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On Minimal Geodetic Hop Domination In Graphs

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Abstract. Let G be a nontrivial connected graph with vertex set $V(G)$ and the edge set $E(G)$. A set $S \subseteq V(G)$ is a geodetic hop dominating set of G if the following two conditions hold for each $x \in V(G) \setminus S$: (1) x lies in some u-v geodesic in G with $u, v \in S$, and (2) x is of distance 2 from a vertex in S. The minimum cardinality $\gamma_{hg}(G)$ of a geodetic hop dominating set of G is the geodetic hop domination number of G.

A geodetic hop dominating set S is a minimal geodetic hop dominating set if S does not contain a proper subset that is itself a geodetic hop dominating set. The maximum cardinality of a minimal geodetic hop dominating set in G is the *upper geodetic hop domination number* of G, and is denoted by $\gamma_{hg}^{+}(G)$. This paper initiates the study of the minimal geodetic hop dominating set and the corresponding upper geodetic hop domination number of nontrivial connected graphs. Interestingly, every pair of positive integers a and b with $2 \leq a \leq b$ is realizable as the geodetic domination number and the upper geodetic hop domination number, respectively, of some graph. Furthermore, this paper investigates the concept in the join, corona and lexicographic product of graphs.

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1. Introduction

In 2021, D. Anusha et al. [17] introduced the concept of geodetic hop domination in graphs, and initially investigated the concept in the complementary prism of graphs. Further investigation of the concept was done by D. Anusha [3] in 2022, by C.J. Saromines and S.R. Canoy Jr. [26, 27] in 2023, and just recently by D. Anusha et al. [4].

This present paper introduces and initiates the study of the concept of minimal geodetic hop domination, a natural variation of the geodetic hop domination.

Throughout this paper, all graphs considered are simple and undirected. Common graph terminologies and notations used here are adapted from [6, 11].

Given two graphs G and H with disjoint vertex sets, the *join* of G and H is the graph $G+H$ with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking $|V(G)|$ copies of H in every vertex of $V(G)$ and then joining the *ith* vertex of $V(G)$ to every vertex of the *ith* copy of H.

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The *lexicographic product* of G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(w, z) \in E(G[H])$ if and only if $uw \in E(G)$ or $u = w$ and $vz \in E(H)$.

Let G be a connected graph. For any two distinct $u, v \in E(G)$, a shortest path joining u and v is called a u-v geodesic. The length of a u-v geodesic is called the *distance* between u and v, which is denoted by $d_G(u, v)$. The *diameter* of the graph G, denoted $diam(G)$, is the maximum distance between any pair of vertices in G. If $diam(G) = 1$, then G is a *complete* graph. A *clique* of a graph G is any complete subgraph of G. The *clique number* of G is the maximum cardinality $\omega(G)$ of $S \subseteq V(G)$ for which $\langle S \rangle$ is a clique.

By the *(open) neighborhood* $N_G(v)$ of a vertex v is meant the set of vertices that are adjacent to v, that is, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of vertex v is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$, while $N_G[S] = N_G(S) \cup S$. A vertex v is an *extreme vertex* if the induced subgraph $\langle N_G(v) \rangle$ is a complete graph. The symbol $Ext(G)$ denotes the set of all extreme vertices in G.

For $u, v \in V(G)$, the set $I_G(u, v)$ refers to the set consisting of all the vertices lying in any $u - v$ geodesic of G and $I_G[u, v] = I_G(u, v) \cup \{u, v\}$. For a subset $S \subseteq V(G)$, the *geodetic closure* $I_G[S]$ is defined by $I_G[S] = \cup\{I_G[u, v] : u, v \in S\}$. A *geodetic set* is any set $S \subseteq V(G)$ with $I_G[S] = V(G)$. The minimum cardinality $g(G)$ of a geodetic set is the *geodetic number* of G. A geodetic set of cardinality $g(G)$ is called a geodetic basis. A geodetic set S in G is a *minimal geodetic set* if S does not have a proper subset that is itself a geodetic set in G. The maximum cardinality of a minimal geodetic set in G is denoted by $g^+(G)$. Geodetic sets and geodetic numbers of graphs are, in fact, among the very well-studied concepts in graph theory (see [1, 2, 7, 8, 10, 12, 15, 16, 18, 19, 28].

