorial illustration of $G_{S_{(4,0)}}$ in Figure 3 which is a trivial graph containing the vertex $\emptyset$. It can be observed that $deg(\emptyset) = 0$. To verify using Lemma 2 with $n = 4$ and $k = 0$, we have $deg(\emptyset) = \binom{4}{0} - 1 = 1 - 1 = 0$.

Moreover, let $k = 4$. By Theorem 3, $G_{S_{(4,4)}}$ is a trivial graph. A pictorial representation of $G_{S_{(4,4)}}$ with a vertex $\{x_1, x_2, x_3, x_4\}$ is shown in Figure 3. It can be observed that $deg(\{x_1, x_2, x_3, x_4\}) = 0$. To verify this using Lemma 2, since $n = 4$ and $k = 4$, it follows that $deg(\{x_1, x_2, x_3, x_4\}) = \binom{4}{4} - 1 = 1 - 1 = 0$.

Now, let $k = 3$. Then $G_{S_{(4,3)}}$ has the vertex set $V(G_{S_{(4,3)}}) = \big\{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\big\}$. A pictorial representation of $G_{S_{(4,3)}}$ is illustrated in Figure 6. It can be observed that $deg(A) = 3$ for all $A \in V(G_{S_{(4,3)}})$. Note that $k = 3$ and $\lfloor \frac{4}{2} \rfloor = 2 < k$. To verify using Lemma 2 with $n = 4$ and $k = 3$, the degree of each

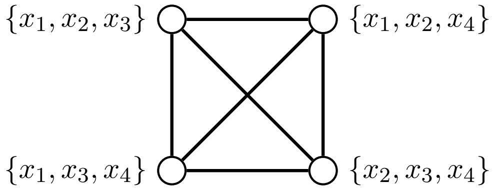


Figure 6: A Pictorial Representation of $G_{S_{(4,3)}}$.

vertex A in $G_{S_{(4,3)}}$ *is equal to*

$$\begin{aligned} deg(A) &= \binom{n}{k} - 1 \\ &= \binom{4}{3} - 1 \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

Lemma 1 and Lemma 2 discusses the degree of every vertex in a $G_{S_{(n,k)}}$. Observe that the vertices in $G_{S_{(n,k)}}$ yield the same degree. By this, then $G_{S_{(n,k)}}$ is a regular graph. Equivalently, we have Theorem 5.

Theorem 5. *Let* $G_{S_{(n,k)}}$ *be a k-restricted intersection graph. Then* $G_{S_{(n,k)}}$ *is an r-regular graph where*

$$r = \begin{cases} \binom{n}{k} - \left[\binom{n-k}{k} + 1\right] & \textit{if and only if} \quad 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor; \\ \binom{n}{k} - 1 & \textit{if and only if} \quad k = 0 \quad \textit{or} \quad \left\lfloor \frac{n}{2} \right\rfloor < k \le n. \end{cases}$$

Proof. This is the direct consequence of Lemma 1 and Lemma 2.

Let $k = 1$. Since $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$, by using Lemma 1, then $deg(A) = \binom{n}{1} - \left[\binom{n-1}{1} + 1\right] = n - (n - 1 + 1) = 0$ for all $A \in V(G_{S_{(n,1)}})$. By Theorem 5, it follows that $G_{S_{(n,1)}}$ is a 0-regular graph. Moreover, observe that $G_{S_{(n,k)}}$ is also a 0-regular graph when $k = 0$ or $k = n$. Equivalently, we have the following remark.

Remark 7. *A k-restricted intersection graph* $G_{S_{(n,k)}}$ *is a* 0*-regular graph if an only if k is* 0, 1, *or n.*

The size of a graph is a parameter that is also helpful in defining a graph. Since $G_{S_{(n,k)}}$ is a regular graph, Equation 1, it follows that $|E(G_{S_{(n,k)}})|$ can be computed by $\frac{\binom{n}{k}(r)}{2}$ where r is equal to either $\binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$ or $\binom{n}{k} - 1$.

Theorem 6. *Let* $G_{S_{(n,k)}}$ *be a k-restricted intersection graph. Then the size of* $G_{S_{(n,k)}}$ *is given by*

$$|E(G_{S_{(n,k)}})| = \begin{cases} \frac{\binom{n}{k}\left\{\binom{n}{k} - \left[\binom{n-k}{k} + 1\right]\right\}}{2} & \text{if} \quad 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor; \\ \frac{\binom{n}{k}\left[\binom{n}{k} - 1\right]}{2} & \text{if} \quad k = 0 \quad \text{or} \quad \left\lfloor \frac{n}{2} \right\rfloor < k \le n. \end{cases}$$

Proof. Note that $G_{S_{(n,k)}}$ is a graph of order $\binom{n}{k}$. By Theorem 5, $G_{S_{(n,k)}}$ is a regular graph. Now, using Equation 1, then the size of $G_{S_{(n,k)}}$ is $\frac{\binom{n}{k}\binom{r}{}}{2}$ where r is the degree of every vertex in $G_{S_{(n,k)}}$. If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $G_{S_{(n,k)}}$ is a $\left\{\binom{n}{k} - \left[\binom{n-k}{k} + 1\right]\right\}$-regular graph. Thus, $|E(G_{S_{(n,k)}})| = \frac{\binom{n}{k}\left\{\binom{n}{k}-\left[\binom{n-k}{k}+1\right]\right\}}{2}$. Furthermore, if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, $G_{S_{(n,k)}}$ is a $\left[\binom{n}{k} - 1\right]$-regular graph. This implies that $|E(G_{S_{(n,k)}})| = \frac{\binom{n}{k}\left[\binom{n}{k}-1\right]}{2}$.

