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Numerical Finite-Difference Approximations of a Coupled Reaction-Diffusion System with Gradient Terms

Manar I. Khalil[1,2,*], Ishak Hashim[1,3], Maan A. Rasheed[4], Faieza Samat[5], Shaher Momani[6,7]

[1] *Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysiay, 43600 UKM Bangi Selangor, Malaysia*
[2] *Department of Applied Geology, College of Science, Tikrit University, Iraq*
[3] *Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman PO Box 346, United Arab Emirates*
[4] *Department of Mathematics,College of Basic Education, Mustansiriyah University, Baghdad, Iraq*
[5] *Pusat GENIUS@Pintar Negara, Universiti Kebangsaan Malaysiay, 43600 UKM Bangi Selangor, Malaysia*
[6] *Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman PO Box 346, United Arab Emirates*
[7] *Department of Mathematics,Faculty of Science, Jordan University, Amman 11942, Jordan*

Abstract. This study focuses on the derivation of explicit and implicit finite difference formulas.The objective of this study is to derive an estimation of the blow-up time for a coupled reaction-diffusion system incorporating gradient terms, employing numerical finite difference approximations. Furthermore, an examination is conducted on the consistency, stability, and convergence of the proposed schemes. Additionally, the study presents two numerical experiments. In each instance, the numerical blow-up time is calculated benefit the suggested methodologies, employing varying space steps and non-fixed time-stepping. The numerical findings obtained demonstrate that the blow-up time sequence exhibits convergence as the space step decreases. Moreover, the numerical orders of convergence for the blow-up time goes well with the theoretical orders observed in the numerical solutions.

2020 Mathematics Subject Classifications: 47H05, 90C33

Key Words and Phrases: Explicit and Implicit finite difference formulas, blow-up time, Convergence analysis, consistency, stability, gradient terms

*Corresponding author.
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Email addresses: p119266@siswa.ukm.edu.my (Manar I. Khalil),
xxxxx@email.com (MISSING AUTHOR EMAILS)

1. Introduction

There are many non-linear partial differential equations of the parabolic type that cannot be extended globally in time for a given initial data and become unbounded in a limited time. The occurrence of this phenomenon is referred to as blow-up, and it can manifest in time-dependent nonlinear equations, provided that the non-linear factors possess sufficient strength, as evidenced by references [9, 10, 14, 17, 21]. This phenomenon can be expanded to coupled systems when each dependent variable has a finite increase within a specific time-frame. In this particular scenario, it is asserted that the dependent variables exhibit synchronous increases [7, 18, 19, 22, 23].

In this paper, we consider a coupled parabolic system:

$$\left.\begin{array}{l} u_t = u_{xx} - |u_x|^{q_1} + |v|^{p_1}, \quad v_t = v_{xx} - |v_x|^{q_2} + |u|^{p_2}, \quad (x,t) \in (0,1) \times (0,T), \\ \quad u(0,t) = u(1,t) = 0, \\ \quad v(0,t) = v(1,t) = 0, \quad t \in (0,T), \\ \quad u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1), \end{array}\right\} \tag{1.1}$$

where $p_1, p_2 > 1, q_1, q_2 \in (1,2), u_0, v_0 \geq 0$. For some details regarding the local existence and uniqueness and blow-up of this system, see [21]. The system (1.1) has been studied in [15] ,where $\Omega \in R^3$ and is a bounded convex domain, and

$$p = p_1 = p_2, \quad \left(\alpha = \beta = \frac{1}{1-p}\right),$$
$$q = q_1 = q_2; \quad p > q > 1.$$

This is shown as the traditional solution of the system blows up (becomes unbounded) in the W-norm, where

$$W(t) = \int_\Omega \left(u^{2p} + v^{2p}\right) \mathrm{d}x,$$

then blow-up time for this problem can be estimated from below as follows:

$$T \geq \frac{1}{2AW_0^2},$$

where $W(0) = W_0 = \int_\Omega \left(u_0^{2p} + v_0^{2p}\right) \mathrm{d}x$ and A is a constant which depends on the data.

In [21], with some restricted conditions on system (1.1), it was shown that the upper blow-up rate estimates for this solution and its gradients terms take the following forms:

$$u(x,t) \leq C_1(T-t)^{-\alpha}, \quad |\nabla u(x,t)| \leq C_1(T-t)^{-\frac{(1+2\alpha)}{2}},$$
$$v(x,t) \leq C_2(T-t)^{-\beta}, \quad |\nabla v(x,t)| \leq C_2(T-t)^{-\frac{(1+2\beta)}{2}},$$

where $(x,t) \in \Omega \times (0,T),\ C_1, C_2 > 0,\ \alpha = \frac{p_1+1}{p_1 p_2 - 1}$ and $\beta = \frac{p_2+1}{p_1 p_2 - 1}$.

The numerical approximation of time-dependent parabolic problems has been studied by many authors, see for instance [1–4, 11–13, 16, 23, 24]. The main aim of these works was to estimate the numerical blow-up time of the considered problems. Blow-up means the solution of the problem becomes Unbounded in finite time. The semi-linear coupled reaction-diffusion system was one of the investigated problems :

$$\left.\begin{array}{l} u_t = u_{xx} + v^p, \quad v_t = v_{xx} + u^q, \quad x \in (0,1),\ t \in (0,T), \\ u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0,\ t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x),\ x \in (0,1), \end{array}\right\} \tag{1.2}$$

where $p, q > 1$ and $u_0(x), v_0(x)$ are both smooth functions.

The semi-discrete approximation problem for system (1.2) is derived in [13], and theoretical notions regarding the rapprochement of blow-up times and blow-up solutions of the semi-discrete issue are proven . Furthermore, this study introduces two alternative approximations to problem (1.2), namely Euler schemes, which incorporate a non-fixed time stepping formula. Furthermore, two numerical experiments were conducted to provide empirical support for the numerical results. Specifically, the authors conducted estimations for a numerical blow-up time, error bounds, CPUt, and numerical order of convergence. The Chipot-Weissler equation [5]: refers to the single equation of system (1.1).

$$\left.\begin{array}{l} u_t = u_{xx} + u^p - |u_x|^q, \quad (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in (0,1). \end{array}\right\} \tag{1.3}$$

System (1.3) is associated with a widely recognized model that has emerged in the field of population dynamics research [20]. It is worth noting that the inclusion of the term may be gradation offer a damping effect that mitigates blow-up. Consequently, numerous scholars have expressed interest in investigating the impact of the gradient term on blow-up characteristics, including blow-up set and blow-up rate estimations [6, 8, 25]. The problem's numerical approximation is examined in reference [24]. The semi-discrete approximation of equation (1.1) was derived by the authors, who also conducted theorems to establish the convergence and blow-up of the semi-discrete problem. Moreover, the authors put forth two approximation equations for (1.1) that are fully discrete: explicit Euler equations and implicit finite difference equations. These equations are formulated using a non-fixed time-stepping algorithm. Furthermore, two numerical experiments are conducted to determine the numerical blow-up time, error limits, and numerical convergence order. The primary objective of this study is to employ numerical finite difference approximations in order to estimate the blow-up time of a coupled reaction-diffusion system that incorporates gradient terms. The explicit and implicit finite difference formulas are developed in parts two and three, respectively. Furthermore, an examination is conducted on the stability ,convergence , and consistency of the proposed schemes. Two numerical experiments are presented in a fourth part. In each instance, the numerical blow-up time is calculated utilizing the suggested methodologies, employing varying space stages and non-fixed time steps. In the final section, we present a set of conclusions and potential avenues for future works.