For $S \subseteq V(G)$, the 2-*path closure* of S, denoted by $P_2[S]_G$, is the set

$$P_2[S]_G = S \cup \{w \in V(G) : w \in I_G[u, v] \ni u, v \in S \text{ with } d_G(u, v) = 2\}$$

A set S is called 2-*path closure absorbing* if $P_2[S]_G = V(G)$. The minimum cardinality of a 2-path closure absorbing set in G is denoted by $\rho_2(G)$. A 2-path closure absorbing set S is a *minimal* 2-*path closure absorbing set* if S does not contain a proper subset that is itself a 2-path closure absorbing. The maximum cardinality of a minimal 2-path closure absorbing set in G is denoted by $\rho_2^+(G)$. The concept of 2-path closure absorbing sets was introduce in [15, 18, 19].

The *(open) hop neighborhood* of a vertex v refers to the set $N_G^2(v) = \{u \in V(G) : d_G(u, v) = 2\}$. The *closed hop neighborhood* of a vertex v is $N_G^2[v] = N_G^2(v) \cup \{v\}$. For $S \subseteq V(G)$, the *(open) hop neighborhood* S is the set $N_G^2(S) = \cup_{v \in S} N_G(v)$. The *closed hop neighborhood* of the set S is $N_G^2[S] = N_G^2(S) \cup S$.

A set $S \subseteq V(G)$ is called a *hop dominating set* if $N_G^2[S] = V(G)$. The *hop domination number* of G, denoted by $\gamma_h(G)$, is the minimum cardinality among all hop dominating sets in G. A hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-*set* of G. The authors refer to [4, 5, 9, 13, 14, 20–25] for the definitions and results on hop domination which are essential in this study.

A subset $S \subseteq V(G)$ is a *geodetic hop dominating set* if S is both a geodetic and a hop dominating set of G. The *geodetic hop domination number* $\gamma_{hg}(G)$ of G is the minimum cardinality among all geodetic hop dominating sets in G. Any geodetic hop dominating set of G with cardinality $\gamma_{hg}(G)$ is called a γ_{hg}-*set*.

2. Main Results

2.1. The minimal geodetic hop domination in graphs

Let G be a graph. A geodetic hop dominating set S is a ***minimal geodetic hop dominating set*** if S does not contain a proper subset that is itself a geodetic hop dominating set. The maximum cardinality of a minimal geodetic hop dominating set of G is the ***upper geodetic hop domination number*** of G denoted by $\gamma^+_{hg}(G)$. A minimal geodetic hop dominating set with cardinality $\gamma^+_{hg}(G)$ is called a γ^+_{hg}***-set***.

Every dominating vertex and every extreme vertex in G is included in any minimal geodetic hop dominating set of G.

Proposition 1. *Let G be a nontrivial connected graph of order n. Then*

$$2 \leq \gamma_{hg}(G) \leq \gamma^+_{hg}(G) \leq n$$

In particular, $\gamma^+_{hg}(G) < n$ if G does not have any dominating vertex.

Proof. Since a γ_{hg}-set is a minimal geodetic hop dominating set, $\gamma_{hg}(G) \leq \gamma^+_{hg}(G)$. Suppose that $\gamma^+_{hg}(G) = n$ and G has no dominating vertex . Since G is not a complete graph, G contains a geodesic of the form $[u, w, v]$. Since $w \in I_G(u, v)$, $S = V(G) \setminus \{w\}$ is a geodetic set of G. Since w is a non-dominating vertex, there exists $z \in V(G)$ such that $d_G(w, z) = 2$. Because $z \in S$, S is a hop dominating set of G. Thus, S is a geodetic hop dominating set of G. Since $V(G)$ is a minimal geodetic hop dominating set, this is impossible. □

The contrapositive of the second statement in Proposition 1 gives the following:

Corollary 1. *If $\gamma^+_{hg}(G) = n$ for a nontrivial connected graph, then G contains at least one dominating vertex.*

Theorem 1. *Let G be a nontrivial connected graph of order n. Then $\gamma^+_{hg}(G) = n$ if and only if one of the following holds:*

(i.) $G = K_n$

(ii.) $G \neq K_n$ *and $V(G) \setminus S$ induces a disconnected graph of complete components where $S \subsetneq V(G)$ is the set of all dominating vertices of G.*