Illustration 6. *Consider $G_{S_{(5,2)}}$ shown in Figure 5. Observe that $k = 2$ which implies that $\lfloor \frac{5}{2} \rfloor = 2 = k$. Since $G_{S_{(5,2)}}$ is a graph of order 10 and a 6-regular graph, by utilizing Theorem 6, it follows that $|E(G_{S_{(5,2)}})| = \frac{10(6)}{2} = 30$.*

Also, given $G_{S_{(4,3)}}$ in Figure 6 that is a 3-regular graph of order $\binom{4}{3} = 4$, note that $k = 3$ which is $\lfloor \frac{4}{2} \rfloor = 2 < k$. So, we have $|E(G_{S_{(4,3)}})| = \frac{4(3)}{2} = 6$.

For any positive integer n, we have identified that $G_{S_{(n,k)}}$ is a 0-regular graph if and only if $k = 0$, $k = 1$, or $k = n$ by Remark 7. With this, then $|E(G_{S_{(n,k)}})| = \frac{\binom{n}{k}(0)}{2} = 0$. Equivalently, we have Remark 8.

Remark 8. *If $k = 0$, $k = 1$, or $k = n$, then $|E(G_{S_{(n,k)}})| = 0$.*

4. k-restricted Intersection Graph as a Special Class of Graph

The k-restricted intersection graph $G_{S_{(n,k)}}$ yields a complete graph and cycle graph depending on the values of the nonnegative integer k. Furthermore, the degree of every vertex for the complement graph of $G_{S_{(n,k)}}$ is given in this section.

Theorem 7. *A k-restricted intersection graph $G_{S_{(n,k)}}$ is a complete graph of order $\binom{n}{k}$ if and only if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.*

Proof. Assume $G_{S_{(n,k)}}$ is a complete graph of order $\binom{n}{k}$. By the definition of a complete graph, $G_{S_{(n,k)}}$ is an $\left[\binom{n}{k} - 1\right]$-regular graph. By Theorem 5, then $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. Conversely, assume $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. By Theorem 5, $G_{S_{(n,k)}}$ is an $\left[\binom{n}{k} - 1\right]$-regular graph. Since $|V(G_{S_{(n,k)}})| = \binom{n}{k}$, it follows that every vertex of $G_{S_{(n,k)}}$ is adjacent to every other vertex of $G_{S_{(n,k)}}$. Hence, $G_{S_{(n,k)}}$ is a complete graph.

Since all of the vertex of $G_{S_{(n,k)}}$ where $\lfloor \frac{n}{2} \rfloor < k < n$ is adjacent to each other, it follows that every $A \in V(G_{S_{(n,k)}})$ bears an intersection to any other vertex in $G_{S_{(n,k)}}$. Note that the trivial graph is a complete graph of order 1. Meaning to say, $G_{S_{(n,0)}}$ and $G_{S_{(n,n)}}$ are complete graphs of order 1.

Illustration 7. *Let $\mathcal{S}_5$ be equal to $\{x_1, x_2, x_3, x_4, x_5\}$ and let $k = 3$. The vertex set of the graph $G_{S_{(5,3)}}$ is given by*

$$\begin{aligned} V(G_{S_{(5,3)}}) = &\big\{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \\ &\{x_1, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}\big\}. \end{aligned}$$

Moreso, $G_{S_{(5,3)}}$ *is pictorially illustrated in Figure 7.*

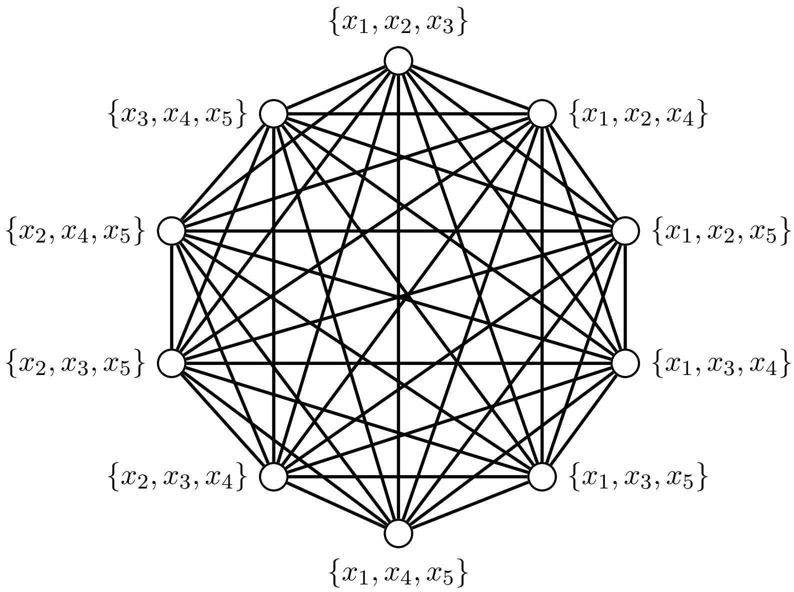


Figure 7: A Pictorial Illustration of $G_{S_{(5,3)}}$.

Observe that every vertex in $G_{S_{(5,3)}}$ *is adjacent to each other. Since* $k = 3$ *and* $\lfloor \frac{5}{2} \rfloor = 2 < 3$, *by Theorem 7, it follows that* $G_{S_{(5,3)}}$ *is a complete graph of order* $\binom{5}{3} = 10$.

The next theorem imparts the necessary and sufficient conditions for $G_{S_{(n,k)}}$ to be a cycle graph.

Theorem 8. *A k-restricted intersection graph* $G_{S_{(n,k)}}$ *is a cycle graph of order* 3 *if and only if* $n = 3$ *and* $k = 2$.

