EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1618-1636
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Closed Geodetic Hop Domination in Graphs

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Abstract. Let G be a simple, undirected and connected graph. A subset $S \subseteq V(G)$ is a geodetic cover of G if $I_G[S] = V(G)$, where $I_G[S]$ is the set of all vertices of G lying on any geodesic between two vertices in S. A geodetic cover S of G is a closed geodetic cover if the vertices in S are sequentially selected as follows: Select a vertex v_1 and let $S_1 = \{v_1\}$. If G is nontrivial, select a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$. Where possible, for $i \geq 3$, successively select vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, ..., v_i\}$. Then there exists a positive integer k such that $S_k = S$.

A geodetic cover S of G is a geodetic hop dominating set if every vertex in $V(G) \setminus S$ is of distance 2 from a vertex in S. A geodetic hop dominating set S is a closed geodetic hop dominating set if S is a closed geodetic cover of G. The minimum cardinality of a (closed) geodetic hop dominating set of G is the (closed) geodetic hop domination number of G. This study initiates the study of the closed geodetic hop domination. First, it characterizes all graphs G of order n whose closed geodetic hop domination numbers are 2 or n, and determines the closed geodetic hop domination number of paths, cycles and multigraphs. Next, it shows that any positive integers a and b with $2 \leq a \leq b$ are realizable as the closed geodetic number and closed geodetic hop domination number of a connected graph. Also, every positive integer n, m and k with $4 \leq m \leq k$ and $2k-m+2 \leq n$ are realizable as the order, geodetic hop domination number and closed geodetic hop domination number, respectively of a connected graph. Furthermore, the study characterizes the closed geodetic hop dominating sets of graphs resulting from the join, corona and edge corona of graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Closed geodetic cover, hop dominating set, geodetic hop dominating set, closed geodetic hop dominating set, closed geodetic hop domination number, join, corona, edge corona

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5241

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1. Introduction

F. Harary in [9] introduced two categories of graphical games called the achievement and avoidance games from which the concept of closed geodetic number evolved. The closed geodetic sets and closed geodetic numbers of connected graphs, which can find applications in location theory and convexity theory, have been extensively studied in [1, 2, 6, 7].

The hop domination in graphs is introduced in [14] by S.K. Ayyaswamy and C.Natarajan. Accordingly, this graph theoretic concept originated from the second electron affinity in Inorganic Chemistry. It has attracted relatively much attention and several further studies including investigations on some of its variations can be found in the existing literature (see [4, 5, 12–17]).

In this present paper, inspired by the above-mentioned concepts, we introduce and initiate the study of closed geodetic hop domination in graphs.

All graphs considered in this study are simple, undirected and connected. All graph terminologies which are not defined but are used here are adopted from [6].

As usual, we write $G = (V(G), E(G))$ for a graph G where $V(G)$ and $E(G)$ are the *vertex set* and *edge set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is the *order* of G.

Let G and H be two graphs with disjoint vertex sets. The *join* of G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G)\dot{\cup}V(H)$ and edge-set $E(G + H) = E(G)\dot{\cup}E(H)\dot{\cup}\ \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H.

For vertices u and v in G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest path in G joining u and v. Any path joining u and v of length $d_G(u, v)$ is called a *u-v geodesic.* The *diameter* $diam(G)$ of a graph G is the length of any longest geodesic of G. For every two vertices u and v of a graph G, the *interval* $I_G[u, v]$ refers the set of all vertices lying in some u-v geodesic. A vertex is called an *end-vertex* or a *leaf* if its degree is 1. The set of all end-vertices of G is denoted by $L(G)$. A vertex v in a connected graph G is an *support vertex* if v is adjacent to a leaf vertex of G. A vertex v in a connected graph G is an *extreme vertex* if for every pair of distinct vertices u and w with $\{uv, wv\} \subseteq E(G)$, $uw \in E(G)$. The set of all extreme vertices in G is denoted by $Ext(G)$. A vertex v in a connected graph G is a *dominating vertex* if $uv \in E(G)$ for all $u \in V(G) \setminus \{v\}$. $Dom(G)$ is the set of all dominating vertices in G.

For $S \subseteq V(G)$, the 2-*path closure* $P_2[S]_G$ of S is the set $P_2[S]_G = S \cup \{w \in V(G) : w \in I_G[u, v]$ for some $u, v \in S$ with $d_G(u, v) = 2\}$. A set S is called 2-*path closure absorbing* if $P_2[S]_G = V(G)$ [7]. We denote by $\rho_2(G)$ the minimum cardinality of a 2-path closure absorbing set of G. A set $S \subseteq V(G)$ is a *pointwise non-dominating set* of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \notin E(G)$. A pointwise non-dominating set $S \subseteq V(G)$ of a graph G is a 2-*path closure absorbing pointwise non-dominating set* if it is a 2-path closure absorbing set. The minimum cardinality of a 2-path closure absorbing pointwise non-dominating set in G is denoted by $\rho_{2pnd}(G)$

A *clique* in G is a complete subgraph of G. A *maximal clique* is a clique which is not a proper subgraph of a larger clique. The *lower clique number* $\omega_L(G)$ is the minimum size of all maximal cliques of G.

For $S \subseteq V(G)$, the *geodetic closure* $I_G[S]$ is the union of intervals between all pairs of vertices from S, that is, $I_G[S] = \bigcup\{I_G[u,v] : u,v \in S\}$. S is a *geodetic set* provided $I_G[S] = V(G)$. The minimum cardinality $gn(G)$ of a geodetic set is the *geodetic number* of G. A geodetic set of cardinality $gn(G)$ is a *geodetic basis.* The introduction and further studies on geodetic sets and geodetic numbers can be found in [6–10].

A geodetic set S of G is a *closed geodetic cover* of G if S is obtained as follows: Choose $v_1 \in V(G)$ and put $S_1 = \{v_1\}$. Where possible, choose $v_2 \in V(G) \setminus \{v_1\}$ and put $S_2 = \{v_1, v_2\}$. For $i \geq 3$, choose $v_i \in V(G) \setminus I_G[S_{i-1}]$, where $S_k = \{v_1, v_2, \ldots, v_i\}$, and there exists a positive integer k for which $S_k = S$. The *closed geodetic number* of G, denoted $cgn(G)$, is the smallest positive integer k for which $I_G[S_k] = V(G)$, where S_k is obtained as illustrated above. If $C^*(G)$ is the collection of all closed geodetic covers of G, then $cgn(G) = min\{|S| : S \in C^*(G)\}$. Any set $S \in C^*(G)$ with $|S| = cgn(G)$ is a *closed geodetic basis* of G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v,u) = 2\}$ is called the *open hop neighborhood of* u. The *closed hop neighborhood of* u in G is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood of* $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The closed hop neighborhood of X in G is the set $N_G^2[X] = N_G^2(X) \cup X$. Let G be a connected graph. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. In particular, a set $S \subseteq V(G)$ is a hop dominating set if $N_G^2[S] = V(G)$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equals to $\gamma_h(G)$ is called a γ_h*-set* of G.

A subset S of vertex set of G is a *geodetic hop dominating set* if it is both a geodetic and a hop dominating set. The *geodetic hop domination number* $\gamma_{hg}(G)$ of G is the minimum cardinality among all geodetic hop dominating sets in G. Any geodetic hop dominating set of G with cardinality $\gamma_{hg}(G)$ is called a γ_{hg}*-set* of G. The geodetic hop dominating set was first introduced by Anusha et al. in [3]. It is further investigated by Saromines et al. in [18, 19].

The following results concerning geodetic hop domination on paths and cycles are found in [19].

