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Hankel and Toeplitz Determinants of Logarithmic Coefficients of Inverse Functions for the Subclass of Starlike Functions with Respect to Symmetric Conjugate Points

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Abstract. This paper focuses on finding the upper bounds of the second Hankel and Toeplitz determinants, whose entries are logarithmic coefficients of inverse functions for a new subclass of starlike functions with respect to symmetric conjugate points associated with the exponential function defined by subordination. Results on initial Taylor coefficients and logarithmic coefficients of inverse functions for a new subclass are also presented. This study may inspire others to focus further to the coefficient functional problems associated with the inverse functions of various classes of univalent functions.

2020 Mathematics Subject Classifications: 30C45, 30C50

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1. Introduction

Let A denote the class of functions defined on the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ which is normalized by the conditions $f(0) = 0$ and $f'(0) - 1 = 0$. The Taylor series of a function $f(z)$ in A has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in E. \tag{1}$$

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The subclass of A that consists of analytic and univalent functions in the open unit disk E is denoted by S.
On the other hand, it is well known that for each function $f(z) \in S$, there is an inverse function $f^{-1}(w)$ in the form of

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n,\ |w| < r_0(f),r_0(f) \geq \frac{1}{4}, \tag{2}$$

where particularly

$$A_2 = -a_2, \tag{3}$$

$$A_3 = -a_3 + 2{a_2}^2, \tag{4}$$

and

$$A_4 = -a_4 + 5a_2a_3 - 5{a_2}^3. \tag{5}$$

Let P denote the class of functions with a positive real part in E. A function $p(z)$ in P has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in E, \tag{6}$$

that is analytic in E and satisfying the condition $\operatorname{Re}(p(z)) > 0$. It is known that

$$p(z) \in P \Leftrightarrow p(z) = \frac{1 + \upsilon(z)}{1 - \upsilon(z)},$$

where $\upsilon(z)$ is a Schwarz function.
Let H denotes the class of Schwarz functions $\upsilon(z)$ which are analytic in E given by

$$\upsilon(z) = \sum_{k=1}^{\infty} b_k z^k,\ z \in E$$

and satisfying $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$. We assume that $g_1(z)$ and $g_2(z)$ are two analytic functions in E, and the symbol $\prec$ is a subordination. We say that the function $g_1(z)$ is subordinate to another function $g_2(z)$, denoted $g_1(z) \prec g_2(z)$, if there exists a Schwarz function $\upsilon(z) \in H$ such that $g_1(z) = g_2(\upsilon(z))$ for all $z \in E$. Furthermore, if $g_1(z)$ is univalent in E, then we have the following equivalence:

$$g_1(z) \prec g_2(z) \Leftrightarrow g_1(0) = g_2(0)$$

and

$$g_1(E) = g_2(E).$$

The topic concerning Taylor coefficients in geometric function theory has stimulated more research into the Hankel and Toeplitz determinants for numerous classes of univalent functions. Because the upper bounds of both determinants for the classes of univalent functions are unknown in general and therefore remain an open problem. There is a close relationship between Toeplitz determinants and Hankel determinants. Constant entries are found along the diagonal of Toeplitz matrices, and along the reverse diagonal of Hankel matrices. The Hankel determinant $H_{q,n}(f)$ and Toeplitz determinant $T_{q,n}(f)$, $n, q \geq 1$ whose elements are Taylor coefficients $a_n, n \geq 2$ of a function $f(z) \in S$ are defined, respectively, by Pommerenke [15, 16] and Thomas and Halim [23, 24] as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q+2} \end{vmatrix}, a_1 = 1 \tag{7}$$

and

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}. \tag{8}$$

The Hankel determinant is a valuable tool in the study of singularities. This is especially essential when investigating power series with integral coefficients [4, 5]. Meanwhile, the Toeplitz determinant has several applications in mathematics, both pure and applied. They appear in algebra, signal processing, partial differential equations, and time series analysis. [28] provides a good description of the applications of Toeplitz matrices across a wide spectrum of pure and applied mathematics.