Proof. Assume that $G_{S_{(n,k)}}$ is a cycle graph of order 3. It is noted that every cycle graph is a 2-regular graph. Suppose that $n \neq 3$ or $k \neq 2$. If $n < 3$, then $\binom{n}{k} \leq 2$, which is a contradiction to the assumption that $|V(G_{S_{(n,k)}})| = 3$. Moreover, if $n > 3$, then $\binom{n}{k} = 1$ or $\binom{n}{k} \geq 4$. This is also a contradiction since the order of $G_{S_{(n,k)}}$ is 3. On the other hand, if $k < 2$, then k is 0 or 1. By Theorem 3 or by Theorem 4, $G_{S_{(n,k)}}$ is a trivial graph or an empty graph of order n, respectively. Hence, this is a contradiction to the assumption that $G_{S_{(n,k)}}$ is a cycle graph of order 3. Furthermore, if $k > 2$, then $\binom{n}{k}$ is either equal to 1 or greater than or equal to 4. This is a contradiction also to the assumption that $G_{S_{(n,k)}}$ is a cycle graph of order 3. Therefore, $n = 3$ and $k = 2$. Conversely, assume that $n = 3$ and $k = 2$. Observe that $\lfloor \frac{3}{2} \rfloor = 1 < 2$, so by Theorem 7, $G_{S_{(3,2)}}$ is a complete graph of order $\binom{3}{2} = 3$. By Remark 5, since $C_3 \simeq K_3$, it follows that $G_{S_{(3,2)}}$ is a cycle graph of order 3.

The only time that a cycle graph is isomorphic to a complete graph is when they have an order equal to 3. It can be verified that when $n = 3$ and $k = 2$, $G_{S_{(3,2)}}$ is a complete graph of order 3 which implies that $G_{S_{(3,2)}}$ is also a cycle graph of order 3. Hence, for any 3-element set $\mathcal{S}_3$, $G_{S_{(3,2)}}$ is a cycle graph as well as a complete graph.

Illustration 8. *Consider the set $\mathcal{S}_3 = \{x_1, x_2, x_3\}$ and let $k = 2$. Then the vertex set of $G_{S_{(3,2)}}$ is given by $\big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\big\}$. The order of $G_{S_{(3,2)}}$ is 3. Since $k = 3$, which means that $\left\lfloor \frac{3}{2} \right\rfloor = 1 < k$, by Theorem 7, it follows that $G_{S_{(3,2)}}$ is a complete graph of order 3. By Remark 5, $C_3 \simeq K_3$. Hence, $G_{S_{(3,2)}}$ is also a cycle graph of order 3. Shown in Figure 8 is a pictorial illustration of $G_{S_{(3,2)}}$.*

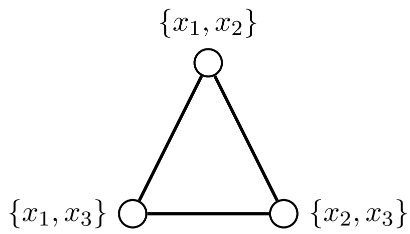


Figure 8: Pictorial Representation of $G_{S_{(3,2)}}$.

Note that $G_{S_{(n,k)}}$ is a simple graph. Now, we consider the complement of $G_{S_{(n,k)}}$, denoted by $\overline{G}_{S_{(n,k)}}$, as a simple graph. The next theorem determines $\overline{G}_{S_{(n,k)}}$ as well as the degree of its vertices.

Theorem 9. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. Then the complement of $G_{S_{(n,k)}}$ is an r-regular graph $\overline{G}_{S_{(n,k)}}$ such that*

$$r = \begin{cases} 0 & \text{if} \quad k = 0 \;\text{ or }\; \left\lfloor \frac{n}{2} \right\rfloor < k \leq n; \\ \binom{n-k}{k} & \text{if} \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \end{cases}$$

Proof. Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. By Theorem 5, $G_{S_{(n,k)}}$ is a regular graph. Note that its complement graph, $\overline{G}_{S_{(n,k)}}$ is also a regular graph. Now, to find the degree of every vertex in $\overline{G}_{S_{(n,k)}}$, consider the following cases:

CASE 1: If $k = 0$ or $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$, by Theorem 7, it follows that $G_{S_{(n,k)}}$ is a complete graph. The complement of a complete graph is an empty graph which implies that $\overline{G}_{S_{(n,k)}}$ is an empty graph. Thus, $deg(A) = 0$ for all $A \in V(\overline{G}_{S_{(n,k)}})$.

CASE 2: Let A be an arbitrary element of $V(G_{S_{(n,k)}})$. If $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, by Lemma 1, $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$ for all $A \in V(G_{S_{(n,k)}})$. It can be observed that A is not adjacent to $\binom{n-k}{k} + 1$ vertices in $G_{S_{(n,k)}}$. Since $\overline{G}_{S_{(n,k)}}$ is a simple graph, it follows that $deg(A) = \binom{n-k}{k}$ for all $A \in V(\overline{G}_{S_{(n,k)}})$.

To illustrate Theorem 9, we utilize the graphs $G_{S_{(4,3)}}$ and $G_{S_{(4,2)}}$ shown in the previous discussions.

Illustration 9. *Let $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ and let $k = 3$. A pictorial representation of $G_{S_{(4,3)}}$ is presented in Figure 9. Since $k = 3$ and $\left\lfloor \frac{4}{2} \right\rfloor = 2 < 3$ it follows that $G_{S_{(4,3)}}$ is a complete graph of order $\binom{4}{3} = 4$. The complement of a complete graph is an empty graph, thus, $\overline{G}_{S_{(4,3)}}$ is an empty graph of order $\binom{4}{3} = 4$. Presented also in Figure 9 is a pictorial representation of $\overline{G}_{S_{(4,3)}}$.*

Figure 9: Pictorial Representations of $G_{S_{(4,3)}}$ and its Complement Graph $\overline{G}_{S_{(4,3)}}$, Respectively.

It can be observed that $deg(A) = 0$ for all $A \in V(\overline{G}_{S_{(4,3)}})$. Therefore, $\overline{G}_{S_{(4,3)}}$ is a 0-regular graph.

For $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, consider also the set $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ and let $k = 2$. The graph of $G_{S_{(4,2)}}$ is previously shown in Figure 2. Now, shown in Figure 10, is a pictorial representation of the complement graph of $G_{S_{(4,2)}}$.