Proposition 1. [19] *Let n be a positive integer.*

(*i*) *For a path P_n on n vertices,*

$$\gamma_{hg}(P_n) = \begin{cases} n & \text{if } n = 1, 2, \\ \frac{n+6}{3} & \text{if } n \equiv 0 (mod\ 3)\ , \\ \frac{n+2}{3} & \text{if } n \equiv 1 (mod\ 3)\ , \\ \frac{n+4}{3} & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

(*ii*) *For a cycle C_n on n vertices,*

$$\gamma_{hg}(C_n) = \begin{cases} 3 & \text{if } n = 3, 4, 5, \\ \frac{n}{3} & \text{if } n \equiv 0 (mod\ 3)\ , \\ \frac{n+2}{3} & \text{if } n \equiv 1 (mod\ 3)\ , \\ \frac{n+4}{3} & \text{if } n \equiv 2 (mod\ 3)\ . \end{cases}$$

2. Results

In this section, we introduce and initiate the study of closed geodetic hop domination in graphs.

2.1. Closed Geodetic Hop Domination

A subset S of vertices of G is a *closed geodetic hop dominating set* if it is both a geodetic hop dominating set and a closed geodetic cover of G. The minimum cardinality among all closed geodetic hop dominating sets in G, denoted by $\gamma_{hcg}(G)$ is called the *closed geodetic hop domination number* of G. A closed geodetic hop dominating set S of G with $|S| = \gamma_{hcg}(G)$ is called a γ_{hcg}-set of G.

We remark that not every graph admits a closed geodetic hop dominating set. Consider, for example, the graph $G = C_{12}$. It is easy to verify that every closed geodetic set of G is not a hop dominating set.

Observation 1. *Let G be a connected graph. If $V(G)$ is a closed geodetic set, then G admits a closed geodetic hop dominating set.*

The following also provides some other conditions under which a graph admits a closed geodetic hop dominating set.

Proposition 2. *If G is a connected graph with $diam(G) \leq 2$, then G admits a closed geodetic hop dominating set.*

Proof. If $diam(G) = 1$, then G is complete so that $V(G)$ is a closed geodetic set. As observed above, G admits a closed geodetic hop dominating set. Suppose $diam(G) = 2$. Then G is not complete. Let G_1 be a maximal clique of G. Let $V(G_1) = S_r =$

$\{u_1, u_2, ..., u_r\}$. Since G is not complete, $r < |V(G)|$. For $i \geq 1$, choose $u_{r+i} \in V(G)$ such that $u_{r+i} \notin I_G[S_{r+i-1}]$ where $S_{r+i-1} = \{u_1, u_2, ..., u_r, u_{r+1}, ..., u_{r+i-1}\}$. Since $V(G)$ is finite, there exists a smallest positive integer $n > r$ for which $I_G[S_n] = V(G)$. If $V(G) \setminus S_n = \varnothing$ then $S_n = V(G)$. Suppose $V(G) \setminus S_n \neq \varnothing$, and let $a \in V(G) \setminus S_n$. By maximality of G_1, there exists b in S_r such that $d_G(a, b) = 2$. Hence S_n is a hop dominating set of G. Therefore, S_n is a closed geodetic hop dominating set of G. □

We denote by $\mathscr{C}_h^*$ the family of all connected graphs that admit a closed geodetic hop dominating set.

Since closed geodetic hop dominating sets are themselves geodetic hop dominating sets,

$$2 \leq \gamma_{hg}(G) \leq \gamma_{hcg}(G) \leq n \tag{1}$$

for all graphs $G \in \mathscr{C}_h^*$.

Theorem 2. *Let $G \in \mathscr{C}_h^*$. Then $\gamma_{hcg}(G) = 2$ if and only if either $G = K_2$ or G has a geodetic set $S = \{u, v\}$ such that $d_G(u, v) = 3$.*

Proof. Suppose that $\gamma_{hcg}(G) = 2$. If $G = K_2$, then we are done. Suppose that $G \neq K_2$. Let $S = \{u, v\}$ be a closed geodetic hop dominating set of G. Since $G \neq K_2$, $V(G) \setminus S \neq \varnothing$ and $w \in I_G(u, v)$ for every $w \in V(G) \setminus S$. Let $[u = x_1, x_2, x_3, \ldots, x_k = v]$ be a u-v geodesic in G. Then $k \geq 3$. Since S is a hop dominating set, in particular, $d_G(x_2, v) = 2$. Necessarily, $k = 4$ and $d_G(u, v) = 3$.

Clearly, if $G = K_2$, then $\gamma_{hcg}(G) = 2$. Suppose that G has a geodesic set $S = \{u, v\}$ with $d_G(u, v) = 3$. Then S is a closed geodetic set of G. Let $w \in V(G) \setminus S$. Being a geodetic set, there exists a u-v geodesic $[u, x, y, v]$ on which w lies. If $w = x$, then $d_G(w, v) = 2$. If $w = y$, then $d_G(u, w) = 2$. Accordingly, S is a hop dominating set of G. Thus, $\gamma_{hcg}(G) \leq |S| = 2$. Equation 1 completes the desired equality. □

Lemma 1. *Let $G \in \mathscr{C}_h^*$ of order n. If $\gamma_{hcg}(G) = n$, then G has a dominating vertex.*

Proof. This is clear for $n = 1, 2$. Let $n \geq 3$. Assume that $\gamma_{hcg}(G) = n$. Suppose that G does not contain a dominating vertex. The assumption implies that there exists a sequence of sets of vertices of G, say $S_k = \{v_1, v_2, \ldots, v_k\}$, $k = 1, 2, \ldots, n$, such that $v_1 \neq v_2$ and $v_k \notin I_G[S_{k-1}]$ for all $k \geq 3$.

Now, since G is not complete and $n \geq 3$, G has vertices u and v such that $d_G(u, v) = 2$. Let $[u, w, v]$ be a u-v geodesic in G. For some distinct $i, j, k \in \{1, 2, \ldots, n\}$, we have $u = v_i$, $w = v_j$ and $v_k = v$. Without loss of generality, assume $i < k$. Since $w \in I_G[u, v]$ and $v_j \notin I_G[S_{j-1}]$, $j < k$. Define $T_l = \{x_1, x_2, \ldots, x_l\}$ for $l = 1, 2, \ldots, m$ with $k \leq m \leq n - 1$ such that

- $x_l = v_l$ for all $l \in \{1, 2, \ldots, j - 1\}$;
- $x_l = v_{l+1}$ for all $l \in \{j, j + 1, \ldots, m\}$.

Then

- $I_G[T_l] = I_G[S_l] = S_l = T_l$ for all $l \in \{1, 2, \dots, k-2\}$;
- $I_G[T_{k-1}] = T_{k-1} \cup \{w\} = S_k$; and
- $I_G[T_l] = S_{l+1}$ for all $l \in \{k, k+1, \dots, m\}$.

This means that $T_m = V(G) \setminus \{w\}$ is a closed geodetic set of G. Finally, since w is not a dominating vertex of G, there exists $z \in V(G) \setminus \{w\} = T_m$ such that $d_G(w, z) = 2$. Thus, T_m is a hop dominating set of G. Hence, $\gamma_{hcg}(G) \leq |T_m| = m < n$, a contradiction. Therefore, G contains a dominating vertex. □

If $G \in \mathscr{C}_h^*$, then a closed geodetic hop dominating set of G contains the extreme vertices. It also contains all dominating vertices.