Furthermore, a recent study has focused on the Hankel and Toeplitz determinants, which involve the use of logarithmic coefficients, but in the direction of inverse functions for some classes of univalent functions, for instance, [2, 11, 12, 18] may provide further insight into this. The idea was that the classic concept of Hankel and Toeplitz determinants is generalized by replacing the entries with the logarithmic coefficients of inverse functions belonging to the classes of univalent functions. The Hankel determinant $H_{q,n}\left(\Gamma_{f^{-1}}\right)$ and Toeplitz determinant $T_{q,n}\left(\Gamma_{f^{-1}}\right)$, $n, q \geq 1$ whose elements are logarithmic coefficients of inverse functions belonging to the class S are defined, respectively, as follows [2, 11, 12, 18]:

$$H_{q,n}\left(\Gamma_{f^{-1}}\right) = \begin{vmatrix} \Gamma_n & \Gamma_{n+1} & \cdots & \Gamma_{n+q-1} \\ \Gamma_{n+1} & \Gamma_{n+2} & \cdots & \Gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \Gamma_{n+q-1} & \Gamma_{n+q} & \cdots & \Gamma_{n+2q-2} \end{vmatrix} \tag{9}$$

and

$$T_{q,n}\left(\Gamma_{f^{-1}}\right) = \begin{vmatrix} \Gamma_n & \Gamma_{n+1} & \cdots & \Gamma_{n+q-1} \\ \Gamma_{n+1} & \Gamma_n & \cdots & \Gamma_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \Gamma_{n+q-1} & \Gamma_{n+q-2} & \cdots & \Gamma_n \end{vmatrix}. \tag{10}$$

The logarithmic coefficients of inverse functions Γ_n are defined in the series form of

$$\log \frac{f^{-1}(w)}{w} = 2\sum_{n=1}^{\infty} \Gamma_n w^n,\ |w| < \frac{1}{4},$$

where particularly

$$\Gamma_1 = -\frac{1}{2}a_2, \tag{11}$$

$$\Gamma_2 = -\frac{1}{2}\left(a_3 - \frac{3}{2}{a_2}^2\right), \tag{12}$$

$$\Gamma_3 = -\frac{1}{2}\left(a_4 - 4a_2a_3 + \frac{10}{3}{a_2}^3\right), \tag{13}$$

and

$$\Gamma_4 = -\frac{1}{2}\left(a_5 - 5a_2a_4 + 15{a_2}^2a_3 - \frac{5}{2}{a_3}^2 - \frac{35}{4}{a_2}^4\right). \tag{14}$$

We now introduce the subclass of starlike functions with respect to symmetric conjugate points associated with the exponential function as follows:

Definition 1. *Let ${S_{SC}}^*(e^z)$ be the class of functions defined by*

$$\frac{zf'(z)}{h(z)} \prec \phi(z), z \in E,$$

where $\phi(z) = e^z$, is an analytic univalent function and $h(z) = \frac{f(z) - \overline{f(-\overline{z})}}{2}$.

Remark 1. *Changing the function $\phi(z)$ in Definition 1 gives us more subclasses of starlike functions with respect to symmetric conjugate points:*

(i) For $\phi(z) = \frac{1+z}{1-z}$, which has been introduced and studied in [8].

(ii) For $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, which has been introduced and studied in [14].

(iii) For $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$ and considering the tilted factor $e^{i\alpha}$, $|\alpha| < \frac{\pi}{2}$, which has been defined and investigated in [25].

(iv) For $\phi(z) = 1 + \sin z$, which has been defined and studied in [26].

It is observed that there have been few studies on Hankel and Toeplitz determinants, whose entries are logarithmic coefficients of inverse functions for the subclass of univalent functions, particularly starlike functions with respect to other points, i.e., symmetric points, conjugate points, and symmetric conjugate points. We can refer the reader to [8, 9], who were among the early researchers who investigated these subclasses. Some researchers, including [14, 20, 21, 25–27], and references therein, have also carried out comprehensive studies related to these subclasses, which may provide diverse insights.

Thus, inspired by the ideas of [11, 12, 18], in this paper, we aim to estimate the upper bounds of the initial Taylor coefficients $|a_n|, n = 2,3,4,5$, logarithmic coefficients of inverse functions $|\Gamma_n|, n = 1,2,3,4$, and the second order Hankel and Toeplitz determinants whose entries are logarithmic coefficients of inverse functions belonging to the new subclass ${S_{SC}}^*(e^z)$, i.e., $\left|H_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$, $\left|H_{2,2}\left(\Gamma_{f^{-1}}\right)\right|$, $\left|T_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$, and $\left|T_{2,2}\left(\Gamma_{f^{-1}}\right)\right|$.