2. Explicit Euler scheme

Here, we derive the explicit fully-discrete finite difference formulas for the problem (1.1),by approximating the time derivative in problem(1.2), using the forward finite difference formula:
We consider that U_i^n and V_i^n, are the approximate values of $u(x_i, t_n)$ and $v(x_i, t_n)$, respectively, where $t_{n+1} = t_n + k_n, I$ be a positive integer, and consider the grid: $x_i =$ ih , $0 \leq i \leq I, h = \frac{1}{I}, t_{n+1} = t_n + k_n, x_{i+1} = x_i + h, h$ is the space-step,and k_n is the time-step .
We approximate u_t, v_t by forward finite difference formulas at the mesh-point (x_i, t_n), as below :

$$u_t|_i^n = \frac{U_i^{n+1} - U_i^n}{k_n} + O(k_n), \qquad v_t|_i^n = \frac{V_i^{n+1} - V_i^n}{k_n} + O(k_n),$$

while u_{xx}, v_{xx} The second-order central finite difference formula is used to approximate the following:

$$u_{xx}|_i^n = \frac{U_i^{n+1} - 2U_i^n + U_{i-1}^n}{h^2} + O\left(h^2\right), \quad v_{xx}|_i^n = \frac{V_i^{n+1} - 2V_i^n + V_{i-1}^n}{h^2} + O\left(h^2\right).$$

Also, u_x, v_x are approximated using first-order central finite difference formula as follows:

$$u_x|_i^n = \frac{U_{i+1}^n - U_{i-1}^n}{2h} + O\left(h^2\right), \quad v_x|_i^n = \frac{V_{i+1}^n - V_{i-1}^n}{2h} + O\left(h^2\right).$$

Substituting in system (1.1) yields

$$\frac{U_i^{n+1} - U_i^n}{k_n} = \frac{U_i^n - 2U_i^n + U_{i-1}^n}{h^2} + F\left(\delta_x U_i^n, V_i^n\right),$$
$$\frac{V_i^{n+1} - V_i^n}{k_n} = \frac{V_i^n - 2V_i^n + V_{i-1}^n}{h^2} + G\left(\delta_x V_i^n, U_i^n\right),$$

where

$$F\left(\delta_x U_i^n, V_i^n\right) = (V_i^n)^{p_1} - \left|\frac{U_{i+1}^n - U_{i-1}^n}{2h}\right|^{q_1}, \quad G\left(\delta_x V_i^n, U_i^n\right) = (U_i^n)^{p_2} - \left|\frac{V_{i+1}^n - V_{i-1}^n}{2h}\right|^{q_2}.$$

These equations can be rewritten as

$$U_i^{n+1} = (1 - 2r_n) U_i^n + r_n \left(U_{i+1}^n + U_{i-1}^n\right) + k_n F\left(\delta_x U_i^n, V_i^n\right), \tag{2.1}$$
$$V_i^{n+1} = (1 - 2r_n) V_i^n + r_n \left(V_{i+1}^n + V_{i-1}^n\right) + k_n G\left(\delta_x V_i^n, U_i^n\right), \tag{2.2}$$

where

$$i = 1, 2, \ldots, I-1, \quad n = 0, 1, 2, \ldots,$$
$$U_h^n = \left(U_1^n, U_2^n, \ldots, U_{1-I}^n\right), \quad V_h^n = \left(V_1^n, V_2^n, \ldots, V_{1-I}^n\right), \quad r_n = \frac{k_n}{h^2}.$$

Furthermore, to ensure that the stability (convergence) criterion of the explicit scheme is satisfied, the non-stationary time step formula is taken as below :

$$k_n = \min\left(\frac{h^2}{3}, \frac{h^\alpha}{\|\mathrm{U}_h^n\|}, \frac{h^\alpha}{\|\mathrm{V}_h^n\|}\right), \quad \alpha \geq 1. \tag{2.3}$$

Lemma 1. *The functions F, G satisfy Lipshtiz condition, that is there exists positive constants $L_1, L_2, L_3, L_4 > 0$, such as*

$$\left|F\left(\delta_x U_i^n, V_i^n\right) - F\left(\delta_x \widetilde{U}_i^n, \tilde{V}_i^n\right)\right| \leq L_1 \left|\delta_x\left(U_i^n - \widetilde{U}_i^n\right)\right| + L_2\left|V_i^n - \tilde{V}_i^n\right|, \tag{2.4}$$

$$\left|G\left(U_i^n, \delta_x V_i^n\right) - G\left(\widetilde{U}_i^n, \delta_x \tilde{V}_i^n\right)\right| \leq L_3 \left|\delta_x\left(V_i^n - \tilde{V}_i^n\right)\right| + L_4\left|U_i^n - \widetilde{U}_i^n\right|, \tag{2.5}$$

where $U_i^n, V_i^n, \widetilde{U}_i^n, \tilde{V}_i^n\ (i = 0, 1, 2, \ldots I)$ are bounded values.

Proof. We denote $f(s) = s^p$ and we use the mean value theorem we get

$$\left|\left(V_i^n\right)^{p_1} - \left(\tilde{V}_i^n\right)^{p_1}\right| \leq \left|f'\left(Z_i\right)\right| \left|V_i^n - \tilde{V}_i^n\right|, \tag{2.6}$$

where Z_i is an intermediate value between V_i^n and $\bar{V}_i^n$. Since V_h^n, V_h^n are bounded, then there exists $C_1 > 0$ such that $|f'(Z_i)| < C_1$. Now, we denote $g(s) = |s|^{q_1}$, with using the mean value theorem, we find

$$\left|\left|\frac{U_{i+1}^n - U_{i-1}^n}{2h}\right|^{q_1} - \left|\frac{\widetilde{U}_{i+1}^n - \widetilde{U}_{i-1}^n}{2h}\right|^{q_1}\right| \leq \left|g'\left(\varepsilon_i\right)\right| \left|\left(\frac{U_{i+1}^n - U_{i-1}^n}{2h}\right) - \left(\frac{\tilde{U}_{i+1}^n - \tilde{U}_{i-1}^n}{2h}\right)\right|, \tag{2.7}$$

where ε_i is an intermediate value between

$$a_i^1 = \left(\frac{U_{i+1}^n - U_{i-1}^n}{2h}\right) \text{ and } a_i^2 = \left(\frac{\tilde{U}_{i+1}^n - \tilde{U}_{i-1}^n}{2h}\right),$$

since $U_h^n \widetilde{U}_h^n$ are bounded in $[0, T]$. Therefore, there exists $C_2 > 0$ such that $|g'(\varepsilon_i)| \leq C_2$. From (2.6) and (2.7), we get (2.4). Similarly, we can show that (2.5) is held.

2.1. The algorithm steps for Euler Explicit method

1. Input $\mathrm{h}, U_h^0, V_h^0, \mathrm{p}_1, \mathrm{p}_2, \mathrm{q}_1, \mathrm{q}_2, \alpha$
2. Put $n = 0$;
3. Choose k_n according to (2.3).
4. Compute the numerical vectors: U_h^{n+1}, V_h^{n+1}, using the explicit formula (2.1) and (2.2).
5. For $n = 1, 2, \ldots..$, repeat steps 3,4 until for $n = m$, we get $\|U_h^n\|_\infty \geq 10^{15}$, or $\|V_h^n\|_\infty \geq 10^{15}$
6. The numerical blow-up time is $t_m = \sum_{n=0}^m k_n$.