Theorem 3. *Let $G \in \mathscr{C}_h^*$ of order n. Then $\gamma_{hcg}(G) = n$ if and only if either*

(*i*) *$G = K_n$; or*

(*ii*) *$G \neq K_n$ such that the set S of dominating vertices is nonempty and each of the components of $\langle V(G) \setminus S \rangle$ is complete.*

Proof. If $G = K_n$, then $V(G)$ is the unique closed geodetic hop dominating set of G. Thus, $\gamma_{hcg}(G) = n$. Suppose that $G \neq K_n$. First, assume $\gamma_{hcg}(G) = n$. By Lemma 1, the set S of dominating vertices of G is nonempty. Let C be a component of $\langle V(G) \setminus S \rangle$. We claim that C is complete. Let $x \in V(C)$ and let $u, v \in N_C(x)$. Suppose, to the contrary, that $uv \notin E(C)$. Following a similar proof to that of Lemma 1, $T = V(G) \setminus \{x\}$ is a closed geodetic hop dominating set of G, a contradiction. Thus, $uv \in E(G)$, showing that C is complete. Conversely, suppose that G is as described in condition (*ii*). Let $T \subseteq V(G)$ be a closed geodetic hop dominating set of G. By the preceding remark, $S \subseteq T$. Let C be a component of $G^* = \langle V(G) \setminus S \rangle$. Let $x \in V(C)$ and $u, v \in N_G(x)$. If $u, v \in V(C)$, then $uv \in E(G)$ since C is complete. Suppose that $u \notin V(C)$. Then $u \in S$, i.e., u is a dominating vertex in G. Thus, $uv \in E(G)$. This shows that $x \in Ext(G) \subseteq T$. Thus, $V(C) \subseteq T$. Since C is arbitrary,

$$V(G) = S \cup (\cup_{C \text{ component of } G^*} V(C)) = T.$$

Since T is arbitrary, $\gamma_{hcg}(G) = |V(G)| = n$. □

The star graph $K_{1,n}$ is an example of the infinite family of graphs described in Theorem 3(*ii*).

2.2. For Paths P_n, Cycles C_n and Multipartite Graphs

Since every geodetic hop dominating set of P_n is a closed geodetic hop dominating set of P_n, we have the following:

Proposition 3. *For a path P_n on n vertices,*

$$\gamma_{hcg}(P_n) = \begin{cases} n & \textit{if } n = 1, 2, \\ \frac{n+6}{3} & \textit{if } n \equiv 0 (\textit{mod } 3)\ , \\ \frac{n+2}{3} & \textit{if } n \equiv 1 (\textit{mod } 3)\ , \\ \frac{n+4}{3} & \textit{if } n \equiv 2 (\textit{mod } 3) \end{cases}$$

Proposition 4. *A cycle graph C_n of order n admits a closed geodetic hop dominating set if and only if $n < 12$. Moreover precisely,*

$$\gamma_{hcg}(C_n) = \begin{cases} 3 & \textit{if } n = 3, 4, 5 \\ \frac{n}{3} & \textit{if } n = 6, 9 \\ \frac{n+2}{3} & \textit{if } n = 7, 10 \\ \frac{n+4}{3} & \textit{if } n = 8, 11 \end{cases} \tag{2}$$

Proof. The case where $3 \leq n \leq 11$ can be readily verified. Suppose that $n \geq 12$. Let $P = [x_1, x_2, ..., x_k]$, $k = \lceil \frac{n}{2} \rceil$, be a path in C_n and $v \in V(C_n) \setminus V(P)$. Then $M = \{x_1, x_2, ..., x_k, v\}$ is a closed geodetic cover of C_n with $|M| = \lceil \frac{n}{2} \rceil + 1$. Since $n \geq 12$, $|V(C_n) \setminus V(P)| \geq 6$. Thus, C_n has at least 2 adjacent vertices which are not hop dominated by $V(P)$. Consequently, C_n has at least one vertex which is not hop dominated by M. This means that M is not a hop dominating set of C_n (see, for example, Figure 1).

We claim that every closed geodetic cover S of C_n is contained in a closed geodetic cover M of C_n as constructed above with $|M| = \lceil \frac{n}{2} \rceil + 1$. Let $S = S_k = \{v_1, v_2, ..., v_k\}$ be a closed geodetic cover of C_n. Note that here, $I_G[S_j] \neq V(C_n)$ for all $j \in \{1, ..., k-1\}$ and $I_G[S_k] = V(C_n)$. If $k = \lceil \frac{n}{2} \rceil + 1$, then by relabelling of vertices where necessary, S_k is the desired M. If $k = 2$, then $d_{C_n}(v_1, v_k) = \lceil \frac{n}{2} \rceil$. Take $M = \{x_1, x_2, ..., x_j\}$, $j = \lceil \frac{n}{2} \rceil + 1$ where $P = [x_1, x_2, ..., x_j]$ is a v_1-v_k geodesic in C_n. Then M is a closed geodetic cover of C_n with $|M| = \lceil \frac{n}{2} \rceil + 1$ and $S \subseteq M$. Now assume $2 < k < \lceil \frac{n}{2} \rceil + 1$. Choose $v \in V(C_n)$ and a v-v_{k-1} geodesic $P = [x_1, x_2, ..., x_m]$, where $m = \lceil \frac{n}{2} \rceil$ such that $d_{C_n}(v, v_{k-1}) = \lceil \frac{n}{2} \rceil - 1$ and $v_k \notin V(P)$. Define $M = \{x_1, x_2, ..., x_m, v_k\}$. Consequently, M is a closed geodetic cover of C_n with $|M| = \lceil \frac{n}{2} \rceil + 1$ as described above. Now, let $j \in \{1, 2, ..., k-1\}$. Then $d_{C_n}(v_j, v_{k-1}) \leq \lceil \frac{n}{2} \rceil - 1$. By the choice of P, $v_j \in V(P)$. Thus, $S \subseteq M$.

Therefore, being a subset of a non-hop dominating set, any closed geodetic set S is not a hop dominating set of C_n. Thus, C_n does not admit a closed geodetic hop dominating set. □

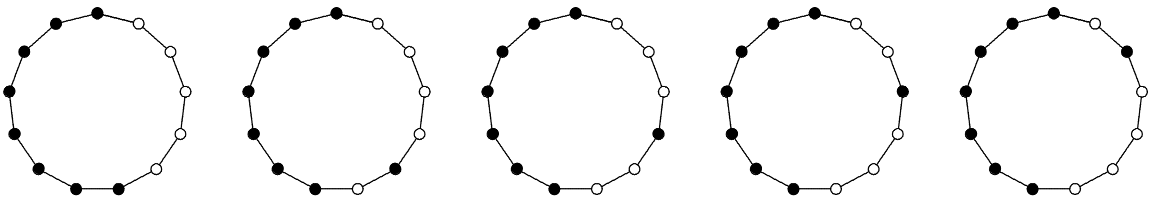

Figure 1: Cycle graph C_{13} illustrating the first part of proof of Proposition 4

Proposition 5. *Let $p \geq 2$, $2 \leq n_1 \leq n_2 \leq ... \leq n_p$ and $G = K_{n_1,n_2,...,n_p}$ with partite sets U_{n_i}, $i = 1, 2, \ldots, p$. Then $S \subseteq V(G)$ is a closed geodetic hop dominating set of G if and only if for some i,*

$$S = U_{n_i} \cup \left(\cup_{k=1;k\neq i}^{p} \{x_{n_k}\} \right), \tag{3}$$

where $x_{n_k} \in U_{n_k}$. Consequently, $\gamma_{hcg}(K_{n_1,n_2,...,n_p}) = n_1+p-1$. In particular, $\gamma_{hcg}(K_{m,n}) = 1 + \min\{m, n\}$ for $m, n \geq 2$.