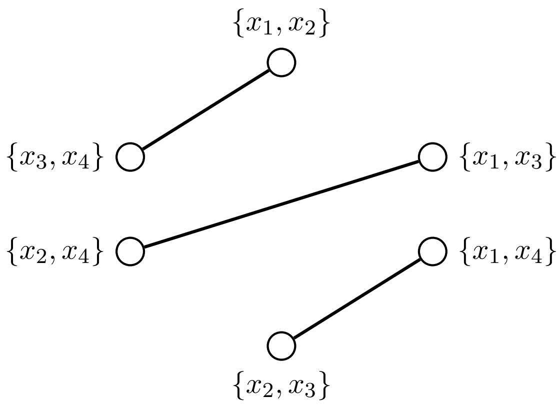


Figure 10: Pictorial Illustration of the Complement Graph $\overline{G}_{S_{(4,2)}}$.

Now, by connecting the vertices in $G_{S_{(4,2)}}$ with empty intersection, we have the edge set of $\overline{G}_{S_{(4,2)}}$ equal to the set $\big\{[\{x_1, x_2\}, \{x_3, x_4\}], [\{x_1, x_3\}, \{x_2, x_4\}], [\{x_2, x_3\}, \{x_1, x_4\}]\big\}$. It can be observed that every element in $V(\overline{G}_{S_{(4,2)}})$ has a degree equal to 1. Note that $k = \left\lfloor \frac{4}{2} \right\rfloor = 2$. Using Theorem 9 with $n = 4$ and $k = 2$, the degree of every vertex A in $\overline{G}_{S_{(4,2)}}$ we have:

$$\begin{aligned} deg(A) &= \binom{n-k}{k} \\ &= \binom{4-2}{2} \\ &= \binom{2}{2} \\ &= 1. \end{aligned}$$

Hence, $\overline{G}_{S_{(4,2)}}$ is a 1-regular graph.

It can be perceived from Case 1 of Theorem 9 $\overline{G}_{S_{(n,k)}}$ is an empty graph of order $\binom{n}{k}$. Consequently, we have Corollary 1 which is a corollary to Theorem 9.

Corollary 1. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, then $\overline{G}_{S_{(n,k)}}$ is an empty graph of order $\binom{n}{k}$.*

Proof. Assume that $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. By Theorem 9, $G_{S_{(n,k)}}$ is a complete graph of order $\binom{n}{k}$. Since the complement of a complete graph is an empty graph, it follows that $\overline{G}_{S_{(n,k)}}$ is an empty graph of order $\binom{n}{k}$.

Moreover, Corollary 2 determines the degree of every vertex in the complement graph of $G_{S_{(n,k)}}$ when $k = 1$. This is a corollary to Theorem 9.

Corollary 2. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 1$, then $\overline{G}_{S_{(n,1)}}$ is a complete graph of order n.*

Proof. Assume that $k = 1$. By Theorem 9, $\overline{G}_{S_{(n,1)}}$ is a $\left[\binom{n-k}{k}\right]$-regular graph. Now, setting $k = 1$, we have $deg(A) = \binom{n-1}{1} = n - 1$ for all $A \in V(\overline{G}_{S_{(n,1)}})$. Since $|G_{S_{(n,1)}}|$ is equal to $|\overline{G}_{S_{(n,1)}}| = n$, it follows that every vertex of $\overline{G}_{S_{(n,1)}}$ is adjacent to each other. Therefore, $\overline{G}_{S_{(n,1)}}$ is a complete graph of order n.

Illustration 10. *Let $\mathcal{S}_4 = \{x_1, x_2, x_3, x_4\}$ and let $k = 1$. If $k = 1$, then by Theorem 4, $G_{S_{(4,1)}}$ is an empty graph of order 4. We know that the complement graph of an empty graph is a complete graph. So, $\overline{G}_{S_{(4,1)}}$ is a complete graph of order 4. A pictorial representation of $G_{S_{(4,1)}}$ and $\overline{G}_{S_{(4,1)}}$ is shown in Figure 11.*

Observe that every $A \in V(\overline{G}_{S_{(4,1)}})$, $deg(A) = 4 - 1 = 3$. Thus, every vertex of $\overline{G}_{S_{(4,1)}}$ is adjacent to each other. Therefore, $\overline{G}_{S_{(4,1)}}$ is a complete graph of order 4.

5. Additional Parameters of $G_{S_{(n,k)}}$

Parameters are numerical values that help define a graph. In this section, other graph parameters such as the independence number, domination number, and isolate domination number are presented with proofs to determine a $G_{S_{(n,k)}}$.

Figure 11: Pictorial Illustrations of $G_{S_{(4,1)}}$ (Left) and its Complement Graph $\overline{G}_{S_{(4,1)}}$ (Right).

5.1. Independence Number of a $G_{S_{(n,k)}}$

This subsection examines the independence number of $G_{S_{(n,k)}}$ given the two cases: when $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. We shall denote the independence number of $G_{S_{(n,k)}}$ as $\alpha(G_{S_{(n,k)}})$.

The lemma below determines the existence of an independent set in $G_{S_{(n,k)}}$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Lemma 3. *Let $\mathcal{S}_n = \{x_1, x_2, ..., x_n\}$ be an n-element set and let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then*

$$T = \Big\{\{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, ..., x_{(\lfloor \frac{n}{k} \rfloor)k}\}\Big\}$$

is an independent set in $G_{S_{(n,k)}}$ where $|T| = \lfloor \frac{n}{k} \rfloor$.