2.2. Local Truncation error of Explicit Euler scheme

Theorem 1. *Let* (T_{ui}^n, T_{vi}^n) *be The local error truncation of the explicit Euler chart (2.1) and (2.2) at the grid-point* (x_i, t_n)*. after that, Here it is*$C_1, C_2, C_3, C_4 > 0$ *like this*

$$|T_{ui}^n| \le C_1 k + C_2 h^2, |T_{vi}^n| \le C_3 k + C_4 h^2,$$

where $k = \max_{n\in N} k_n$*, i.e.* $T_{ui}^n = O\left(k + h^2\right)$ *and* $T_{vi}^n = O\left(k + h^2\right)$*.*

Proof. Replace the precise solution $u_i^n = u\left(x_i, t_n\right)$ and $v_i^n = v\left(x_i, t_n\right)$ into the explicit (2.1) yields

$$\begin{aligned}
T_{ui}^n &= \left(u_i^{n+1} - u_i^n\right) - \frac{k_n}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - k_n \left(v_i^n\right)^{p_1} + k_n \left|\frac{U_{i+1}^n - U_{i-1}^n}{2h}\right|^{q_1} \\
&= k_n\left[\frac{\partial u_i^n}{\partial t} + O(k_n)\right] - k_n\left[\frac{\partial^2 u_i^n}{\partial x^2} + O\left(h^2\right)\right] - k_n \left(v_i^n\right)^{p_1} + k_n\left[\left|\frac{\partial u_i^n}{\partial x}\right|^{q_1} + O\left(h^2\right)\right] \\
&= k_n\left[\frac{\partial u_i^n}{\partial t} - \frac{\partial^2 u_i^n}{\partial x^2} - \left(v_i^n\right)^{p_1} + \left|\frac{\partial u_i^n}{\partial x}\right|^{q_1}\right] + k_n\left[O(k_n) + O\left(h^2\right)\right].
\end{aligned}$$

From system (1.1), assuming all the partial derivatives are bounded at the grid-point (x_i, t_n), we obtain $|T_{ui}^n| \le C_1 k_n + C_2 h^2 \le C_1 k + C_2 h^2$, where $C_1, C_2 > 0$, i.e. $T_{ui}^n = O\left(k + h^2\right), C > 0$.

Likewise, we show that $T_{vi}^n = O\left(k + h^2\right), C > 0$.

2.3. The stability analysis of explicit Euler method

The stability condition for explicit Euler scheme can be obtained based on the the following definition

Definition 1. *[13]. For* $i = 1, 2, \dots, I-1$*, set* $E_u^n = (e_{ui}^n)$*,* $E_v^n = (e_{vi}^n)$*,* $e_{ui}^n = u_i^n - U_i^n$ *and* $e_{vi}^n = v_i^n - V_i^n$*, where* $u_i^n = u(x_i, t_n)$*,* $v_i^n = v(x_i, t_n)$ *and* (U_i^n, V_i^n) *are a accurate and numerical solutions of a one-dimensional time-dependent coupled system of two PDEs, Consecutively. For any hypothetical initial rounding error* (E_u^0, E_v^0)*, we say that the numerical solution is stable, If a positive number is found*μ *freelance on the space-step* (h) *and time-step* (k)*, as follows*

$$\|E_u^n\| \le \mu \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\} \text{ and } \|E_v^n\| \le \mu \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\},$$

where $\|E_u^n\| = \max_{1\le i\le I} |e_{ui}^n|$ *and* $\|E_v^n\| = \max_{1\le i\le I} |e_{vi}^n|$*,* $n = 0, 1, 2, \dots$*.*

Theorem 2. *The explicit formulae(2.1), (2.2) is stable, if* $(1 - 2r) \ge 0$*, where* $k = \max_{n\in N} k_n$*,* $r = k/h^2$*.*

Proof. For proof this theorem, we use the maximum error stability-technicality [27].

Let $e^n_{ui} = u^n_i - U^n_i$, $e^n_{vi} = v^n_i - V^n_i$, $(u^n_i = u(x_i,t_n), v^n_i = v(x_i,t_n))$ be Accurate solution to the issue (1.1).

To finish this, we implement mathematical induction. For $n = 1$, set $\left\|E^1_u\right\| = \max_{1\le i\le I}\left|e^1_{ui}\right| = \left|e^1_{uj}\right|, \left\|E^1_v\right\| = \max_{1\le i\le I}\left|e^1_{vi}\right| = \left|e^1_{um}\right|$. By substituting e^n_{uj} in the explicit formula (2.1), we obtain

$$e^1_{uj} = (1-2r_1)e^0_{uj} + r_1\left(e^0_{uj+1} + e^0_{uj-1}\right) + k_1\left(F\left(\delta_x u^0_j, v^0_j\right) - F\left(\delta_x U^0_j, V^0_j\right)\right).$$

Since $(1-2r_1) \ge 0$, we have

$$\left|e^1_{uj}\right| \le (1-2r_1)\left|e^0_{uj}\right| + r_1\left(\left|e^0_{uj+1}\right| + \left|e^0_{uj-1}\right|\right) + k_1\left|F\left(\delta_x u^0_j, v^0_j\right) - F\left(\delta_x U^0_j, V^0_j\right)\right|.$$

From (1.2), it follows that

$$\begin{aligned}
\left|e^1_{uj}\right| &\le (1-2r_1)\left|e^0_{uj}\right| + r_1\left(\left|e^0_{uj+1}\right| + \left|e^0_{uj-1}\right|\right) + k_1L_1\left|\delta_x\left(u^0_j - U^0_j\right)\right| + k_1L_2\left|v^0_j - V^0_j\right| \\
\left|e^1_{uj}\right| &\le (1-2r_1)\left\|E^0_u\right\| + 2r_1\left\|E^0_u\right\| + k_1L_1\left|\delta_x e^0_{uj}\right| + k_1L_2\left|e^0_{vj}\right| \\
\left\|E^1_u\right\| &\le \left\|E^0_u\right\| + 2k_1L_1\left\|E^0_u\right\| + k_1L_2\left\|E^0_v\right\| \\
&\le (1+2k_1L_1+k_1L_2)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\} \\
&\le (1+3kL)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\}.
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\left\|E^1_v\right\| &\le (1+2kL_3+kL_4)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\} \\
&\le (1+3kL)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\},
\end{aligned}$$

where $L = \mathrm{Max}\left\{L_1, L_2, L_3, L_4\right\}$. Now, we suppose that

$$\begin{aligned}
\|E^s_u\| &\le (1+3kL)^s\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\}, \; s = 1,2,3,\ldots n, \\
\|E^s_v\| &\le (1+3kL)^s\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\}, \; s = 1,2,3,\ldots n.
\end{aligned}$$

For $n+1$, set $\left\|E^{n+1}_u\right\| = \max_{1\le i\le I}\left|e^{n+1}_{ui}\right| = \left|e^{n+1}_{uj}\right|, \left\|E^{n+1}_v\right\| = \max_{1\le i\le l}\left|e^{n+1}_{vi}\right| = \left|e^{n+1}_{um}\right|$. By substituting e^n_{uj} in the explicit formula (2.1), we obtain

$$e^{n+1}_{uj} = (1-2r_n)e^n_{uj} + r_n\left(e^n_{uj+1} + e^n_{uj-1}\right) + k_n\left(F\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right).$$

Since $(1-2r) \ge 0$, we have

$$\left|e^{n+1}_{uj}\right| \le (1-2r_n)\left|e^n_{uj}\right| + r_n\left(\left|e^n_{uj+1}\right| + \left|e^n_{uj-1}\right|\right) + k_n\left|F\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right|.$$

Thus, from (1.2), it follows that

$$\left|e^{n+1}_{uj}\right| \le (1-2r_n)\left\|E^n_u\right\| + 2r_n\left\|E^n_u\right\| + k_nL_1\left|\delta_x\left(u^n_j - U^n_j\right)\right| + k_nL_2\left|v^n_j - V^n_j\right|$$

$$
\begin{aligned}
& \leq \left\|E_u^n\right\|+k_n L_1\left|\delta_x e_{uj}^n\right|+k_n L_2\left|e_{vj}^n\right| \\
\left\|E_u^{n+1}\right\| & \leq \left\|E_u^n\right\|+2 k_n L_1\left\|E_u^n\right\|+k_n L_2\left\|E_v^n\right\| \\
& \leq (1+3 k L) \operatorname{Max}\left\{\left\|E_u^n\right\|,\left\|E_v^n\right\|\right\} \\
& \leq (1+3 k L)(1+3 k L)^n \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\} \\
& \leq (1+3 k L)^{n+1} \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\} \\
& \leq \exp (3(n+1) k L) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\} \\
& = \exp \left(3 t_{n+1} L\right) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\} .
\end{aligned}
$$

Likewise,

$$
\left\|E_v^{n+1}\right\| \leq \exp \left(3 t_{n+1} L\right) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\} .
$$

So, this implies the stability for explicit scheme, if $r \leq 1/2$.