Proof. Clearly, if $S \subseteq V(G)$ satisfies Equation 3, then S is a closed geodetic hop dominating set of G. Conversely, let S be a closed geodetic hop dominating set of G. Since S is a hop dominating set, $S \cap U_{n_j} \neq \varnothing$ for all $j = 1, 2, ..., p$. Since S is a closed geodetic set, $U_{n_i} \subseteq S$ for some i and $|S \cap U_{n_k}| = 1$ for all $k \neq i$.

The remaining statements follow immediately. □

2.3. Realization Problems

Theorem 4. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $cgn(G) = a$ and $\gamma_{hcg}(G) = b$.*

Proof. Let $m = b - a + 1$. Consider the tree G in Figure 2 below obtained from the $P_{3m} = [y_1, y_2, \ldots, y_{3m}]$ on $3m$ vertices by adding $(a - 1)$ pendant edges $x_k y_1$, $k = 1, 2, \ldots, a - 1$.

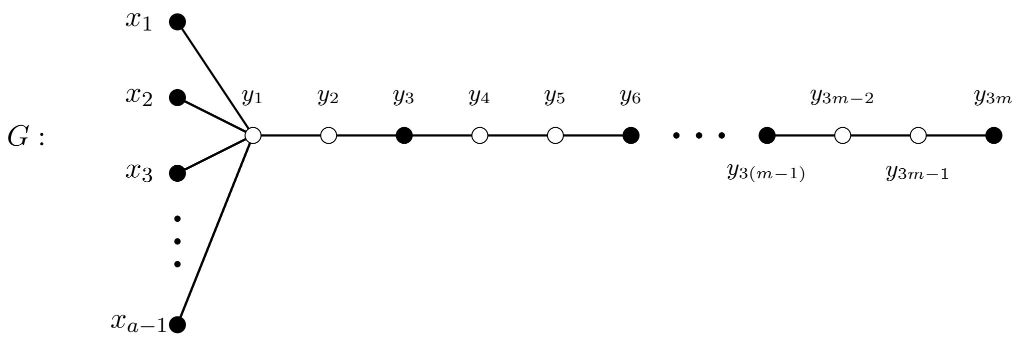

Figure 2: Graph G complying with the specifications of Theorem 4

Then set $Ext(G) = \{x_1, x_2, ..., x_{a-1}, y_{3m}\}$ is a closed geodetic basis of G. Hence $cgn(G) = a - 1 + 1 = a$. On the other hand, the set $\{x_1, x_2, ..., x_{a-1}, y_3, y_6, ..., y_{3m}\}$ is a γ_{hcg}-set of G. Hence $\gamma_{hcg}(G) = a - 1 + m = b$. □

Theorem 5. *If n, m, and k are integers with $4 \leq m \leq k$ and $2k - m + 2 \leq n$, then there exists a connected graph G such that $|V(G)| = n$, $\gamma_{hg}(G) = m$ and $\gamma_{hcg}(G) = k$.*

Proof. Let $r = k - m + 3$ and $s = n - k + 1$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of $K_{r,s}$. Obtain G as in Figure 3 by adding to $K_{r,s}$ $(m-4)$ new pendant edges $w_j v_1$, $j = 1, 2, \ldots, m-4$. Then $|V(G)| = r + s + (m-4) = n$. The

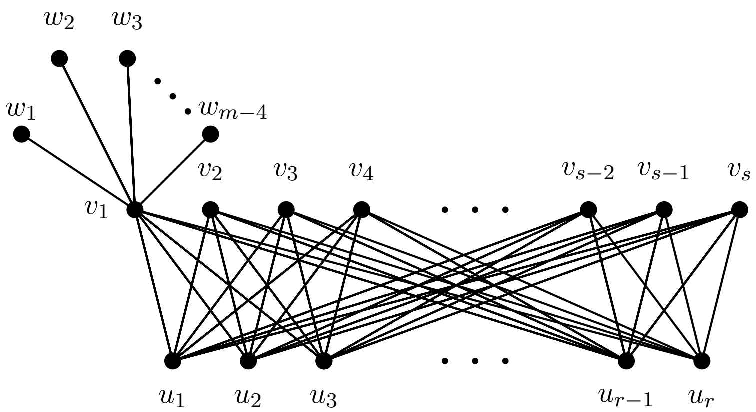

Figure 3: Graph G complying with the specifications of Theorem 5

vertices $w_1, w_2, ..., w_{m-4}$ are extreme vertices, thus are in any geodetic cover of G. Since the set $\{w_1, w_2, ..., w_{m-4}, u_1, u_r, v_1, v_s\}$ is a γ_{hg}-set of G, it follows that $\gamma_{hg}(G) = m$. Since the set $\{v_1, w_1, w_2, ..., w_{m-4}, u_1, ..., u_r\}$ is a γ_{hcg}-set of G, we have $\gamma_{hcg}(G) = 1 + m - 4 + r = 1 + m - 4 + k - m + 3 = k$. □

2.4. In the Join of Graphs

Since $diam(G + H) \leq 2$, $G + H \in \mathscr{C}_h^*$ for any graphs G and H.

A set $S \subseteq V(G)$ is a ***closed*** ***2-path closure absorbing set*** of G if $P_2[S] = V(G)$ and $S = S_k = \{v_1, v_2, ..., v_k\}$ where $v_1 \neq v_2$ and $v_i \notin P_2[S_{i-1}]$ for $3 \leq i \leq k$. The minimum cardinality of a closed 2-path closure absorbing set in G is denoted by $\rho_{c2}(G)$. A 2-path closure absorbing set of G with cardinality $\rho_{c2}(G)$ is called ρ_{c2}*-set.* A set $S \subseteq V(G)$ is a ***closed 2-path closure absorbing pointwise non-dominating set*** of G provided S is a closed 2-path closure absorbing set and at the same time pointwise non-dominating set of G. The minimum cardinality of a closed 2-path closure absorbing pointwise non-dominating set of G is denoted by $\rho_{c2pnd}(G)$. A closed 2-path closure absorbing pointwise non-dominating set of G with cardinality $\rho_{c2pnd}(G)$ is called ρ_{c2pnd}*-set.*

Since any closed 2-path closure absorbing pointwise non-dominating set is a 2-path closure absorbing pointwise non-dominating set and a closed 2-path closure absorbing set, $\rho_{2pnd}(G) \leq \rho_{c2pnd}(G)$ and $\rho_{c2}(G) \leq \rho_{c2pnd}(G)$for all connected graphs.

Example 1. *Consider the graph $K_{5,6}$ in Figure 4, the sets $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_3, u_1, u_2\}$, and $\{u_1, v_1, v_2, v_3, v_4, v_5\}$ are ρ_{2c}-set, ρ_{2pnd}-set and ρ_{c2pnd}-set of $K_{5,6}$, respectively. Therefore, $\rho_{c2}(K_{5,6}) = 5$ $\rho_{2pnd}(K_{5,6}) = 4$ and $\rho_{c2pnd}(K_{5,6}) = 6$.*

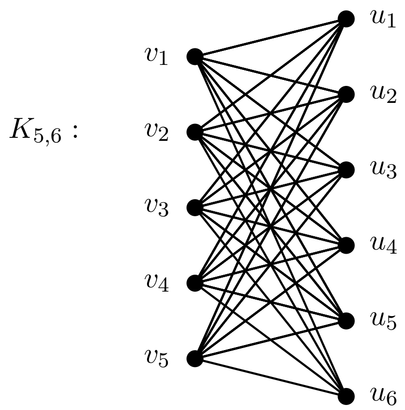

Figure 4: The bipartite graph $K_{5,6}$

Observation 6. *Let n be a positive integer. Then*

(i) $\rho_{c2pnd}(K_n) = n$ *and* $\rho_{c2}(K_n) = n$;