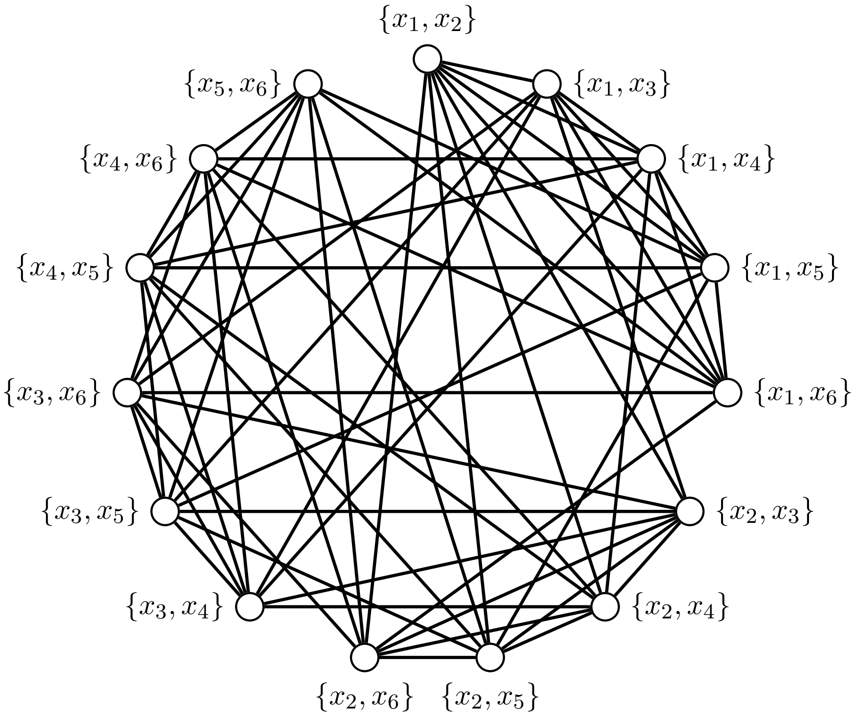
Proof. Let $T = \Big\{\{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, ..., x_{(\lfloor \frac{n}{k} \rfloor)k}\}\Big\}$. It can be observed that T contains some of the partitions of $\mathcal{S}_n$ where for every $A \in T$, $|A| = k$. This implies that $T \subseteq S_{(n,k)}$ and thus, $T \subseteq V(G_{S_{(n,k)}})$. Since for all $A, B \in T$ and $A \neq B$, $A \cap B = \emptyset$, it follows that $[A, B] \notin E(G_{S_{(n,k)}})$. Hence, T is an independent set in $G_{S_{(n,k)}}$. Now, if n is divisible by k, then $|T| = \frac{n}{k}$. On the other hand, if n is not divisible by k, then by Division Algorithm, there exist unique integers a and b with $0 < b < k$ such that $n = ka + b$. Meaning to say, there are b elements of $\mathcal{S}_n$ that are not in T. We cannot form a k-element subset from the b remaining elements, so $|T| = \lfloor \frac{n}{k} \rfloor$. Either way $|T| = \lfloor \frac{n}{k} \rfloor$.

Lemma 3 identifies one independent set in $G_{S_{(n,k)}}$. We will utilize this to determine $\alpha(G_{S_{(n,k)}})$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Illustration 11. *Let $\mathcal{S}_6 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $k = 2$. Then the vertex set of a $G_{S_{(6,2)}}$ is given by*

$$\begin{aligned} V(G_{S_{(6,2)}}) =& \big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_1, x_6\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \\ & \{x_2, x_6\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_3, x_6\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_5, x_6\}\big\} \end{aligned}$$

A pictorial illustration of $G_{S_{(6,2)}}$ is shown in Figure 12.



$G_{S_{(6,2)}}$

Figure 12: A Pictorial Illustration of $G_{S_{(6,2)}}$.

Now, we let $T_i \subseteq V(G_{S_{(6,2)}})$ *for* $1 \leq i \leq 3$. *Notice that* $T_1 = \big\{\{x_i, x_j\}\big\}$ *where* $1 \leq i, j \leq 6$ *and* $i \neq j$ *is an independent set since* $G_{S_{(6,2)}}$ *is a simple graph. Moreover, the set* $T_2 = \big\{\{x_1, x_2\}, \{x_3, x_4\}\big\}$ *is an independent set also since* $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$ *it follows that* $[\{x_1, x_2\}, \{x_3, x_4\}] \notin E(G_{S_{(6,2)}})$. *This implies that there exists an independent set with the cardinality equal to* 2. *On the other hand,* $T_3 = \big\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\big\}$ *is also an independent set since* $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$, $\{x_1, x_2\} \cap \{x_5, x_6\} = \emptyset$, *and* $\{x_3, x_4\} \cap \{x_5, x_6\} = \emptyset$ *it follows that* $[\{x_1, x_2\}, \{x_3, x_4\}], [\{x_1, x_2\}, \{x_5, x_6\}]$, $[\{x_3, x_4\}, \{x_5, x_6\}]$ *are not edges in* $G_{S_{(6,2)}}$. *Observe that* $\lfloor \frac{6}{2} \rfloor = 3$ *and* $|T_3| = 3$. *Hence, there exists an independent set in* $G_{S_{(6,2)}}$ *with the cardinality equal to* 3.

Note that there exists an independent set in $G_{S_{(n,k)}}$. With this, we can now compute for $\alpha(G_{S_{(n,k)}})$. Theorem 10 determines the independence number for a $G_{S_{(n,k)}}$ if $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 10. *Let* $\mathcal{S}_n = \{x_1, x_2, ..., x_n\}$ *be an* n*-element set and let* $G_{S_{(n,k)}}$ *be a* k*-restricted intersection graph. If* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, *then* $\alpha(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$.

Proof. Let $T = \Big\{\{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, ..., x_{(\lfloor \frac{n}{k} \rfloor)k}\}\Big\}$. By Lemma 3, T is an independent set in $G_{S_{(n,k)}}$ with $|T| = \lfloor \frac{n}{k} \rfloor$. Since there exists an independent set in $G_{S_{(n,k)}}$ with a cardinality equal to $\lfloor \frac{n}{k} \rfloor$, it follows that $\alpha(G_{S_{(n,k)}}) \geq \lfloor \frac{n}{k} \rfloor$. We claim that $\alpha(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$. Now, suppose $\alpha(G_{S_{(n,k)}}) > \lfloor \frac{n}{k} \rfloor$. This implies that there exists $W \subseteq V(G_{S_{(n,k)}})$ with $|W| > \lfloor \frac{n}{k} \rfloor$ that is an independent set in $G_{S_{(n,k)}}$. Without loss of generality, we say that n is not divisible by k. Hence, by the division algorithm, there exist integers a and b where $0 < b < k$ such that $n = ka + b$. However $|W| > |T|$. Thus, the remaining b elements are now in W in the form of k-subsets. Since $b < k$, it follows that these b elements are paired with some elements in $\mathcal{S}_n$ to form a k-element subset. By this, it can be observed that there will exist $A, B \in W$ where $A \neq B$ such that $A \cap B \neq \emptyset$. This is a contradiction to the fact that W is an independent set. Therefore, $\alpha(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$.