2. Preliminary results

In this section, we present certain lemmas that are essential to verify our main findings.

Lemma 1. ([6]) *For a function $p(z) \in P$ of the form* (6)*, the sharp inequality $|p_n| \leqslant 2$ holds for each $n \geqslant 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2. ([7]) *Let $p(z) \in P$ be a function of the form* (6) *and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\}, \; 1 \leqslant k \leqslant n-1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 3. ([10]) *Let $p(z) \in P$ be a function of the form* (6) *and $\alpha, \beta, \gamma \in \Re$. Then*

$$\left|\alpha {p_1}^3 - \beta p_1 p_2 + \gamma p_3\right| \leqslant 2\,|\alpha| + 2\,|\beta - 2\alpha| + 2\,|\alpha - \beta + \gamma|\,.$$

3. Main results

This section is devoted to the proof of our main results. We will now determine the coefficient estimates for functions belonging to ${S_{SC}}^*(e^z)$, followed by logarithmic coefficients of inverse functions and the second Hankel and Toeplitz determinants of logarithmic coefficients of inverse functions for the new subclass ${S_{SC}}^*(e^z)$, as follows:

3.1. Coefficient estimates

Theorem 1. *Let $f(z) \in {S_{SC}}^*(e^z)$. Then*

$$|a_2| \leq \frac{1}{2},$$

$$|a_3| \leq \frac{1}{2},$$

$$|a_4| \leq \frac{25}{96},$$

and

$$|a_5| \leq \frac{7}{24}.$$

Proof. If $f(z) \in {S_{SC}}^*(e^z)$ and is the form of (1), then according to subordination relationship, there exists a Schwarz function $\upsilon(z)$ such that

$$\frac{zf'(z)}{h(z)} = e^{\upsilon(z)}, \tag{15}$$

where $h(z) = \frac{f(z) - \overline{f(\overline{-z})}}{2}$.

Define a function

$$p(z)=\frac{1+\upsilon(z)}{1-\upsilon(z)}=1+\sum_{n=1}^{\infty}p_nz^n\in P.$$

This leads to

$$\upsilon(z)=\frac{p(z)-1}{p(z)+1}.$$

Hence, from the right-hand side of (15), we obtain

$$\begin{aligned}e^{\upsilon(z)}&=1+\tfrac{1}{2}p_1z+\left(\tfrac{p_2}{2}-\tfrac{{p_1}^2}{8}\right)z^2+\left(\tfrac{p_3}{2}-\tfrac{p_1p_2}{4}+\tfrac{{p_1}^3}{48}\right)z^3\\&+\left(\tfrac{p_4}{2}-\tfrac{p_1p_3}{4}-\tfrac{{p_2}^2}{8}+\tfrac{{p_1}^2p_2}{16}+\tfrac{{p_1}^4}{384}\right)z^4+\cdots.\end{aligned}$$

On the other hand, since $f(z)$ is in the form of (1), this gives

$$zf'(z)=z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots$$

and

$$h(z)=z+a_3z^3+a_5z^5+\cdots.$$

Further, we have from (15) that

$$\begin{aligned}&z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots\\&=\left(z+a_3z^3+a_5z^5+\cdots\right)\left[\begin{array}{l}1+\tfrac{1}{2}p_1z+\left(\tfrac{p_2}{2}-\tfrac{{p_1}^2}{8}\right)z^2+\left(\tfrac{p_3}{2}-\tfrac{p_1p_2}{4}+\tfrac{{p_1}^3}{48}\right)z^3\\+\left(\tfrac{p_4}{2}-\tfrac{p_1p_3}{4}-\tfrac{{p_2}^2}{8}+\tfrac{{p_1}^2p_2}{16}+\tfrac{{p_1}^4}{384}\right)z^4+\cdots\end{array}\right].\end{aligned}\tag{16}$$