(ii) $\rho_{c2pnd}(P_n) = \begin{cases} n & \text{if } n = 1,2,3, \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 4, \end{cases}$ *and* $\rho_{c2}(P_n) = \begin{cases} 2 & \text{if } n = 3, \\ \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 4; \end{cases}$

(iii) $\rho_{c2pnd}(C_n) = \begin{cases} 3 & \text{if } n = 3,4, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 5 \end{cases}$ *and* $\rho_{c2}(C_n) = \begin{cases} 3 & \text{if } n = 3, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 4; \end{cases}$

(iv) $\rho_{c2pnd}(K_{m,n}) = \begin{cases} m+n & \text{if } m = 1 \text{ or } n = 1 \\ min\{m,n\}+1 & \text{if } m,n \geq 2 \end{cases}$

Lemma 2. [11] *Let G be a connected noncomplete graph, and let $S \subseteq V(G)$. If S is a 2-path closure absorbing set of G, then $\langle S \rangle$ is not complete.*

Theorem 7. *Let G be a noncomplete connected graph and $n \geq 1$. Then $S \subseteq V(G + K_n)$ is a closed geodetic hop dominating set of $G + K_n$ if and only if*

$$S = V(K_n) \cup C,$$

where $C \subseteq V(G)$ and is a closed 2-path closure absorbing pointwise non-dominating set in G.

Proof. Let $S \subseteq V(G+K_n)$. Suppose that S is a closed geodetic hop dominating set of $G+K_n$, say $S = S_k = \{x_1, x_2, \ldots, x_k\}$ with $x_1 \neq x_2$ and for $k \geq 3$, $x_k \notin I_{G+K_n}[S_{k-1}]$. Since $V(K_n) \subseteq Dom(G+K_n)$, $V(K_n) \subseteq S$. Let $j = |S \cap V(H)|$. Write $A_i = \{x_{n_1}, x_{n_2}, \ldots, x_{n_i}\}$ for $i = 1, 2, \ldots, j$ such that $n_1 < n_2 < \cdots < n_j$. First, we claim that $C = A_j$ is a closed 2-path closure absorbing set of G. Suppose that for some $3 \leq l \leq j$, $x_{n_l} \in P_2[A_{l-1}]$. This means that there exist $r < s < l$ such that $[x_{n_r}, x_{n_l}, x_{n_s}]$ is a geodesic in G. Since $diam(G+K_n) = 2$, $[x_{n_r}, x_{n_l}, x_{n_s}]$ is also a geodesic in $G+K_n$. Thus, $x_{n_l} \in I_{G+K_n}[S_{n_l-1}]$, a contradiction to the definition of $S = S_k$. Hence, $x_{n_l} \notin P_2[A_{l-1}]$ for each $3 \leq l \leq j$. Let $x \in V(G) \setminus A_j$. There exist $a, b \in \{1, 2, \ldots, k\}$ such that $x \in I_{G+K_n}(x_a, x_b)$. Necessarily, $x_a, x_b \in V(G) \cap S = A_j$ and each x_a-x_b geodesic containing x lies entirely in G. Since $diam(G+K_n) = 2$, $d_G(x_a, x_b) = 2$. Therefore, $P_2[A_j] = V(G)$, and the first claim is done. We next claim that C is a pointwise non-dominating set of G. Let $x \in V(G) \setminus C$. Since S is a hop dominating set of $G+K_n$, there exists $v \in S$ such that $d_{G+K_n}(x, v) = 2$. Clearly, $v \in V(G) \cap S = C$ and $d_G(x, v) = 2$. This shows that the second claim holds.

Conversely, suppose that $S = V(K_n) \cup C$, where $C \subseteq V(G)$ and is a closed 2-path closure absorbing pointwise non-dominating set in G. Let $k = |C|$. Being a closed 2-path closure absorbing set, there is a sequence of sets $A_j = \{v_1, v_2, \ldots, v_j\}$ $(j = 1, 2, \ldots, k)$ such that $v_1 \neq v_2$, $v_j \notin P_2[A_{j-1}]$ for $2 \leq j \leq k$ and $P_2[A_k] = V(G)$. For $i = 1, 2, \ldots, n+k$, write $S_i = \{x_1, x_2, \ldots, x_i\}$, where $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and $x_{n+j} = v_j$ for all $j = 1, 2, \ldots, k$. Observe that

- $I_{G+K_n}[S_i] = S_i$ for all $i = 1, 2, \ldots, n$;
- $x_{n+1} \notin I_{G+K_n}[S_n]$ and $x_{n+2} \notin I_{G+K_n}[S_{n-1}]$;
- $x_{n+i} \notin I_{G+K_n}[S_{n+i-1}] = V(K_n) \cup P_2[A_{i-1}]$ for all $i = 1, 2, \ldots, k$; and
- $I_{G+K_n}[S] = V(G+K_n)$.

This means that S is a closed geodetic set of $G+K_n$. Finally, let $x \in V(G+K_n) \setminus S$. Then $x \notin C$. Since C is a pointwise non-dominating set, there exists $y \in C \subseteq S$ such that $d_{G+K_n}(x, y) = d_G(x, y) = 2$. Therefore, S is a closed geodetic hop dominating set of $G+K_n$. □

Corollary 1. *Let G be a noncomplete connected graph and $n \geq 1$. Then*

$$\gamma_{hcg}(G+K_n) = n + \rho_{c2pnd}(G).$$

Example 2.

$$(i)\ \gamma_{hcg}(P_n+K_p) = \begin{cases} p+3 & \text{if } n = 3, \\ p + \lceil \frac{n+1}{2} \rceil & \text{if } n \geq 4, \end{cases}$$

$$(ii)\ \gamma_{hcg}(C_n+K_p) = \begin{cases} p+3 & \text{if } n = 3, 4, \\ p + \lceil \frac{n}{2} \rceil & \text{if } n \geq 5. \end{cases}$$

Theorem 8. [18, Theorem 4] *Let G and H be any graphs. A set $S \subseteq V(G+H)$ is geodetic hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively, such that*

(*i*) *S_G is a 2-path closure absorbing set in G whenever $\langle S_H \rangle$ is a complete subgraph of H and*

(*ii*) *S_H is a 2-path closure absorbing set in H whenever $\langle S_G \rangle$ is a complete subgraph of G.*

Part of Theorem 9 asserts that under the same condition, S is a hop dominating set of $G+H$ if and only if S_G and S_H are pointwise non-dominating sets of G and H, respectively.