2.4. Convergence analysis of explicit Euler scheme

Theorem 3. *The explicit Euler formulas (2.1), (2.2) are convergent with:* $O\left(k+h^2\right)$, *if* $(1-2r) \geq 0$, *where* $k=\max_{n \in N} k_n, r=k/h^2$.

Proof. Set $e_{ui}^n=u_i^n-U_i^n, e_{vi}^n=v_i^n-V_i^n$, $(u_i^n=u(x_i, t_n), v_i^n=v(x_i, t_n))$ is the accurate solution to the problem (1.1). suppose that $e_{ui}^0=0, e_{vi}^0=0, \forall i=0,1, \ldots, I$. We seek for $C>0$, such that

$$
e_{ui}^{n+1} \leq C\left(k+h^2\right), \quad e_{vi}^{n+1} \leq C\left(k+h^2\right), \quad n=0,1, \ldots .
$$

To finish this, We use the Tech of mathematical induction For $n=1$, we set $\left|e_{uj}^1\right|= \max_{1 \leq i \leq I-1}\left|e_{ui}^1\right|$. Substituting e_{uj}^1 in the explicit formula (2.1) yields that

$$
\begin{aligned}
e_{uj}^1=(1-2 r_n) e_{uj}^0+r_n\left(e_{uj+1}^0+e_{uj-1}^0\right)+k_n\left(F\left(\delta_x u_j^0, v_j^0\right)-F\left(\delta_x U_j^0, V_j^0\right)\right)+T_u i^0 \\
=k_n\left(F\left(\delta_x u_j^0, v_j^0\right)-F\left(\delta_x U_j^0, V_j^0\right)\right)+T_u i^0 .
\end{aligned}
$$

From Lemma (1) we obtain

$$
\left|e_{uj}^1\right| \leq k_n L_1\left|\delta_x\left(u_j^0-U_j^0\right)\right|+k_n L_2\left|v_j^0-V_j^0\right|+\left|T_{ui}^0\right|=\left|T_{ui}^0\right| \leq C\left(k+h^2\right) .
$$

Hence $\left|e_{ui}^1\right| \leq C\left(k+h^2\right), i=1,2, \ldots I-1$. Likewise, we show that

$$
\left|e_{vi}^1\right| \leq C\left(k+h^2\right), \quad i=1,2, \ldots, I-1 .
$$

assume $\left|e_{ui}^S\right| \leq C_s\left(k+h^2\right)$, $\left|e_{vi}^S\right| \leq C_s\left(k+h^2\right)$, $s=0,1,2, \ldots, n$, $C_s>0$. Let $C^*= \max_{0 \leq s \leq n} C_s$. For $n+1$, we set
$\left|e_{uj}^{n+1}\right|=\max_{1 \leq i \leq I-1}\left|e_{ui}^{n+1}\right|$.
Replace e_{uj}^{n+1} in the explicit scheme (2.1) yields

$$
e_{uj}^{n+1}=(1-2 r_n) e_{uj}^n+r_n\left(e_{uj+1}^n+e_{uj-1}^n\right)+k_n\left(F\left(\delta_x u_j^n, v_j^n\right)-F\left(\delta_x U_j^n, V_j^n\right)\right)+T_{ui}^n .
$$

Thus

$$\begin{aligned}\left|e_{uj}^{n+1}\right| &\leq (1-2r_n)\left\|E_u^n\right\| + 2r_n\left\|E_u^n\right\| + k_n\left|F\left(\delta_x u_j^n, v_j^n\right) - F\left(\delta_x U_j^n, V_j^n\right)\right| + \left|T_{uj}^n\right| \\ &\leq (1-2r_n)\left\|E_u^n\right\| + 2r_n\left\|E_u^n\right\| + k_n L_1\left|\delta_x\left(u_j^n - U_j^n\right)\right| + k_n L_2\left|v_j^n - V_j^n\right| + \left|T_{uj}^n\right|.\end{aligned}$$

From Lemma (1), we have

$$\begin{aligned}\left\|E_u^{n+1}\right\| &\leq \left\|E_u^n\right\| + k_n L_1\left|\delta_x e_{uj}^n\right| + k_n L_2\left|e_{vj}^n\right| + \left|T_j^n\right| \\ &\leq \left\|E_u^n\right\| + 2k_n L_1\left\|E_u^n\right\| + k_n L_2\left\|E_v^n\right\| + \left|T_j^n\right| \\ &\leq (1+2k_nL_1+k_nL_2)\,C^*\left(k+h^2\right) + C\left(k+h^2\right) \\ &= \left[(1+2kL_1+kL_2)\,C^* + C\right]\left(k+h^2\right).\end{aligned}$$

Next $\left\|E_u^{n+1}\right\| \leq C\left(k+h^2\right)$, $n = 0, 1, \ldots$. Likewise, we show that

$$\left\|E_v^{n+1}\right\| \leq C\left(k+h^2\right), \quad n = 0, 1, \ldots.$$

3. Implicit Euler scheme

Now, in order to obtain the implicit fully-discrete finite difference formulae for problem (1.1), we will employ the backward finite difference formula to approximate the time derivative in problem (1.1). In this analysis, we define U_i^n and V_i^n, as the estimated values of $u(x_i, t_n)$ and $v(x_i, t_n)$, consecutively, let $t_{n+1} = t_n + k_n$, where I represent a positive integer, and proceed to examine the grid:: $x_i = ih$, $0 \leq i \leq I$, $h = \frac{1}{I}$, $t_{n+1} = t_n + k_n$, $x_{i+1} = x_i + h$, h is the space-step, k_n is the time-step.

At the grid point (x_i, t_{n+1}),we employ backward finite difference formulas to approximate the values of u_t, v_t as below :

$$\begin{aligned}\left.\frac{\partial u}{\partial t}\right|_i^{n+1} &= \frac{1}{k_n}\left(U_i^{n+1} - U_i^n\right) + O(k_n), \\ \left.\frac{\partial v}{\partial t}\right|_i^{n+1} &= \frac{1}{k_n}\left(V_i^{n+1} - V_i^n\right) + O(k_n).\end{aligned}$$

While u_{xx}, v_{xx} are approximated as using the second-order central finite difference formulas below:

$$\begin{aligned}\left.u_{xx}\right|_i^{n+1} &= \frac{U_i^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + O\left(h^2\right), \\ \left.v_{xx}\right|_i^{n+1} &= \frac{V_i^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} + O\left(h^2\right).\end{aligned}$$

Moreover, the non-linear terms are approximated as follows:

$$\begin{aligned}F\left(\delta_x U_j^{n+1}, V_j^{n+1}\right) &= F\left(\delta_x U_j^n, V_j^n\right) + O(k), \\ G\left(\delta_x V_i^{n+1}, U_i^{n+1}\right) &= G\left(\delta_x V_i^n, U_i^n\right) + O(k).\end{aligned}$$

Substituting all of these formulas into system (1.1) gives

$$\frac{U_i^{n+1}-U_i^n}{k_n}=\frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{h^2}+F\left(\delta_x U_j^n, V_j^n\right), \quad 1\le i\le I-1,$$
$$\frac{V_i^{n+1}-V_i^n}{k_n}=\frac{V_{i+1}^{n+1}-2V_i^{n+1}+V_{i-1}^{n+1}}{h^2}+G\left(\delta_x V_i^n, U_i^n\right), \quad 1\le i\le I-1.$$

Thus

$$(1+2r_n)\,U_i^{n+1}-r_n\left(U_{i+1}^{n+1}+U_{i-1}^{n+1}\right)=U_i^n+k_nF\left(\delta_x U_j^n, V_j^n\right), \tag{3.1}$$
$$(1+2r_n)\,V_i^{n+1}-r_n\left(V_{i+1}^{n+1}+V_{i-1}^{n+1}\right)=V_i^n+k_nG\left(\delta_x V_i^n, U_i^n\right), \tag{3.2}$$

where $r_n=\frac{k_n}{h^2}$, $1\le i\le I-1$, $n=0,1,2,\ldots$.