Now, equating the coefficients of $z^n, n=1,2,3,4,$ on both sides of (16) yields

$$a_2=\frac{p_1}{4},\tag{17}$$

$$a_3=\frac{1}{16}\left(4p_2-{p_1}^2\right),\tag{18}$$

$$a_4=\frac{1}{6144}\left(768p_3-192p_1p_2-16{p_1}^3\right),\tag{19}$$

and

$$a_5=\frac{1}{384}\left({p_1}^4-24p_1p_3+48p_4\right).\tag{20}$$

Using Lemma 1 in (17), we get

$$|a_2|\le\frac{1}{2}.$$

Applying Lemma 2 in (18) and Lemma 3 in (19), respectively, implies

$$|a_3|=\frac{1}{16}\left|4p_2-{p_1}^2\right|\le\frac{1}{4}\left[2\max\left\{1,\left|2\left(\frac{1}{4}\right)-1\right|\right\}\right]=\frac{1}{2}$$

and

$$\begin{aligned} |a_4| &= \tfrac{1}{6144} \left|-\left(16{p_1}^3 - (-192p_1p_2) + (-768p_3)\right)\right| \\ &\leq \tfrac{1}{6144} \left[2\,|16| + 2\,|-192 - 2\,(16)| + 2\,|16 - (-192) + (-768)|\right] \\ &= \tfrac{25}{96}. \end{aligned}$$

Rearranging the terms and taking modulus on both sides of (20), we can rewrite it as

$$|a_5| = \frac{1}{384}\left|48\left(p_4 - \nu p_1 p_3\right) + {p_1}^4\right|,$$

where $\nu = \frac{1}{2}$.
Consequently, by applying Lemma 1 and Lemma 2 as well as the triangle inequality, we obtain

$$|a_5| \leq \frac{7}{24}.$$

This completes the proof of Theorem 1.

3.2. Logarithmic coefficients of inverse functions for $\mathbf{S_{SC}}^*\left(\mathbf{e^z}\right)$

Theorem 2. *Let $f(z) \in {S_{SC}}^*\left(e^z\right)$. Then*

$$|\Gamma_1| \leq \frac{1}{4},$$

$$|\Gamma_2| \leq \frac{1}{4},$$

$$|\Gamma_3| \leq \frac{41}{192},$$

and

$$|\Gamma_4| \leq \frac{197}{256}.$$

Proof. Putting (17)-(20) in (11)-(14), we obtain

$$\Gamma_1 = -\frac{p_1}{8}, \tag{21}$$

$$\Gamma_2 = -\frac{1}{64}\left(8p_2 - 5{p_1}^2\right), \tag{22}$$

$$\Gamma_3 = -\frac{1}{768}\left(43{p_1}^3 - 108p_1p_2 + 48p_3\right), \tag{23}$$

and

$$\Gamma_4 = -\frac{1}{256}\left(-\frac{99}{8}{p_1}^4 - 28p_1p_3 + 16p_4 + 45{p_1}^2p_2 - 20{p_2}^2\right). \tag{24}$$

The upper bounds of $|\Gamma_1|$, $|\Gamma_2|$, and $|\Gamma_3|$ follow from applying Lemma 1, Lemma 2, and Lemma 3, respectively.

On the other hand, we write (24) as

$$|\Gamma_4| = \frac{1}{256}\left|p_1\left(\alpha p_1{}^3 - \beta p_1 p_2 + \gamma p_3\right) + 16\left(p_4 - \mu p_2{}^2\right)\right|, \tag{25}$$

where $\alpha = \frac{99}{8}, \beta = 45, \gamma = 28$, and $\mu = \frac{5}{4}$.
Hence, implementing Lemma 2 and Lemma 3, we get the desired bound of $|\Gamma_4|$. This completes the proof of Theorem 2.

3.3. Hankel determinant of logarithmic coefficients of inverse functions for $\mathbf{S_{SC}}^*\left(\mathbf{e^z}\right)$

Theorem 3. *Let $f(z) \in S_{SC}{}^*\left(e^z\right)$. Then*

$$\left|H_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{95}{768}.$$

Proof. Using (21)-(23), we can establish

$$\begin{aligned} H_{2,1}\left(\Gamma_{f^{-1}}\right) &= \Gamma_1\Gamma_3 - \Gamma_2{}^2 \\ &= \tfrac{p_1}{4096}\left(\tfrac{86}{3}p_1{}^3 - 72p_1p_2 + 32p_3\right) - \tfrac{1}{4096}\left(64p_2{}^2 - 80p_1{}^2 + 25p_1{}^4\right) \\ &= -\tfrac{1}{4096}\left(-8p_1{}^2p_2 - \tfrac{11}{3}p_1{}^4 - 32p_1p_3 + 64p_2{}^2\right). \end{aligned} \tag{26}$$