Proof. Clearly, if $G = K_n$, then $\gamma^+_{hg}(G) = n$. Suppose $G \neq K_n$ with a nonempty set S of dominating vertices of G and each of the components of $\langle V(G) \setminus S\rangle$ is complete. Note first that $V(G)$ is a geodetic hop dominating set of G. Suppose $T \subseteq V(G)$ is a geodetic hop dominating set of G. By the above remark, $S \subseteq T$. Let C be a component of $\langle V(G) \setminus S\rangle$. We claim that $V(C) \subseteq Ext(G)$. Let $x \in V(C)$, and let $u, v \in N_G(x)$ with $u \neq v$. Then $u, v \in S \cup V(C)$. If $u, v \in V(C)$, then since C is complete, $uv \in E(G)$. If $u \in S$ or $v \in S$, then $uv \in E(G)$. Accordingly, $x \in Ext(G)$. Since x is arbitrary, $V(C) \subseteq Ext(G)$. Thus, $V(C) \subseteq T$. Since C is arbitrary, $V(G) \setminus S \subseteq T$. Therefore, $T = V(G)$. This means G does not have a proper subset that is itself a geodetic hop dominating set. In other words, $V(G)$ is a minimal geodetic hop dominating set of G. Therefore, $\gamma^+_{hg}(G) = n$.

Conversely, suppose $\gamma^+_{hg}(G) = n$. If $G = K_n$, then we are done. Now, suppose $G \neq K_n$. By Corollary 1, the set $S \subseteq V(G)$ consisting of the dominating vertices of G is nonempty. Suppose that $\langle V(G) \setminus S\rangle$ has a component C that is not complete. Then C contains a geodesic of the form $[u, w, v]$. We claim that $W = V(G) \setminus \{w\}$ is a geodetic hop dominating set of G. Since

$w \in I_G(u,v)$, W is a geodetic set of G. Since $w \notin S$, there exists $z \in V(G)$ such that $wz \notin E(G)$. Necessarily, $z \notin S$. Pick $t \in S$. Then $[w,t,z]$ is a geodesic in G. Thus, $d_G(w,z)=2$. This means that W is a geodetic hop dominating set of G. Consequently, $V(G)$ is not a minimal geodetic hop dominating set, implying that $\gamma^+_{hg}(G) < n$, a contradiction. Therefore, C is complete. Finally, suppose $\langle V(G)\setminus S\rangle = C$, and let $u \in V(C)$. Then $uv \in E(G)$ for each $v \in V(G)\setminus S = V(C)$ with $u \neq v$. Also, $uv \in E(G)$ for each (dominating) vertex $v \in S$. Hence, u is a dominating vertex of G, a contradiction. Therefore, $\langle V(G)\setminus S\rangle$ is a disconnected graph of at least two complete components.

Theorem 2. *Let G be a nontrivial graph connected graph. Then $\gamma^+_{hg}(G)=2$ if and only if G satisfies the following graphs:*

(*i*) $G = P_2$;

(*ii*) $G = C_6$;

(*iii*) *G has a geodetic set $S=\{u,v\}$ such that $d_G(u,v)=3$ and $u,v \in Ext(G)$.*

Proof. It is easy to verify that if $G=P_2$ or $G=C_6$, then $\gamma^+_{hg}(G)=2$. Suppose that G satisfies condition (*iii*). Let $w \in V(G)\setminus S$. Then w lies on a u-v geodesic. Thus, either $d_G(u,w)=2$ or $d_G(w,v)=2$. Since w is arbitrary, S is a hop dominating set of G. Therefore, S is a (minimal) geodetic hop dominating set of G. Now, $S \subseteq T$ for every geodetic hop dominating set T of G. Hence, S is a γ^+_{hg}-set of G. Therefore, $\gamma^+_{hg}(G)=2$.

Conversely, assume $\gamma^+_{hg}(G)=2$. Suppose that $G \notin \{P_2, C_6\}$. Let $S=\{u,v\}$ be a γ^+_{hg}-set of G. Since $G \neq P_2$, $uv \notin E(G)$. For each $a \in V(G)\setminus S$, a lies on a u-v geodesic. Choose $a \in V(G)\setminus S$ such that $ua \in E(G)$. Since S is a hop dominating set of G, $d_G(a,v)=2$. Necessarily, $d_G(u,v)=3$. Suppose that $u \notin Ext(G)$. Then G contains a geodesic $[x,u,y]$ containing u. Let $[u,y,z,v]$ be a u-v geodesic containing y. We consider two cases:

Case 1: Suppose $deg_G(u) \geq 3$, and let $w \in N_G(u)\setminus\{x,y\}$. If $xz \in E(G)$, then a minimal geodetic hop dominating set T of G can be constructed containing x,y,z. Clearly, in this case, $|T| \geq 3$. On the other hand, if $xz \notin E(G)$, a minimal geodetic hop dominating set T can be constructed containing x,z. Since $w \notin N_G[\{x,z\}]$, $|T| \geq 3$. Either subcase yields a contradiction.