Moreover, The non-fixed time-stepping formula is employed due to the lack of condition stability in the implicit Euler chart.

$$k_n=\min\left(\frac{h^\alpha}{\left\|\mathrm{U}_h^n\right\|},\frac{h^\alpha}{\left\|\mathrm{V}_h^n\right\|}\right), \alpha\ge 1. \tag{3.3}$$

We write (3.1)–(3.2) in matrix form as below:

$$(I-r_h^nH)\,U_h^{n+1}=U_h^n+k_nF\left(V_h^n\right), \tag{3.4}$$
$$(I-r_h^nH)\,V_h^{n+1}=V_h^n+k_nG\left(U_h^n\right), \tag{3.5}$$

where

$$H=\begin{bmatrix} -2 & 1 & 0 & \cdots & 0\\ 1 & -2 & 1 & \cdots & 0\\ & & \ddots & & \\ & \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 & -2\end{bmatrix}_{(I-1)\times(I-1)},$$

$F\left(V_h^n\right)=\left(f\left(V_1^n\right),f\left(V_2^n\right),\ldots,f\left(V_{I-1}^n\right)\right)^T$, $G\left(U_h^n\right)=\left(g\left(U_1^n\right),g\left(U_2^n\right),\ldots,g\left(U_{I-1}^n\right)\right)^T$.

3.1. The algorithm steps for Euler implicit method

1. Input $\mathrm{h}, U_h^0, V_h^0, \mathrm{p}_1, \mathrm{p}_2, \mathrm{q}_1, \mathrm{q}_2, \alpha$
2. Put $n=0$;
3. Choose k_n according to (3.3).
4. Compute the numerical vectors: U_h^{n+1}, V_h^{n+1}, by solving the linear systems (3.4) and (3.5).
5. For $n=1,2,\ldots..$, repeat steps 3,4 until for $n=m$, we get $\|U_h^n\|_\infty\ge 10^{15}$, or $\|V_h^n\|_\infty\ge 10^{15}$

6. The numerical blow-up time is $t_m = \sum_{n=0}^{m} k_n$.

7.

3.2. Local Truncation error of implicit Euler scheme

Theorem 4. *Let* (T_{ui}^n, T_{vi}^n) *be the local truncation error of the implicit Euler scheme (3.1)–(3.2) at the mesh point* (x_i, t_{n+1})*. Then, there exist* $C_1, C_2, C_3, C_4 > 0$*, such that* $\left|T_{ui}^{n+1}\right| \le C_1 k + C_2 h^2$ *and* $\left|T_{vi}^{n+1}\right| \le C_3 k + C_4 h^2$*, where* $k = \max_{n\in N} k_n$*, i.e.* $T_{ui}^{n+1} = O\left(k + h^2\right)$ *and* $T_{vi}^{n+1} = O\left(k + h^2\right)$*.*

Proof. Replace the precise solution $u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right)$ into the implicit Euler scheme (3.1) yields

$$\begin{aligned} T_{ui}^{n+1} &= \left(u_i^{n+1} - u_i^n\right) - \frac{k_n}{h^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right] - k_n F\left(\delta_x U_j^n, V_j^n\right) \\ &= k_n\left[\frac{\partial u_i^{n+1}}{\partial t} + O(k_n)\right] - k_n\left[\frac{\partial^2 u_i^{n+1}}{\partial x^2} + O\left(h^2\right)\right] \\ &= -k\left[\left(v_i^{n+1}\right)^{p_1} - \left|\frac{\partial u_i^{n+1}}{\partial x}\right|^{q_1} + O(k_n) + O\left(h^2\right)\right]. \end{aligned}$$

It follows that

$$T_{ui}^{n+1} = k_n\left[\frac{\partial u_i^{n+1}}{\partial t} - \frac{\partial^2 u_i^{n+1}}{\partial x^2} - \left(v_i^{n+1}\right)^{p_1} + \left|\frac{\partial u_i^{n+1}}{\partial x}\right|^{q_1}\right] + k_n\left[O(k_n) + O\left(h^2\right)\right].$$

From equation (1.1) and Presuming all the partial derivatives are bounded at the mesh-point (x_i, t_{n+1}), we obtain $\left|T_{ui}^{n+1}\right| \le C_1 k_n + C_2 h^2 \le C_1 k + C_2 h^2$, where $C_1, C_2 > 0.$, i.e. $\left|T_i^{n+1}\right| = O\left(k + h^2\right), C > 0$. Similarly, we show that there exist $C_3, C_4 > 0$ such that

$$\left|T_{vi}^{n+1}\right| \le C_3 k + C_4 h^2.$$

3.3. The stability analysis of implicit Euler method

Theorem 5. *The implicit Euler scheme (3.1)–(3.2) is unconditionally stable.*

Proof. To demonstrate the validity of this theorem, the maximum error stability technique can be employed. [6].

Let $e_{ui}^n = u_i^n - U_i^n$, $\quad e_{vi}^n = v_i^n - V_i^n$, $(u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right))$ is the accurate solution to the problem (1.1).

To finish this, We implement the mathematical induction. For $n = 1$ and setting $\left\|E_u^1\right\| = \max_{1\le i\le I}\left|e_{ui}^1\right| = \left|e_{uj}^1\right|$, $k = \max_{n\in N} k_n, r = k/h^2$, we have

$$\left|e_{uj}^1\right| = \left(1 + 2r_1\right)\left|e_{uj}^1\right| - r_1\left(\left|e_{uj}^1\right| + \left|e_{uj}^1\right|\right)$$

$$\begin{aligned}
&\leq (1+2r_1)\left|e^1_{uj}\right| - r_1\left(\left|e^1_{uj+1}\right| + \left|e^1_{uj-1}\right|\right)\\
&\leq \left|(1+2r_1)e^1_{uj} - r_1\left(e^1_{uj+1} + e^1_{uj-1}\right)\right|\\
&= \left|e^0_{uj} + k_1\left(F\left(\delta_x u^0_j, v^0_j\right) - F\left(\delta_x U^0_j, V^0_j\right)\right)\right|.
\end{aligned}$$

Lemma (1) then gives

$$\begin{aligned}
\left|e^1_{uj}\right| &\leq \left|e^0_j\right| + k_1\left|F\left(\delta_x u^0_j, v^0_j\right) - F\left(\delta_x U^0_j, V^0_j\right)\right|\\
&\leq \left|e^0_j\right| + k_1 L_1\left|\delta_x\left(u^0_j - U^0_j\right)\right| + k_1 L_2\left|v^0_j - V^0_j\right|\\
&= \left|e^0_j\right| + k_1 L_1\left|\delta_x e^0_{uj}\right| + k_1 L_2\left|e^0_{vj}\right|.
\end{aligned}$$

It follows that

$$\left\|E^1_u\right\| \leq \left\|E^0_u\right\| + 2kL_1\left\|E^0_u\right\| + kL_2\left\|E^0_v\right\| \leq (1+3kL)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\}.$$

Similarly, we can show that

$$\left\|E^1_v\right\| \leq \left\|E^0_v\right\| + 2kL_3\left\|E^0_v\right\| + kL_4\left\|E^0_u\right\| \leq (1+3kL)\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\},$$

where $L = \mathrm{Max}\{L_1, L_2, L_3, L_4\}$.