Illustration 12. *Consider $\mathcal{S}_6 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $k = 2$. A pictorial representation of $G_{S_{(6,2)}}$ is shown in Figure 12. Recall that from Illustration 11, $T = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ is an independent set in $G_{S_{(6,2)}}$ with $|T| = 3 = \lfloor \frac{6}{2} \rfloor$.*

Note that $\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}$ are 2-element set partitions of $\mathcal{S}_6$. Thus adding another element of $V(G_{S_{(6,2)}})$ will produce an intersection to any of $\{x_1, x_2\}, \{x_3, x_4\}$, or $\{x_5, x_6\}$. This entails that there are no independent sets in $G_{S_{(6,2)}}$ of cardinality greater than 3. Therefore, $\alpha(G_{S_{(6,2)}}) = 3$. Now, using Theorem 10, setting $n = 6$ and $k = 2$, we have

$$\begin{aligned} \alpha(G_{S_{(6,2)}}) &= \left\lfloor \frac{n}{k} \right\rfloor \\ &= \left\lfloor \frac{6}{2} \right\rfloor \\ &= \lfloor 3 \rfloor \\ &= 3. \end{aligned}$$

Now, by Theorem 4, if $k = 1$, then $G_{S_{(n,1)}}$ is an empty graph of order n. Note that there is no edge connecting every vertex of $G_{S_{(n,1)}}$ and $V(G_{S_{(n,1)}}) \subseteq V(G_{S_{(n,1)}})$, so the set $V(G_{S_{(n,1)}})$ is an independent set in $G_{S_{(n,1)}}$. Since $|V(G_{S_{(n,1)}})| = n$, it follows that $\alpha(G_{S_{(n,1)}}) = n$. It can be verified from Theorem 10 that when $k = 1$, $\alpha(G_{S_{(n,1)}}) = \lfloor \frac{n}{1} \rfloor = n$. Correspondingly, we have Remark 9.

Remark 9. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 1$, then $\alpha(G_{S_{(n,1)}})$ is equal to n.*

We have now identified the $\alpha(G_{S_{(n,k)}})$ whenever $1 \leq k \leq \lfloor \frac{n}{k} \rfloor$. Now, Theorem 11 determines the independence number of $G_{S_{(n,k)}}$ when $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.

Theorem 11. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, then $\alpha(G_{S_{(n,k)}}) = 1$.*

Proof. By Theorem 7, $G_{S_{(n,k)}}$ is a complete graph of order $\binom{n}{k}$. Since the independence number of a complete graph is 1, it follows that $\alpha(G_{S_{(n,k)}}) = 1$.

5.2. Domination Number of $G_{S_{(n,k)}}$

This subsection further exposes the domination number of a $G_{S_{(n,k)}}$ when $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $k = 0$ or $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$. We will denote $\gamma(G_{S_{(n,k)}})$ as the domination number of $G_{S_{(n,k)}}$.

The next lemma establishes the existence of a dominating set in $G_{S_{(n,k)}}$ whenever $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Lemma 4. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, then*

$$T = \left\{ \{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)k+1}, ..., x_{\left(\left\lfloor \frac{n}{k} \right\rfloor\right)k}\} \right\},$$

is a dominating set in $G_{S_{(n,k)}}$ where $|T| = \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. Let $T = \left\{ \{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)k+1}, ..., x_{\left(\left\lfloor \frac{n}{k} \right\rfloor\right)k}\} \right\}$. It can be observed that T contains some of the k-element partitions of $\mathcal{S}_n$. Thus, $T \subseteq S_{(n,k)}$ which implies that $T \subseteq V(G_{S_{(n,k)}})$. If n is divisible by k, then T contains all the elements of $\mathcal{S}_n$ that are partitioned into k-subsets. By this, then for every $A \in V(G_{S_{(n,k)}}) \setminus T$, there exists $B \in T$ such that $A \cap B \neq \emptyset$. By Definition 7, $[A, B] \in E(G_{S_{(n,k)}})$. Hence, T is a dominating set in $G_{S_{(n,k)}}$. Since n is divisible by k, it follows that $|T| = \frac{n}{k}$. On the other hand, if n is not divisible by k, then by the division algorithm, there exist integers a and b with $0 < b < k$ such that $n = ka + b$. Meaning to say, there are b remaining elements that are not in T since we cannot form a k-subset from these elements. Observe that the remaining b elements can be paired to other elements of $\mathcal{S}_n$ to form a k-subset such that these subsets are elements of $V(G_{S_{(n,k)}}) \setminus T$. Thus, for all $A \in V(G_{S_{(n,k)}}) \setminus T$, there exists $B \in T$ where $A \cap B \neq \emptyset$ that implies $[A, B] \in E(G_{S_{(n,k)}})$. By this, T is a dominating set in $G_{S_{(n,k)}}$ with $|T| = \left\lfloor \frac{n}{k} \right\rfloor$. Either way, $|T| = \left\lfloor \frac{n}{k} \right\rfloor$.

The set T discussed in Lemma 4 is one of the dominating sets in $G_{S_{(n,k)}}$. This set shall be employed to find to find $\gamma(G_{S_{(n,k)}})$ whenever $1 \leq k \leq \left\lfloor \frac{n}{k} \right\rfloor$.