Theorem 9. *Let G and H be connected noncomplete graphs. Then S is a closed geodetic hop dominating set of G if and only if $S = S_G \cup S_H$ where S_G and S_H are pointwise non-dominating sets of G and H, respectively, such that either*

(*i*) *$\langle S_G \rangle$ is complete and S_H is a closed 2-path closure absorbing set of H; or*

(*ii*) *$\langle S_H \rangle$ is complete and S_G is a closed 2-path closure absorbing set of G.*

Proof. Suppose that S is a closed geodetic hop dominating set of $G+H$. Then S_G and S_H are pointwise non-dominating sets of G and H, respectively. Since S is a closed geodetic set of $G+H$, there is a positive integer k and sequence of sets $S_j = \{x_1, x_2, \dots, x_j\}$ $(j = 1, 2, \dots, k)$ such that $x_1 \neq x_2$, $I_{G+H}[S_k] = I_{G+H}[S] = V(G)$ and $x_j \notin I_{G+H}[S_{j-1}]$ for $3 \leq j \leq k$. First, we claim that $\langle S_G \rangle$ is a complete subgraph of G or $\langle S_H \rangle$ is a complete subgraph of H. Suppose this claim is false. If $\langle S_G \rangle$ and $\langle S_H \rangle$ are noncomplete, then there exist distinct integers i, j, l, r such that $x_i, x_j \in S_G$ with $d_G(x_i, x_j) = 2$ and $x_l, x_r \in S_H$ with $d_G(x_l, x_r) = 2$. Without loss of generality, assume that $l = \max\{i, j, l, r\}$. Since $x_l \in I_{G+H}(x_i, x_j)$, $x_l \in I_{G+H}[S_{l-1}]$, a contradiction. The claim, therefore, is true. Next, suppose $\langle S_G \rangle$ is a complete subgraph of G. Write $S_H = \{x_{n_1}, x_{n_2}, \dots, x_{n_l}\} \subseteq S_k$ with $n_1 < n_2 < \cdots < n_l$, and let $A_j = \{x_{n_1}, x_{n_2}, \dots, x_{n_j}\}$ for each $j = 1, 2, \dots, l$. As shown in the proof of Theorem 7, $x_{n_j} \notin P_2[A_{j-1}]$ for $3 \leq j \leq l$, and $P_2[A_l] = V(H)$. Therefore, S_H is a closed 2-path closure absorbing set of H. Similarly, if $\langle S_H \rangle$ is complete, then S_G is a closed 2-path closure absorbing set of G. In view of Lemma 2, conditions (i) and (ii) cannot hold at the same time.

Conversely, suppose that S_G and S_H are pointwise non-dominating sets of G and H, respectively. Then $S = S_G \cup S_H$ is a hop dominating set of $G+H$. Suppose further that condition (i) holds, i.e., $\langle S_G \rangle$ is complete and S_H is a closed 2-path closure absorbing set of H. Let $k = |S_G| = k$ and $j = |S_H|$. There is a sequence of sets $C_i = \{v_1, v_2, \dots, v_i\}$ $(i = 1, 2, \dots, j)$ such that $v_1 \neq v_2$, $v_i \notin P_2[C_{i-1}]$ for $2 \leq i \leq j$ and $P_2[C_j] = V(H)$. For $i = 1, 2, \dots, k+j$, write $S_i = \{x_1, x_2, \dots, x_i\}$, where $S_G = \{x_1, x_2, \dots, x_k\}$ and $x_{k+i} = v_i$ for all $i = 1, 2, \dots, j$. As observed in the proof of Theorem 7, S is a closed geodetic set of $G+H$. Similarly, if condition (ii) holds, then S is a closed geodetic set of $G+H$. □

Lemma 3. *Let G be a connected noncomplete graph and $S \subseteq V(G)$ such that $\langle S\rangle$ is complete. Then $\langle S\rangle$ is a maximal clique if and only if S is a pointwise non-dominating set of G.*

Proof. Assume that $\langle S\rangle$ is a maximal clique of G. Let $v \in V(G) \setminus S$. Suppose that $uv \in E(G)$ for all $u \in S$. Then $\langle S \cup \{u\}\rangle$ is a complete subgraph of G, contradicting the maximality of $\langle S\rangle$. Thus, there exists $u \in S$ for which $d_G(u, v) \geq 2$. Since v is arbitrary, S is pointwise non-dominating.

Conversely, suppose that S is pointwise non-dominating set of G. Let $C \subseteq V(G)$ for which $\langle C\rangle$ is a complete subgraph of G and $S \subseteq C$. Suppose that $C \setminus S \neq \varnothing$, say $x \in C \setminus S$. Since S is pointwise non-dominating, there exists $y \in S$ such that $xy \notin E(G)$. However, $y \in C$ since $S \subseteq C$. This is a contradiction since $\langle C\rangle$ is complete. □

In view of Lemma 3, Theorem 9 can be rephrased as follows:

Theorem 10. *Let G and H be connected noncomplete graphs. Then S is a closed geodetic hop dominating set of G if and only if $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ such that either*

(i) $\langle S_G\rangle$ is a maximal clique of G and S_H is a closed 2-path closure absorbing pointwise non-dominating set of H; or

(ii) $\langle S_H\rangle$ is maximal clique of H and S_G is a closed 2-path closure absorbing pointwoise non-dominating set of G.

Corollary 2. *Let G and H be connected noncomplete graphs. Then*

$$\gamma_{hcg}(G + H) = min\{\rho_{c2pnd}(G) + \omega_L(H), \rho_{c2pnd}(H) + \omega_L(G)\}. \tag{4}$$

Example 3.

(i) $$\gamma_{hcg}(P_r+K_{m,n}) = \begin{cases} 5 & \text{if } r = 3 \text{ and } m, n \geq 2 \\ min\{5, m + n + 2\} & \text{if } r = 3 \text{ and } m = 1 \text{ or } n = 1 \\ min\{\lceil \frac{r+1}{2} \rceil + 2, m + n + 2\} & \text{if } r \geq 4 \text{ and } m = 1 \text{ or } n = 1 \\ min\{\lceil \frac{r+1}{2} \rceil + 2, min\{m, n\} + 3\} & \text{if } r \geq 4 \text{ and } m, n \geq 2, \end{cases}$$

(ii) $$\gamma_{hcg}(C_r + K_{m,n}) = \begin{cases} 5 & \text{if } r = 4 \text{ and } m, n \geq 2 \\ min\{5, m + n + 2\} & \text{if } r = 4 \text{ and } m = 1 \text{ or } n = 1 \\ min\{\lceil \frac{r}{2} \rceil + 2, m + n + 2\} & \text{if } r \geq 5 \text{ and } m = 1 \text{ or } n = 1 \\ min\{\lceil \frac{r}{2} \rceil + 2, min\{m, n\} + 3\} & \text{if } r \geq 5 \text{ and } m, n \geq 2 \end{cases}.$$

2.5. In the Corona and Edge Corona of Graphs

For the purpose of this section, a sequence of subsets $S_k = \{v_1, v_2, \ldots, x_k\}$ $(k = 1, 2, \ldots, n)$ of $V(G)$ is said to be a ***closed geodetic sequence of sets*** if $v_1 \neq v_2$ and $v_k \notin I_G[S_{k-1}]$ for $3 \leq k \leq n$. A closed geodetic sequence of sets $S_k = \{v_1, v_2, \ldots, x_k\}$ $(k = 1, 2, \ldots, n)$ is a ***maximal closed geodetic sequence*** if $I_G[S_n] = V(G)$. More precisely, $S \subseteq V(G)$ is a closed geodetic set of G if and only if there exists a positive integer n and a maximal closed geodetic sequence of sets $S_k = \{v_1, v_2, \ldots, x_k\}$ $(k = 1, 2, \ldots, n)$ such that $S = S_n$. Parallel definitions are adopted for a ***closed 2-path closure absorbing sequence of sets*** and ***maximal closed 2-path closure absorbing sequence of sets***.

Theorem 11. *Let G and H be connected graphs where G is nontrivial, and let $S \subseteq V(G \circ H)$. Then S is a closed geodetic hop dominating set of $G \circ H$ if and only if*

$$S = A \cup \left(\cup_{v \in V(G)} S_v \right), \tag{5}$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ satisfying the following conditions:

(i) S_v is a pointwise non-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$;

(ii) S_v is a closed 2-path closure absorbing set of H^v; and

(iii) The vertices in A constitute a closed geodetic sequence of sets of G

Proof. Assume S is a closed geodetic hop dominating set of $G \circ H$. Let $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup \left(\cup_{v \in V(G)} S_v \right)$. Let $v \in V(G) \setminus N_G(A)$, and let $u \in V(H^v) \setminus S_v$. Since S is a hop dominating set of $G \circ H$, there exists $w \in S$ such that $d_{G \circ H}(u, w) = 2$. If $w \in V(G)$, then $w \in A$ and $wv \in E(G)$, which is impossible. Thus, $w \notin V(G)$ so that $w \in S_v$. In this case, $d_{G \circ H}(u, w) = d_{H^v}(u, w) = 2$. This means that S_v is pointwise non-dominating in H^v, showing (i).