Now, we suppose that

$$\begin{aligned}
&\|E^s_u\| \leq (1+3kL)^s\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\},\ s = 1,2,3,\ldots n,\\
&\|E^s_v\| \leq (1+3kL)^s\,\mathrm{Max}\left\{\left\|E^0_u\right\|, \left\|E^0_v\right\|\right\},\ s = 1,2,3,\ldots n.
\end{aligned}$$

For $n+1$ and setting $\left\|E^{n+1}_u\right\| = \max_{1\leq i\leq I}\left|e^{n+1}_{ui}\right| = \left|e^{n+1}_{uj}\right|$ and $\left\|E^{n+1}_v\right\| = \max_{1\leq i\leq I}\left|e^{n+1}_{vi}\right|$, $k = \max_{n\in N} k_n$, $r = k/h^2$ yields

$$\begin{aligned}
\left|e^{n+1}_{uj}\right| &= (1+2r_n)\left|e^{n+1}_{uj}\right| - r_n\left(\left|e^{n+1}_{uj}\right| + \left|e^{n+1}_{uj}\right|\right)\\
&\leq (1+2r_n)\left|e^{n+1}_{uj}\right| - r_n\left(\left|e^{n+1}_{uj+1}\right| + \left|e^{n+1}_{uj-1}\right|\right)\\
&\leq \left|(1+2r_n)e^{n+1}_{uj} - r_n\left(e^{n+1}_{uj+1} + e^{n+1}_{uj-1}\right)\right|\\
&= \left|e^n_{uj} + k_n\left(F\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right)\right|.
\end{aligned}$$

By Lemma (1), we obtain

$$\begin{aligned}
\left|e^{n+1}_{uj}\right| &\leq \left|e^n_{uj}\right| + k_n\left|fF\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right|\\
&\leq \left|e^n_{uj}\right| + k_n L_1\left|\delta_x\left(u^n_j - U^n_j\right)\right| + k_n L_2\left|v^n_j - V^n_j\right|\\
&= \left|e^n_j\right| + k_n L_1\left|\delta_x e^n_{uj}\right| + k_n L_2\left|e^n_{vj}\right|.
\end{aligned}$$

It follows that

$$\left\|E^{n+1}_u\right\| \leq \left\|E^n_u\right\| + 2k_n L\left\|E^n_u\right\| + k_n L\left\|E^n_v\right\|$$

$$\leq (1+3kL)^n \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}+3kL(1+3kL)^n \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}$$
$$\leq (1+3kL)^{n+1} \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}$$
$$\leq \exp(3(n+1)kL) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}$$
$$= \exp\left(3t_{n+1}L\right) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}.$$

Similarly, we can show that

$$\left\|E_v^{n+1}\right\| \leq \exp\left(3t_{n+1}L\right) \operatorname{Max}\left\{\left\|E_u^0\right\|,\left\|E_v^0\right\|\right\}. \text{ where } L=\operatorname{Max}\left\{L_1,L_2,L_3,L_4\right\}$$

So, the implicit Euler scheme (2.4)–(2.5) is unconditionally stable.

3.4. Convergence analysis of implicit Euler scheme

Theorem 6. *The implicit Euler formula (3.1)–(3.2) is convergent with $r=k/h^2>0$, if $r>0$, where $k=\max_{n\in N} k_n$.*

Proof. Set $e_{ui}^n=u_i^n-U_i^n$, $e_{vi}^n=v_i^n-V_i^n$, and let $(u_i^n=u(x_i,t_n)$ and $v_i^n=v(x_i,t_n))$ be Accurate solution to the issue (1.1). suppose that $e_{ui}^0=0, e_{vi}^0=0, \forall i=0,1,\ldots,I$. We aim to show that there exists $C>0$ such that

$$e_{ui}^{n+1} \leq C\left(k+h^2\right), \quad e_{vi}^{n+1} \leq C\left(k+h^2\right), \quad n=0,1,\ldots.$$

To finish this, we use the mathematical induction tech. For $n=1$, we set $|e_{uj}^1|=\max_{1\leq i\leq I-1}|e_{ui}^1|$. Thus

$$\begin{aligned}
|e_{uj}^1| &= (1+2r_1)|e_{uj}^1| - r_1(|e_{uj}^1|+|e_{uj}^1|)\\
&\leq (1+2r_1)\left|e_{uj}^1\right| - r_1\left(\left|e_{uj+1}^1\right|+\left|e_{uj-1}^1\right|\right)\\
&\leq \left|(1+2r_1)e_{uj}^1 - r_1\left(e_{uj+1}^1+e_{uj-1}^1\right)\right|\\
&= \left|e_{uj}^0 + k_1\left(F\left(\delta_x u_j^0, v_j^0\right)-F\left(\delta_x U_j^0, V_j^0\right)\right)+T_i^0\right|\\
&= \left|k_1\left(F\left(\delta_x u_j^0, v_j^0\right)-F\left(\delta_x U_j^0, V_j^0\right)\right)+T_j^0\right|.
\end{aligned}$$

From Lemma (1), we have

$$\begin{aligned}
\left|e_{uj}^1\right| &\leq k_1L_1\left|\delta_x\left(u_j^0-U_j^0\right)\right| + k_1L_2\left|v_j^0-V_j^0\right| + \left|T_j^0\right|,\\
&= k_1L_1\left|\delta_x e_{uj}^0\right| + k_1L_2\left|e_{vj}^0\right| + \left|T_j^0\right|
\end{aligned}$$

where $L=\operatorname{Max}\left\{L_1,L_2\right\}$. So,

$$\left|e_{uj}^1\right| \leq 2kL_1\left\|E_u^0\right\| + kL_2\left\|E_v^0\right\| + \left|T_i^0\right| = \left|T_i^0\right| \leq C\left(k+h^2\right).$$

Thus $|e_{ui}^1| \leq C(k+h^2)$, $i=1,2,\ldots I-1$.

Likewise,

$$\left|e_{vi}^1\right| \leq C\left(k+h^2\right),\ i=1,2,\ldots,I-1.$$

Now assume that $|e^s_{ui}| \leq C_s\left(k+h^2\right)$ and $|e^s_{vi}| \leq C_s\left(k+h^2\right)$, where $s = 0,1,2,\ldots,n$, $C_s > 0$, $C^* = \max_{0\leq s\leq n} C_s$.

For $n+1$, we set $|e^{n+1}_{uj}| = \max_{1\leq i\leq I-1}|e^{n+1}_{ui}|$, and so

$$\begin{aligned}
\left|e^{n+1}_{uj}\right| &= (1+2r_n)\left|e^{n+1}_{uj}\right| - r_n\left(\left|e^{n+1}_{uj}\right| + \left|e^{n+1}_{uj}\right|\right) \\
&\leq (1+2r_n)\left|e^{n+1}_{uj}\right| - r_n\left(\left|e^{n+1}_{uj+1}\right| + \left|e^{n+1}_{uj-1}\right|\right) \\
&\leq \left|(1+2r_n)e^{n+1}_{uj} - r_n\left(e^{n+1}_{uj+1} + e^{n+1}_{uj-1}\right)\right| \\
&= \left|e^n_{uj} + k_n\left(F\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right) + T^{n+1}_j\right|.
\end{aligned}$$

Thus

$$\left|e^{n+1}_{uj}\right| \leq \left|e^n_{uj}\right| + k_n\left|F\left(\delta_x u^n_j, v^n_j\right) - F\left(\delta_x U^n_j, V^n_j\right)\right| + \left|T^{n+1}_j\right|.$$

By Lemma (1), we obtain

$$\begin{aligned}
\left|e^{n+1}_{uj}\right| &\leq \left|e^n_{uj}\right| + k_n L_1\left|\delta_x\left(u^n_j - U^n_j\right)\right| + k_n L_2\left|v^n_j - V^n_j\right| + \left|T^{n+1}_j\right| \\
&\leq \left|e^n_{uj}\right| + kL_1\left|\delta_x e^n_{uj}\right| + kL_2\left|e^n_{vj}\right| + \left|T^{n+1}_j\right|.
\end{aligned}$$

By setting $L = \text{Max}\{L_1, L_2\}$, we get

$$\begin{aligned}
\left\|E^{n+1}_u\right\| &\leq \|E^n_u\| + 2kL_1\|E^n_u\| + kL_2\|E^n_v\| + \left|T^{n+1}_j\right| \\
&\leq C^*\left(k+h^2\right) + 3kLC^*\left(k+h^2\right) + C\left(k+h^2\right) \\
&\leq (1+3kL)C^*\left(k+h^2\right) + C\left(k+h^2\right) \\
&= \left[(1+3kL)C^* + C\right]\left(k+h^2\right).
\end{aligned}$$

It follows that $\left\|E^{n+1}_u\right\| \leq C\left(k+h^2\right)$, $n = 0,1,\ldots$, $i = 1,2,\ldots,I-1$.