Illustration 13. *Let $\mathcal{S}_6 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $k = 2$. To pictorially illustrate $G_{S_{(6,2)}}$, refer to Figure 12. Also, let $T = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$. It can be observed that $T \subseteq V(G_{S_{(6,2)}})$. Now, the set $V(G_{S_{(6,2)}}) \setminus T$ is given by*

$$\begin{aligned} V(G_{S_{(6,2)}}) \setminus T = &\{ \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_1, x_6\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \\ &\{x_2, x_6\}, \{x_3, x_5\}, \{x_3, x_6\}, \{x_4, x_5\}, \{x_4, x_6\} \} \end{aligned}$$

The vertices in $V(G_{S_{(6,2)}}) \setminus T$ that are adjacent to $\{x_1, x_2\}$ are $\{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_1, x_6\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_2, x_6\}$. Moreover, $\{x_3, x_5\}, \{x_3, x_6\}, \{x_4, x_5\}, \{x_4, x_6\}$ are the vertices adjacent to $\{x_3, x_4\}$. All of the elements in $V(G_{S_{(6,2)}}) \setminus T$ are adjacent to either $\{x_1, x_2\}$ or $\{x_3, x_4\}$. Thus, T is a dominating set. Note that $|T| = 3 = \left\lfloor \frac{6}{2} \right\rfloor$. Hence, there exists a dominating set in $G_{S_{(6,2)}}$ with a cardinality of $\left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{6}{2} \right\rfloor$.

By Lemma 4, it can be observed that there exists a dominating set in $G_{S_{(n,k)}}$. Hence, we can now compute for $\gamma(G_{S_{(n,k)}})$ where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 12. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, then $\gamma(G_{S_{(n,k)}}) = \left\lfloor \frac{n}{k} \right\rfloor$.*

Proof. Let $T = \Big\{\{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, ..., x_{(\lfloor \frac{n}{k} \rfloor)k}\}\Big\}$. By Lemma 4, T is a dominating set in $G_{S_{(n,k)}}$ where $|T| = \left\lfloor \frac{n}{k} \right\rfloor$. Since there exists a dominating set in $G_{S_{(n,k)}}$ with a cardinality equal to $\left\lfloor \frac{n}{k} \right\rfloor$, it follows that $\gamma(G_{S_{(n,k)}}) \leq \left\lfloor \frac{n}{k} \right\rfloor$. We claim that $\gamma(G_{S_{(n,k)}}) = \left\lfloor \frac{n}{k} \right\rfloor$. Now, suppose $\gamma(G_{S_{(n,k)}}) < \left\lfloor \frac{n}{k} \right\rfloor$. Thus, there exists $W \subseteq V(G_{S_{(n,k)}})$ such that W is a dominating set in $G_{S_{(n,k)}}$ and $|W| < \left\lfloor \frac{n}{k} \right\rfloor$. Without loss of generality, assume that n is not divisible by k. So by the division algorithm, there exist unique integers a and b where $n = ka + b$. But note that $|W| < |T|$. Hence, $k \leq b$ to make $|W| < |T|$. This is a contradiction to the fact that $0 < b < k$. Therefore, $\gamma(G_{S_{(n,k)}}) = \left\lfloor \frac{n}{k} \right\rfloor$.

Illustration 14. *Let $\mathcal{S}_6 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $k = 2$. In Illustration 13, we have identified that $T = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ is a dominating set. We cannot remove $\{x_5, x_6\}$ from T since $\{x_1, x_2\} \cap \{x_5, x_6\} = \emptyset$ and $\{x_3, x_4\} \cap \{x_5, x_6\} = \emptyset$ which implies that they are not adjacent to each other if $\{x_5, x_6\}$ becomes an element of $V(G_{S_{(6,2)}}) \setminus T$. Thus, there are no dominating sets in $G_{S_{(6,2)}}$ with cardinality less than 3. Hence, $\gamma(G_{S_{(6,2)}}) = 3$.*

Now, by using Theorem 12, setting $n = 6$ and $k = 2$, we have:

$$\begin{aligned} \gamma(G_{S_{(6,2)}}) &= \left\lfloor \frac{n}{k} \right\rfloor \\ &= \left\lfloor \frac{6}{2} \right\rfloor \\ &= \lfloor 3 \rfloor \\ &= 3. \end{aligned}$$

By Theorem 4, $G_{S_{(n,1)}}$ is an empty graph of order n. Since every vertex of an empty graph is isolated, or has no adjacency to every other vertex, it follows that $V(G_{S_{(n,1)}})$ is a dominating set in $G_{S_{(n,1)}}$. It can be verified that there are no existing dominating set in $G_{S_{(n,1)}}$ with cardinality less than n, therefore, $\gamma(G_{S_{(n,1)}}) = n$. To verify using Theorem 12 with $k = 1$, then we have $\gamma(G_{S_{(n,1)}}) = \left\lfloor \frac{n}{1} \right\rfloor = n$. Equivalently, we have Remark 10.

Remark 10. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 1$, then $\gamma(G_{S_{(n,1)}})$ is equal to n.*

Moreover, Theorem 13 determines the domination number of $G_{S_{(n,k)}}$ whenever $k = 0$ or $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$.

Theorem 13. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $k = 0$ or $\left\lfloor \frac{n}{2} \right\rfloor < k \leq n$ then $\gamma(G_{S_{(n,k)}}) = 1$.*

Proof. Note that by Theorem 7, $G_{S_{(n,k)}}$ is a complete graph. Since $G_{S_{(n,k)}}$ is a complete graph, since the domination number of a complete graph is 1, it follows that $\gamma(G_{S_{(n,k)}}) = 1$.

5.3. Isolate Domination Number of $G_{S_{(n,k)}}$

This subsection analyzes the isolate domination of $G_{S_{(n,k)}}$ given the two cases: if $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. We will utilize the notion $\gamma_0(G_{S_{(n,k)}})$ to denote the isolate domination number of $G_{S_{(n,k)}}$.

The next theorem determines the γ_0 of a $G_{S_{(n,k)}}$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. The set T discussed in Lemma 3 and Lemma 4 will be used to define the existence of an isolate dominating set in $G_{S_{(n,k)}}$.