To show, (ii), let $v \in V(G)$. Let $n = |S|$. There exists a closed geodetic sequence of sets $S_k = \{x_1, x_2, \ldots, x_k\}$, $3 \leq k \leq n$, such that $I_{G \circ H}[S_n] = V(G \circ H)$. Write $S_v = \{x_{n_1}, x_{n_2}, \ldots, x_{n_j}\}$ with $n_1 < n_2 < \cdots < n_j$. Define $T_i = \{x_{n_1}, x_{n_2}, \ldots, x_{n_i}\}$ for $i = 1, 2, \ldots, j$. Suppose that for $a < b < c$, $[x_{n_a}, x_{n_c}, x_{n_b}]$ is a geodesic in H^v. Then $[x_{n_a}, x_{n_c}, x_{n_b}]$ is a geodesic in $G \circ H$ so that $x_{n_c} \in I_{G \circ H}[S_{n_b}]$, a contradiction. Therefore, $x_{n_i} \notin P_2[T_{i-1}]$ for $3 \leq i \leq j$, and therefore, $T_i = \{x_{n_1}, x_{n_2}, \ldots, x_{n_i}\}$, $i = 1, 2, \ldots, j$, is a closed 2-path closure absorbing sequence of sets in H^v. Let $x \in V(H^v) \setminus T_j$. Since $I_{G \circ H}[S_n] = V(G \circ H)$, there exist $1 \leq a, b \leq n$ such that $x \in I_{G \circ H}(x_a, x_b)$. Because $yv \in E(G \circ H$ for all $y \in V(H^v)$, any x_a-x_b geodesic lies completely in $V(H^v)$. Thus, $a, b \in \{n_1, n_2, \ldots, n_j\}$. This means that $P_2[T_j] = V(H^v)$ and $T_j = S_v$ is a closed 2-path closure absorbing set of H^v.

Statement (iii) is done similarly. The sequence $A_i = \{x_{k_1}, x_{k_2}, \ldots, x_{k_i}\}$, $i = 1, 2, \ldots, j$, such that $A_j = A$ is a closed geodetic sequence of sets of G.

To prove the converse, assume that Equation 3 holds for S together with conditions (i), (ii) and (iii). Let $n = |S|$, $j = |A|$, and for each $v \in V(G)$, let $S_v^j = \{x_v^1, x_v^2, \ldots, x_v^j\} \subseteq S_v$,

$j = 1, 2, \ldots, k_v$, be a maximal closed 2-path closure absorbing sequence of sets in H^v. Define for each $k = 1, 2, \ldots, n$, $S_k = \{x_1, x_2, \ldots, x_k\} \subseteq S$ such that

- $A = \{x_1, x_2, \ldots, x_j\}$;
- If for $i < k$, $x_i, x_k \in S_v$, say $x_i = x_v^s$ and $x_k = x_v^r$, then $s < r$.

Then $S_k = \{x_1, x_2, \ldots, x_k\}$ $(k = 1, 2, \ldots, n)$ is a closed geodetic sequence of sets in $G \circ H$. Let $w \in V(G \circ H) \setminus S$, and let $v \in V(G)$ for which $w \in V(H^v + v)$. If $w \in V(H^v)$, then $w \in P_2[S_v] = I_{G \circ H}[S_v]$. Suppose that $w = v$. Let $z \in V(G) \cap N_G(v)$. Pick $u \in S_v$ and $y \in S_z$. Then $w \in I_{G \circ H}[u, z] \subseteq I_{G \circ H}[S]$. Hence, S is a closed geodetic set of $G \circ H$.

Finally, we show S is a hop dominating set of $G \circ H$. Let $w \in V(G \circ H) \setminus S$, and let $v \in V(G)$ for which $w \in V(H^v + v)$. If $w = v$, then for any $z \in N_G(v)$, $d_{G \circ H}(w, y) = 2$ for all $y \in S_z$. Suppose that $w \in V(H^v)$. If $v \in N_G(A)$, then $d_{G \circ H}(w, y) = 2$ for all $y \in A \cap N_G(v)$. If $v \notin N_G(A)$, then since S_v is pointwise non-dominating, there exists $y \in S_v$ for which $wy \notin E(H^v)$. Then $d_{G \circ H}(w, y) = 2$. □

Corollary 3. *Let G and H be connected graphs where G is nontrivial of order n. Then*

$$n \cdot \rho_2(H) \leq \gamma_{hcg}(G \circ H) \leq n \cdot \rho_{c2pnd}(H),$$

and these bounds are sharp.

Proof. Let $S \subseteq V(G \circ H)$ be a γ_{hcg}-set of $G \circ H$. By Theorem 11, $S = A \cup \left(\cup_{v \in V(G)} S_v\right)$, where S_v is a closed 2-path closure absorbing set of H^v. Thus,

$$n \cdot \rho_2(H) \leq \sum_{v \in V(G)} |S_v| \leq |S| = \gamma_{hcg}(G \circ H).$$

To get the other inequality, for each $v \in V(G)$, let $S_v \subseteq V(H^v)$ be a closed 2-path closure absorbing pointwise non-dominating set of H^v. By Theorem 11, $S = \cup_{v \in V(G)} S_v$ is a closed geodetic hop dominating set of $G \circ H$. Hence,

$$\gamma_{hcg}(G \circ H) \leq |S| = n \cdot \rho_{c2pnd}(H).$$

□

For a graph G, let $\tau(G)$ be the set of all support vertices v of G for which $N_G(x) = \{v\}$ for all $x \in N_G(v)\}$. In particular, if $G = K_{1,n}$ with central vertex v, then $\tau(G) = \{v\}$.

Theorem 12. *Let G be a nontrivial connected graph and $n \geq 1$, and let $S \subseteq V(G \diamond K_n)$. Then S is a closed geodetic hop dominating set of $G \diamond K_n$ if and only if*

$$S = A \cup \left(\cup_{uv \in E(G)} V(H^{uv})\right), \tag{6}$$

where $A \subseteq V(G)$ such that $L(G) \cup \tau(G) \subseteq A$ and the elements of A constitute a closed geodetic sequence of sets of G.

Proof. Put $H = K_n$. Suppose that S is a closed geodetic hop dominating set of $G \diamond H$. Since $L(G) \cup V(H^{uv}) \subseteq Ext(G \diamond H)$, $L(G) \cup V(H^{uv}) \subseteq S$ for each $uv \in E(G)$. Let $A = S \cap V(G)$. Since the vertices in S constitute a closed geodetic sequence of sets in $G \diamond H$, it follows that the vertices in A constitute a closed geodetic sequence of sets in G. Let $w \in \tau(G)$. Suppose that $w \notin A$. Since S is a hop dominating set, there exists $x \in S$ such that $d_{G\diamond H}(w, x) = 2$. If $x \in V(G)$, then $d_G(x, w) = 2$. Thus, G contains a geodesic $[x, v, w]$. This means that there exists $v \in N_G(w)$ with $N_G(v) \neq \{w\}$, a contradiction since $w \in \tau(G)$. Suppose there exist $uv \in E(G)$ such that $x \in S_{uv}$. Then either $wu \in E(G)$ or $wv \in E(G)$. Assume $wv \in E(G)$. Then there exists $v \in N_G(w)$ with $N_G(v) \neq \{w\}$, a contradiction. Hence, $\tau(G) \subseteq A$.