Similarly, we can show that

$$\left\|E^{n+1}_v\right\| \leq C\left(k+h^2\right),\ n = 0,1,\ldots.$$

Definition 2. *If the specified conditions are satisfied, a solution of both the Explicit and Implicit Euler schemes will together blow up within a finite time denoted as T_h:*

- $\|U^n_h\|_\infty \to \infty, \|V^n_h\|_\infty \to \infty$, *as* $n \to \infty$,
- $T_h = \sum_n^\infty k_n$.

Remark 1.

- *The matrix $A = (I - r^n_h H)$ exhibits diagonal dominance, characterized by the presence of positive real diagonal components. This observation suggests that matrix A is positive-definite and non singular [26], Therefore, it is possible to solve the linear systems(3.4)-(3.5) and get a unique solution.*

- *The finite difference schemes, explicit Euler schemes, and implicit Euler schemes that have been developed demonstrate consistency, stability, and convergence.*

- *Explicit (implicit) Euler numerical systems provide approximate solutions for each set time interval* $l\,[0,T]$*, with a convergence rate of* $\mathrm{O}\left(k+h^2\right)$ *where* $k=\max_n k_n$*, However, when using the time-steps formulas (2.3) and (3.3), the convergence rate becomes* $\mathrm{o}\left(h^\alpha\right)$*, as* $h\to 0$*, for* $\alpha\leq 2$*. The numerical blow-up time is likely to exhibit a similar pattern of convergence.*

- *The numerical blow-up time of a fully discrete formula, such as explicit Euler or implicit Euler, is considered the blow-up time for system (1.1).*

- *The duration of numerical blow-up achieved through the utilization of a discrete scheme is contingent upon both the space-step* h *and the selection of time-steps* k_n*.Nevertheless, it should be noted that the time-stepping formulas (2.3) and (3.3) are also contingent upon the space-step. Hence, in the examination of the numerical convergence of the aforementioned schemes, namely explicit Euler and implicit Euler, it is sufficient to enhance the space-stepping technique in order to calculate the error boundaries and numerical order of convergence.*

4. Numerical experiments

In this section, we present an estimation of the numerical blow-up times for two numerical experiments, employing distinct space steps, utilizing the finite difference techniques described in this study, namely explicit Euler and implicit Euler. The Matlab (R2020a) software is used to write all the numerical computing codes.We use h to denote the space-step, and k_n. is the time-step, $h=\frac{1}{I}$. Furthermore, we assess the rate at which the numerical blow-up time increases for each of these strategies. The numerical blow-up time is determined by identifying a value of $m\in N$ such that the condition:$\|U_h^m\|_\infty\geq 10^{15}$ or $\|V_h^n\|_\infty\geq 10^{15}$ holds . Furthermore, the numerical blow-up time to the examined problem is denoted as $T_h=t_m=\sum_{n=0}^{m}k_n$ The error bounds between T_{2h} and T_h are denoted as . $E_h=|T_{2h}-T_h|$, .The numerical results obtained using the suggested schemes (explicit Euler and implicit Euler) are presented in tables for each case. The mesh sizes used are $I=\{20,\ 40,80,160,320\}$, The tables display the iteration count, occurrence of numerical blow-up, numerical blow-up durations, central processing unit (CPUT) durations in seconds, and the error limits of numerical blow-up durations. In order to empirically investigate the rate of numerical convergence for numerical blow-up durations, it is necessary to consider various mesh sizes with specific values ofα,The formula employed for this purpose is $S_h=\frac{\log(E_{2h}/E_h)}{\log 2}$. In addition, numerical simulations are conducted to corroborate the numerical findings.

4.1. Numerical Experiments

Example 1.

$$\left.\begin{array}{l} u_t = u_{xx} + v^5 - |u_x|^{1.6}, \quad v_t = v_{xx} + u^6 - |v_x|^{1.6}, \quad x \in (0,1),\ t \in (0,T), \\ \quad u(0,t) = u(1,t) = 0, \\ \quad v(0,t) = v(1,t) = 0, \quad t \in (0,T), \\ \quad u(x,0) = 80\left(x - x^2\right), \quad v(x,0) = 90\left(x - x^2\right), \quad x \in (0,1). \end{array}\right\} \tag{4.1}$$

Example 2.

$$\left.\begin{array}{l} u_t = u_{xx} + v^3 - |u_x|^{1.1}, \quad v_t = v_{xx} + u^4 - |v_x|^{1.2}, \quad x \in (0,1),\ t \in (0,T), \\ \quad u(0,t) = u(1,t) = 0, \\ \quad v(0,t) = v(1,t) = 0, \quad t \in (0,T), \\ \quad u(x,0) = 100(\sin \pi x), \quad v(x,0) = 90(\sin \pi x), \quad x \in (0,1). \end{array}\right\} \tag{4.2}$$

Table 1: Example 1, Explicit formula, $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	3	6.81E-04	0.088252	$\cdots$	$\cdots$
1/40	4	2.09E-04	0.082713	4.72E-04	$\cdots$
1/80	4	5.29E-05	0.127637	1.56E-04	1.5971
1/160	4	1.42E-05	0.154282	3.87E-05	2.0095
1/320	4	4.73E-06	0.261390	9.44E-06	2.0372

Table 2: Example 1, Explicit formula, $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	3.45E-05	0.096988	$\cdots$	$\cdots$
1/40	4	6.49E-06	0.081515	2.80E-05	$\cdots$
1/80	5	1.44E-06	0.104819	5.05E-06	2.4727
1/160	7	5.16E-07	0.151543	9.26E-07	2.4473
1/320	15	2.88E-07	0.486987	2.28E-07	2.0231

Table 3: Example 1, Implicit formula, $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	3	6.81E-04	0.333453	$\cdots$	$\cdots$
1/40	4	2.09E-04	0.242359	4.72E-04	$\cdots$
1/80	4	5.29E-05	0.447964	1.56E-04	1.5972
1/160	4	1.42E-05	1.003357	3.87E-05	2.0094
1/320	5	4.73E-06	2.699985	9.44E-06	2.0373

Table 4: Example 1, Implicit formula, $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	3.45E-05	0.202292	$\cdots$	$\cdots$
1/40	4	6.49E-06	0.219009	2.80E-05	$\cdots$
1/80	5	1.44E-06	0.259753	5.05E-06	2.4727
1/160	7	5.16E-07	0.847733	9.26E-07	2.4473
1/320	15	2.88E-07	2.355910	2.28E-07	2.0231

Table 5: Example 2, Explicit formula, $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	1.59E-04	0.337957	$\cdots$	$\cdots$
1/40	5	5.73E-05	0.086509	1.02E-04	$\cdots$
1/80	6	2.135E-05	0.135266	3.59E-05	1.5083
1/160	7	9.15E-06	0.165087	1.22E-05	1.5590
1/320	9	5.01E-06	0.316164	4.14E-06	1.5566

Table 6: Example 2, Explicit formula, $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	7	1.01E-05	0.785978	$\cdots$	$\cdots$
1/40	12	4.15E-06	0.102545	5.96E-06	$\cdots$
1/80	36	3.145E-06	0.113383	1.01E-06	2.5626
1/160	163	2.93E-06	0.155124	2.17E-07	2.2135
1/320	879	2.89E-06	0.424556	4.31E-08	2.3346

4.2. Discussion

By the numerical results in Examples 1 and 2, we refer to the following comments:

- The numerical blow-up can happen simultaneously at one time and one point ($x = 0.5$), and that agreed with the known theoretical blow-up results of the single equation of system (1.1), [25].
- When the space-steps are refined, the blow-up time errors-bounds decrease. This refers that the numerical blow-up times sequence T_h, is convergent, as the space-step approaches to 0