Taking modulus and rearranging the terms in (26), it becomes

$$\left|H_{2,1}\left(\Gamma_{f^{-1}}\right)\right| = \frac{1}{4096}\left|-p_1\left(\chi p_1{}^3 - \lambda p_1p_2 + \eta p_3\right) + 64p_2{}^2\right|, \tag{27}$$

where $\chi = \frac{11}{3}, \lambda = -8$, and $\eta = 32$.
By Lemma 3, we get

$$\left|\chi p_1{}^3 - \lambda p_1p_2 + \eta p_3\right| \leq 2\left|\frac{11}{3}\right| + 2\left|-8 - 2\left(\frac{11}{3}\right)\right| + 2\left|\frac{11}{3} - (-8) + 32\right| = \frac{376}{3}.$$

Thus, from (27), in view of the triangle inequality as well as Lemma 1, we get the desired inequality. This completes the proof of Theorem 3.

Theorem 4. *Let $f(z) \in S_{SC}{}^*\left(e^z\right)$. Then*

$$\left|H_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{7691}{36864}.$$

Proof. In view of (22)-(24), we obtain

$$\begin{aligned} H_{2,2}\left(\Gamma_{f^{-1}}\right) &= \Gamma_2\Gamma_4 - \Gamma_3{}^2 \\ &= \tfrac{1}{131072}\left(\begin{array}{l} -2592p_1{}^4p_2 - 1792p_1p_2p_3 + 1024p_2p_4 + 3680p_1{}^2p_2{}^2 \\ -1280p_2{}^3 + 495p_1{}^6 + 1120p_1{}^3p_3 - 640p_1{}^2p_4 \end{array}\right) \\ &\quad - \tfrac{1}{589824}\left(\begin{array}{l} 11664p_1{}^2p_2{}^2 - 9288p_1{}^4p_2 - 10368p_1p_2p_3 \\ +1849p_1{}^6 + 4128p_1{}^3p_3 + 2304p_3{}^2 \end{array}\right) \\ &= \tfrac{1}{1179648}\left(\begin{array}{l} -4752p_1{}^4p_2 + 4608p_1p_2p_3 + 9216p_2p_4 + 9792p_1{}^2p_2{}^2 \\ -11520p_2{}^3 + 757p_1{}^6 + 1824p_1{}^3p_3 - 5760p_1{}^2p_4 - 4608p_3{}^2 \end{array}\right). \end{aligned} \tag{28}$$

Further, we can write (28) in the following expression:

$$\left|H_{2,2}\left(\Gamma_{f^{-1}}\right)\right| = \frac{1}{1179648}\left|\begin{array}{l} p_1{}^3\left(757p_1{}^3 - 4752p_1p_2 + 1824p_3\right) - 4608p_3\left(p_3 - p_1p_2\right) \\ +9216p_4\left(p_2 - \frac{5}{8}p_1{}^2\right) - 11520p_2{}^2\left(p_2 - \frac{17}{20}p_1{}^2\right) \end{array}\right|. \tag{29}$$

Hence, by Lemma 2 and Lemma 3, we obtain that

$$\left|757p_1{}^3 - 4752p_1p_2 + 1824p_3\right| \leq 12332,$$

$$\left|p_3 - p_1p_2\right| \leq 2,$$

$$\left|p_2 - \frac{5}{8}p_1{}^2\right| \leq 2,$$

and

$$\left|p_2 - \frac{17}{20}p_1{}^2\right| \leq 2.$$

Thus, from (29), making use of Lemma 1 and the triangle inequality yields the desired bound. This completes the proof of Theorem 4.

3.4. Toeplitz determinant of logarithmic coefficients of inverse functions for $\mathbf{S_{SC}}^*\left(\mathbf{e^z}\right)$

Theorem 5. *Let $f(z) \in S_{SC}{}^*\left(e^z\right)$. Then*

$$\left|T_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{9}{32}.$$

Proof. It follows from (21) and (22) that

$$\begin{aligned} T_{2,1}\left(\Gamma_{f^{-1}}\right) &= \Gamma_1{}^2 - \Gamma_2{}^2 \\ &= \tfrac{1}{4096}\left(64p_1{}^2 - 64p_2{}^2 + 80p_1{}^2p_2 - 25p_1{}^4\right). \end{aligned} \tag{30}$$

According to Lemma 2, we write

$$\left|T_{2,1}\left(\Gamma_{f^{-1}}\right)\right| = \frac{1}{4096}\left|64p_1{}^2 - 64p_2{}^2 + 80p_1{}^2\left(p_2 - \frac{5}{16}p_1{}^2\right)\right|. \tag{31}$$

From (31), we find that

$$\left|p_2 - \frac{5}{16}p_1{}^2\right| \leq 2\max\left\{1, \left|2\left(\frac{5}{16}\right) - 1\right|\right\} = 2.$$

Hence, applying Lemma 1 and triangle inequality implies

$$\left|T_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{9}{32}.$$

This completes the proof of Theorem 5.