Conversely, suppose that S is as described in Equation 3 together with the indicated properties. Let $n = |S|$ and $|A| = k$. For each $j = 1, 2, \ldots, k$, let $A_j = \{x_1, x_2, \ldots, x_j\} \subseteq A$ be a closed geodetic sequence in G. Extend the sequence by defining for each $i = 1, 2, \ldots, n$,

$$S_i = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_i\} \subseteq S.$$

This means that $S_n \setminus A_k = \cup_{uv\in E(G)} V(H^{uv})$. Thus, $S_i = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_i\}$ $(i = 1, 2, \ldots, n)$ is a closed geodetic sequence in $G \diamond H$. Let $w \in V(G) \setminus A$. Since $w \notin L(G)$, there exist distinct $x, y \in V(G) \cap N_G(w)$. Pick $z \in V(H^{xw})$ and $t \in V(H^{yw})$. Then $z, t \in S$ and $w \in I_{G\diamond H}(z, y)$. Since w is arbitrary, $I_{G\diamond H}[S] = V(G \diamond H)$ and S is a closed geodetic set of $G \diamond H$. To show that S is a hop dominating set, let $w \in V(G) \setminus A$. Since $w \notin \tau(G)$, there exists $v \in N_G(w)$ such that $N_G(v) \setminus \{w\} \neq \varnothing$, say $u \in N_G(v) \setminus \{w\}$. Pick $z \in S_{uv}$. Then $z \in S$ and $d_{G\diamond H}(w, z) = 2$. Accordingly, S is a hop dominating set of $G \diamond H$. □

Corollary 4. *Let G be a nontrivial connected graph of size n and $p \geq 1$. Then*

$$\gamma_{hcg}(G \diamond K_p) = np + |L(G)| + |\tau(G)|. \tag{7}$$

In particular, if $L(G) = \varnothing$ and $\tau(G) = \varnothing$, then

$$\gamma_{hcg}(G \diamond K_p) = np. \tag{8}$$

If $G = K_2$, then $G \diamond H = K_2 + H$. This case is taken in Theorem 7. In what follows, we consider G of order $n \geq 3$.

Theorem 13. *Let G and H be connected graphs where $|V(G)| \geq 3$ and H is not complete, and let $S \subseteq V(G \diamond H)$. Then S is a closed geodetic hop dominating set of $G \diamond H$ if and only if*

$$S = A \cup \left(\cup_{uv\in E(G)} S_{uv}\right), \tag{9}$$

where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ satisfying the following:

(i) The elements in A constitute a closed geodetic sequence of sets of G and $\tau(G) \subseteq A$;

(ii) S_{uv} is a closed 2-path closure absorbing set of H^{uv} for each $uv \in E(G)$.

Proof. Assume S is a closed geodetic set of $G \circ H$. Let $A = S \cap V(G)$, and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then $S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right)$. At this far, showing that the elements of A and S_{uv} constitute a closed geodetic sequence and a closed 2-path closure absorbing sequence of sets in G and H^{uv}, respectively, for each $uv \in E(G)$, is already a routine. Since S is a hop dominating set, $\tau(G) \subseteq A$. Thus, (i) holds. To completely show (ii), observe that for each $z \in V(H^{uv})$, every x-y geodesic (with $x \neq z \neq y$) in $G \diamond H$ containing z lies entirely in H^{uv}. Thus, since $I_{G \diamond H}[S] = V(G \diamond H)$, $P_2[S_{uv}] = V(H^{uv})$. This makes S_{uv} a closed 2-path absorbing set of H^{uv}.

Conversely, suppose that S is as given in Equation 4 and satisfies conditions (i) and (ii). Assume $|S| = n$. Obtain from S a closed geodetic sequence of sets in $G \diamond H$ as follows: Construct $S_k = \{x_1, x_2, \ldots, x_k\}$ for $k = 1, 2, \ldots, n$ such that S_k for $k \in \{1, 2, \ldots, |A|\}$ is a closed geodetic sequence constituted by the vertices in A and $S_n \setminus A = \cup_{uv \in E(G)} S_{uv}$. Then S_k, $k = 1, 2, \ldots, n$, is a closed geodetic sequence of sets in $G \diamond H$. Let $w \in V(G \diamond H) \setminus S$ and let $uv \in E(G)$ such that $w \in V(H^{uv} + uv)$. Suppose that $u = w$. If $S_{uv} = V(H^{uv})$, then since H^{uv} is not complete, there exist $x, y \in S_{uv}$ such that $xy \notin E(H^{uv})$. Then $d_{G \diamond H}(x, y) = 2$ and $w \in I_{G \diamond H}(x, y)$. Suppose that $S_{uv} \neq V(H^{uv})$. Since S_{uv} is a 2-path closure absorbing set of H^{uv}, for $w \in V(H^{uv}) \setminus S_{uv}$, there exist $x, y \in S_{uv}$ such that $[x, w, y]$ is a geodesic in H^{uv}. This means that $d_{G \diamond H}(x, y) = 2$ and $w \in I_{G \diamond H}(x, y)$. The case where $w = v$ is handled similarly. Now, suppose that $w \in V(H^{uv})$. Since S_{uv} is 2-path closure absorbing, there exist $x, y \in S_{uv}$ such that $d_{H^{uv}}(x, y) = 2$ and $w \in I_{H^{uv}}(x, y)$. This means that $d_{G \diamond H}(x, y) = 2$ and $w \in I_{G \diamond H}(x, y)$. We have just shown that S is a closed geodetic set of $G \diamond H$.

Finally, to show that S is a hop dominating set, let $w \in V(G \diamond H) \setminus S$. If $w \in V(G)$, then since $w \notin \tau(G)$, there exists $v \in N_G(w)$ such that $N_G(v) \setminus \{w\} \neq \varnothing$. Let $u \in N_G(v) \setminus \{w\}$. Pick $z \in S_{uv}$. Then $d_{G \diamond H}(w, z) = 2$. Suppose that $w \in V(H^{uv})$ for some $uv \in V(G)$. Then $w \in V(H^{uv}) \setminus S_{uv}$. Since $|V(G)| \geq 3$ and G is connected, there exists $z \in V(G)$ such that uz or vz is an edge in G. Let $uz \in E(G)$. Take $x \in S_{uz}$. Then $d_{G \diamond H}(x, w) = 2$. Same goes for the case where $vz \in E(G)$. Therefore, S is a hop dominating set of $G \diamond H$. □

Corollary 5. *Let G and H be connected graphs where G is of order $n \geq 3$ and H is not complete. Then*

$$\gamma_{hcg}(G \diamond H) = n \cdot \rho_{c2}(H) + |\tau(G)|. \tag{10}$$

3. Conclusion

The concept of closed geodetic hop domination in graphs has been introduced and initially investigated in this study. As shown, not all graphs admit this concept. Some conditions under which a graph admits a closed geodetic hop dominating set are provided. Realizations results involving closed geodetic number, geodetic hop domination number and closed geodetic hop domination number are also provided. The closed geodetic hop dominating sets of the join corona, and edge corona of two graphs have been obtained. These characterizations have been used to obtain bounds or exact values of the closed geodetic hop domination number of each of these graphs. Exploring necessary and sufficient conditions for a graph to admit closed geodetic hop dominating set may be interesting and worthwhile to possibly provide insightful results.

Acknowledgements

The authors would like to thank the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines, and MSU-Iligan Institute of Technology for funding this research.

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