Theorem 14. *Let $\mathcal{S}_n = \{x_1, x_2, ..., x_n\}$ be an n-element set and let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then $\gamma_0(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$.*

Proof. Let $T = \Big\{\{x_1, x_2, ..., x_k\}, \{x_{k+1}, ..., x_{2k}\}, ..., \{x_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, ..., x_{(\lfloor \frac{n}{k} \rfloor)k}\}\Big\}$. Note that by Lemma 3, T is an independent set in $G_{S_{(n,k)}}$ where $|T| = \lfloor \frac{n}{k} \rfloor$. It can be verified that $< T >$ is an empty graph of order $\lfloor \frac{n}{k} \rfloor$. Thus, for all $A \in V(< T >)$, we have $deg(A) = 0$ which implies that every vertex in $< T >$ is an isolated vertex. Moreover, by Lemma 4, the set T is also a dominating set in $G_{S_{(n,k)}}$. Since T is an independent set and dominating set in $G_{S_{(n,k)}}$, it follows that T is an isolate dominating set in $G_{S_{(n,k)}}$ with $|T| = \lfloor \frac{n}{k} \rfloor$. Hence, there exist an isolate dominating set in $G_{S_{(n,k)}}$ with cardinality equal to $\lfloor \frac{n}{k} \rfloor$. This implies that $\gamma_0(G_{S_{(n,k)}}) \leq \lfloor \frac{n}{k} \rfloor$. By Theorem 12, $\gamma(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$. Note that $\gamma(G_{S_{(n,k)}}) \leq \gamma_0(G_{S_{(n,k)}})$. Therefore, $\gamma_0(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$.

Theorem 15. *Let $G_{S_{(n,k)}}$ be a k-restricted intersection graph. Then $\gamma_0(G_{S_{(n,k)}}) = 1$ if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.*

Proof. By Theorem 7, $G_{S_{(n,k)}}$ is a complete graph of order $\binom{n}{k}$. Since $G_{S_{(n,k)}}$ is a complete graph, since the isolate domination number of a complete graph is 1, it follows that $\gamma_0(G_{S_{(n,k)}}) = 1$.

6. Summary, Conclusion, and Recommendations

This study introduces and discusses a k-restricted intersection graph including some of the graph's parameters.A k-restricted intersection graph is a simple graph whose vertex set is equal to $S_{(n,k)}$ and two vertices A, B in $G_{S_{(n,k)}}$ are adjacent whenever $A \cap B \neq \emptyset$ and $A \neq B$. The case $\mathcal{S}_0$ is a trivial case for $G_{S_{(n,k)}}$.

The order of $G_{S_{(n,k)}}$ is given by $\binom{n}{k}$. It was determined that $G_{S_{(n,k)}}$ is a trivial graph if and only if $k = 0$ or $k = n$. Also, if $k = 1$, then $G_{S_{(n,k)}}$ is an empty graph of order n. In addition, it was established that $G_{S_{(n,k)}}$ is a regular graph such that for any $A \in V(G_{S_{(n,k)}})$, $deg(A) = \binom{n}{k} - \left[\binom{n-k}{k} + 1\right]$ if and only if $1 \leq k \leq \lfloor \frac{n}{k} \rfloor$. Furthermore, $deg(A) = \binom{n}{k} - 1$ if and only if $k = 0$ or $\lfloor \frac{n}{k} \rfloor < k \leq n$. With these, the size of $G_{S_{(n,k)}}$ is proven to be equal to

$$|E(G_{S_{(n,k)}})| = \begin{cases} \frac{\binom{n}{k}\left\{\binom{n}{k} - \left[\binom{n-k}{k}+1\right]\right\}}{2} & \text{if} \quad 1 \leq k \leq \lfloor \frac{n}{2} \rfloor; \\ \frac{\binom{n}{k}\left[\binom{n}{k}-1\right]}{2} & \text{if} \quad k = 0 \quad \text{or} \quad \lfloor \frac{n}{2} \rfloor < k \leq n. \end{cases}$$

Additionally, the study disclosed that $G_{S_{(n,k)}}$ was a complete graph if and only if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$. More so, $G_{S_{(n,k)}}$ is a cycle graph of order 3 if and only if $n = 3$ and $k = 2$.

The complement graph of $G_{S_{(n,k)}}$ was also determined. Considering that $G_{S_{(n,k)}}$ is a simple graph, then its complement graph is also a simple graph. It was further established that $\overline{G}_{S_{(n,k)}}$ is a regular graph having the following degree for all $A \in V(\overline{G}_{S_{(n,k)}})$, depending on each cases: if $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $deg(A) = \binom{n-k}{k}$; on the other hand if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, $deg(A) = 0$.

Finally, other parameters of $G_{S_{(n,k)}}$ such as the independence number, domination number, and the isolate domination number were specified. The independence number of $G_{S_{(n,k)}}$ when $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is equal to $\lfloor \frac{n}{k} \rfloor$ while if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$, then the independence number is 1. Further, it was found out that for any value of k, we have $\gamma(G_{S_{(n,k)}}) = \gamma_0(G_{S_{(n,k)}})$. If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then $\gamma(G_{S_{(n,k)}}) = \gamma_0(G_{S_{(n,k)}}) = \lfloor \frac{n}{k} \rfloor$. Meanwhile, $\gamma(G_{S_{(n,k)}}) = \gamma_0(G_{S_{(n,k)}}) = 1$ if $k = 0$ or $\lfloor \frac{n}{2} \rfloor < k \leq n$.

In conclusion, we determine the order, size, independence number, domination number, and isolated domination number of a k-restricted intersection graph in this study. Also, we provided the necessary and sufficient conditions for a $G_{S_{(n,k)}}$ to be isomorphic to a cycle graph and a complete graph.

From the results drawn, the researchers believed that a parallel study may be done to further characterize $G_{S_{(n,k)}}$. In particular, we recommend that future studies find other parameters of a $G_{S_{(n,k)}}$ such as its girth, clique number, chromatic number, and locating domination number to name a few, for this will also be helpful in determining the graph. Moreover, it is recommended to further investigate $G_{S_{(n,k)}}$ proposing the utilization of binary graph operations wherein some of these are the sum of joint, cartesian product, composition, edge gluing, and vertex gluing may be imperatively conducted. Lastly, future researchers may consider adapting $G_{S_{(n,k)}}$ in solving real-world problems for some of the results established are based on the binomial coefficient or the combination formula that has several real-world applications.

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