Theorem 6. *Let* $f(z) \in S_{SC}{}^{*}(e^{z})$. *Then*

$$\left|T_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{7165}{9216}.$$

Proof. Making use of (22) and (23), and after some calculations and simplifications, we obtain

$$\begin{aligned} T_{2,2}\left(\Gamma_{f^{-1}}\right) &= {\Gamma_2}^2 - {\Gamma_3}^2 \\ &= \tfrac{1}{4096}\begin{pmatrix} 64{p_2}^2 - 80{p_1}^2 p_2 - \frac{86}{3}{p_1}^3 p_3 + 72 p_1 p_2 p_3 - 16{p_3}^2 \\ -81{p_1}^2{p_2}^2 + \frac{129}{2}{p_1}^4 p_2 + 25{p_1}^4 - \frac{1849}{144}{p_1}^6 \end{pmatrix}. \end{aligned} \tag{32}$$

Considering (32) can be expressed as

$$\left|T_{2,2}\left(\Gamma_{f^{-1}}\right)\right| = \frac{1}{4096}\left|\begin{array}{c} 64p_2\left(p_2 - \frac{5}{4}{p_1}^2\right) + p_3\left(-\frac{86}{3}{p_1}^3 + 72p_1p_2 - 16p_3\right) \\ -81{p_1}^2 p_2\left(p_2 - \frac{43}{54}{p_1}^2\right) + 25{p_1}^4 - \frac{1849}{144}{p_1}^6 \end{array}\right|. \tag{33}$$

Applying Lemma 2 and Lemma 3, from (33), we find that

$$\left|p_2 - \frac{5}{4}{p_1}^2\right| \leq 2\max\left\{1, \left|2\left(\frac{5}{4}\right) - 1\right|\right\} = 3,$$

$$\left|p_2 - \frac{43}{54}{p_1}^2\right| \leq 2\max\left\{1, \left|2\left(\frac{43}{54}\right) - 1\right|\right\} = 2,$$

and

$$\left|-\frac{86}{3}{p_1}^3 + 72p_1p_2 - 16p_3\right| \leq 2\left|-\frac{86}{3}\right| + 2\left|-72 - 2\left(-\frac{86}{3}\right)\right| + 2\left|-\frac{86}{3} - (-72) + (-16)\right| = \frac{424}{3}.$$

Hence, applying Lemma 1 and in view of the triangle inequality, (33) implies

$$\left|T_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{7165}{9216}.$$

This completes the proof of Theorem 6.

4. Conclusion

Recent studies have provided strong motivation to find the upper bounds related to the Hankel and Toeplitz determinants whose entries are logarithmic coefficients of inverse functions for a new subclass $S_{SC}{}^{*}(e^{z})$. This paper specifically presents $\left|H_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$, $\left|H_{2,2}\left(\Gamma_{f^{-1}}\right)\right|$, $\left|T_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$, and $\left|T_{2,2}\left(\Gamma_{f^{-1}}\right)\right|$ which also include estimates on initial Taylor coefficients $|a_n|$, $n = 2, 3, 4, 5$ and logarithmic coefficients of inverse functions $|\Gamma_n|$, $n = 1, 2, 3, 4$, which extends the existing knowledge in the field of geometric function theory. It appears that we may determine the upper bounds associated with the coefficient problems by using the lemma from the preliminary section. The obtained results of this study will prompt readers to further investigate other properties, such as Fekete-Szegö functional [25], Zalcman inequality [11, 13, 19], as well as the higher-order Hankel and Toeplitz determinants [1, 3, 10, 17, 22]. Moreover, further research that may help to understand more properties of the inverse functions for other subclasses of starlike functions with respect to other points (symmetric points, conjugate points, and symmetric conjugate points) by considering different analytic univalent functions $\phi(z)$ (trigonometric function, exponential function, hyperbolic function) could also be done.

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