Table 7: Example 2, Implicit formula, $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	1.59E-04	0.360841	$\cdots$	$\cdots$
1/40	5	5.73E-05	0.237088	1.02E-04	$\cdots$
1/80	6	2.13E-05	0.329224	3.59E-05	1.5084
1/160	7	9.15E-06	0.855710	1.22E-05	1.5589
1/320	9	5.01E-06	2.501314	4.14E-06	1.5567

Table 8: Example 2, Implicit formula, $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	6	1.01E-05	0.459823	$\cdots$	$\cdots$
1/40	12	4.15E-06	0.246959	5.96E-06	$\cdots$
1/80	36	3.15E-06	0.305770	1.01E-06	2.5627
1/160	163	2.93E-06	0.858916	2.17E-07	2.2136
1/320	879	2.89E-06	2.554193	4.31E-08	2.3346

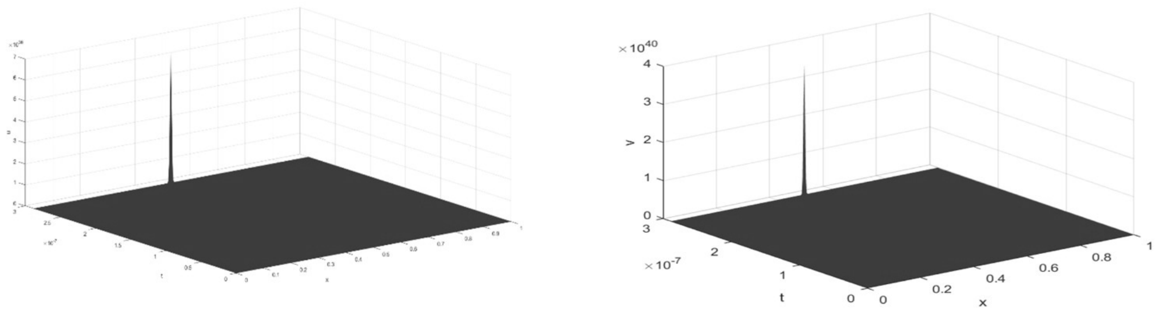

Figure 1: development of numerical blow-up solution with passage of time emerging from using Explicit for Example 1, with$h = 320, \alpha = 2$

- Convergence order of the numerical blow-up times, S_h is close to or larger than the value of $\boldsymbol{\alpha}$, this refers, the numerical order of convergence is: $0\left(h^{\alpha+\epsilon}\right)$, where $\epsilon > 0$
- With the time-stepping formulas (2.3) and (3.3), the number of iteration that is required to achieve blow-up is increasing as the value of $\boldsymbol{\alpha}$ increases.
- When we refine the spatial step, or compare CPUT of implicit method with that of explicit method, we find the CPUT times are increasing.

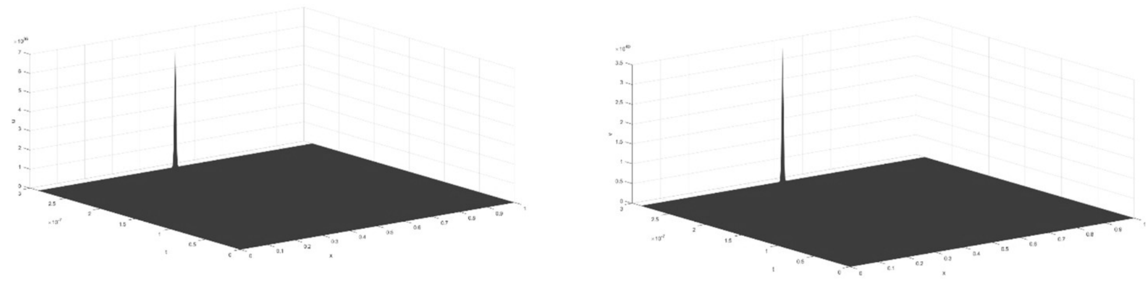

Figure 2: development of numerical blow-up solution with passage of time emerging from using Implicit for Example 1, with $h = 320, \alpha = 2$

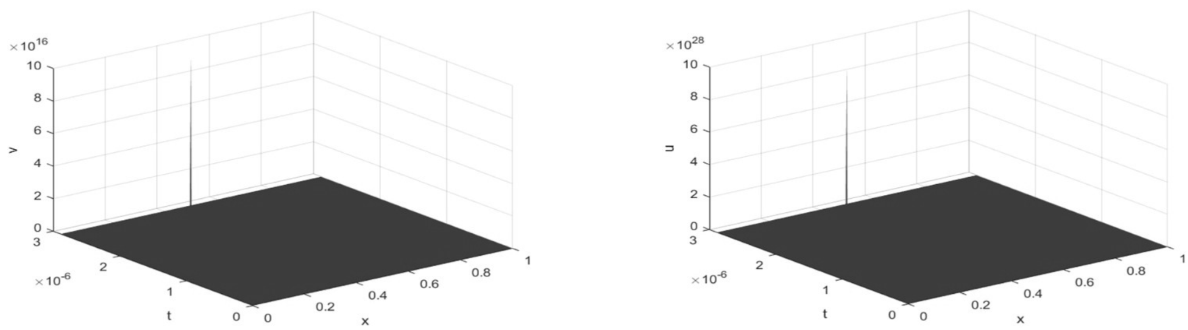

Figure 3: development of numerical blow-up solution with passage of time emerging from using Explicit for Example 2, with $h = 320, \alpha = 2$

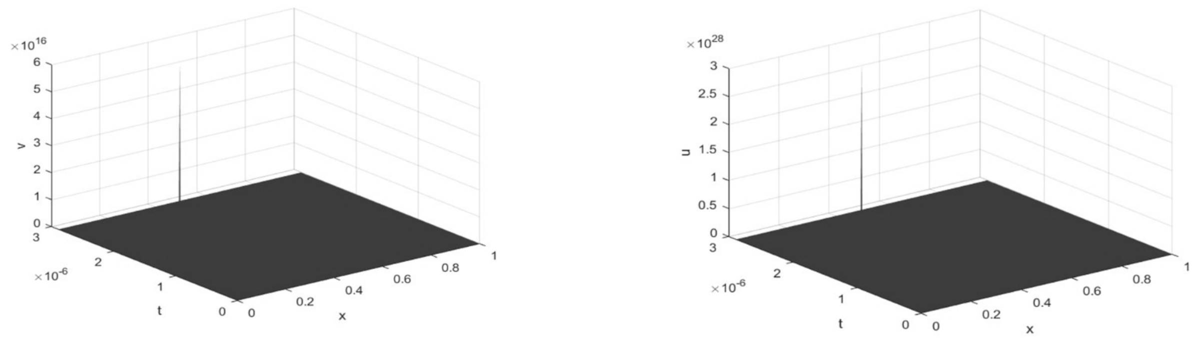

Figure 4: development of numerical blow-up solution with passage of time emerging from using Implicit for Example 2— with $h = 320, \alpha = 2$

- From tow figures (1) and (2), we find in each of studied problem, the numerical blow-up growth-rates, obtained by use explicit Euler scheme, is almost the same as that obtained by use implicit Euler scheme.

5. Conclusions

This paper deals with numerical approximations of a one-dimensional coupled reaction-diffusion system with gradient terms. Namely, we propose both Euler explicit and implicit finite difference methods with a non-fixed time-stepping procedure to estimate the numerical blow-up time of the considered problem. Moreover, some numerical experiments are given to illustrate the efficiency, accuracy, and numerical order of convergence of the proposed technique. The obtained numerical results show that the used finite difference schemes with the proposed non-fixed time-stepping procedure can give accurate results with high order of numerical convergence. Furthermore, the numerical results show that the blow-up can only occur at the center point. The two proposed schemes are effective and can be easily used comprised with other numerical techniques. As future work, we may use Crank-Nicolson technique to computer the numerical solution of the considered system.

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