Case 2: Suppose that $deg_G(u)=2$. If $xz \in E(G)$, then $T=\{x,y,z,v\}$ makes a minimal geodetic hop dominating set of G. Suppose $xz \notin E(G)$. Then there exists a u-v geodetic in G of the form $[u,x,w,v]$ with $w \neq z$. Consequently, $|V(G)| \geq 6$. Suppose that $|V(G)| \geq 7$. Then a minimal geodetic hop dominating set can be constructed containing x and y. In this case, $|T| \geq 3$. Finally, suppose that $|V(G)|=6$. Since $G \neq C_6$, G is obtained from C_6 by adding at least one edge to join a pair of nonadjacent vertices. In this case, a minimal geodetic hop dominating set can be constructed with $|T| \geq 3$. All possibilities yield to contradiction.

The above cases imply that $u \in Ext(G)$. Similarly, $v \in Ext(G)$ □

Proposition 2. *For the complete graph K_n, path P_n, cycle C_n and Petersen graph P, and for $k \geq 1$,*

(*i*) $\gamma^+_{hg}(K_n) = n$;

(*ii*) $$\gamma^+_{hg}(P_n) = \begin{cases} n & \text{if } n = 1,2,3 \\ 2k+1 & \text{if } n = 4k+1 \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$$

$$(iii)\ \gamma^+_{hg}(C_n) = \begin{cases} 3 & \textit{if } n = 3,4,5 \\ 2 & \textit{if } n = 6 \\ 2k+1 & \textit{if } n = 4k+3 \\ 2\left\lceil \frac{n}{4} \right\rceil & \textit{otherwise} \end{cases}$$

(iv) $\gamma^+_{hg}(P) = 6$.

Proof. Since $\gamma_{hg}(K_n) = n$, Proposition 1 yields $\gamma^+_{hg}(K_n) = n$.

Let $P_n = [x_1, x_2, \dots, x_n]$. The case where $n = 1, 2, 3$ is trivial. Suppose $n = 4k+1$, where $k \geq 1$. Since $S = \{x_1, x_2, x_5, x_6, \dots, x_n\}$ is a minimal geodetic hop dominating set, $2k+1 = |S| \leq \gamma^+_{hg}(G)$. Conversely, let S be a γ^+_{hg}-set of G. Being a geodetic set, $x_1, x_n \in S$. For every $1 \leq j \leq n-3$, S contains at most two vertices in $x_j, x_{j+1}, x_{j+2}, x_{j+3}$. Thus, $|S| \leq 2k+1$. Hence, $\gamma^+_{hg}(P_n) = 2k+1$.

For the third case, assume $n \neq 4k+1$. If $n = 4k$, then assume $S = \{x_1, x_2, x_5, x_6, \dots, x_{n-3}, x_n\}$. By the same argument provided above, we have S be a minimal geodetic hop dominating set so that S is a γ^+_{hg}-set. Hence, $|S| = \frac{4k}{2} = 2k = 2\lceil \frac{n}{4} \rceil$. If $n > 4k+1$, choose the set $S = \{x_1, x_2, x_5, x_6, \dots, x_{n-1}, x_n\}$ to be minimal geodetic hop dominating set. By the same argument, we have S to be a γ^+_{hg}-set with $|S| = 2\lceil \frac{n}{4} \rceil$. Therefore, $\gamma^+_{hg}(G) = 2\lceil \frac{n}{4} \rceil$.

Let $G = C_n = [v_1, v_2, \dots, v_n, v_1]$. The first and second case is trivial. Let $k \geq 1$. For the third case, assume $n = 4k+3$. Suppose $S = \{v_1, v_2, v_5, v_6, \dots, v_{n-2}\}$. Note that for every $v_i, v_{i+1} \notin S$ where $i \geq 3$, there exist $v_{i-1}, v_{i+2} \in S$ such that $v_i, v_{i+1} \in I_G(v_{i-1}, v_{i+2})$ with $v_i \in N^2_G(v_{i+2})$ and $v_{i+1} \in N^2_G(v_{i-1})$ so that S is both a geodetic set and hop dominating set. Thus, S is a minimal geodetic hop dominating set so that $|S| = \frac{4k}{2} + 1 = 2k+1 \leq \gamma^+_{hg}(G)$. Conversely, by the same argument provided in above, S is a minimal geodetic hop dominating set and since S is arbitrary we have $\gamma^+_{hg}(G) \leq 2k+1$. Therefore, $\gamma^+_{hg}(G) = 2k+1$. For the fourth case, assume $n < 4k+3$ with $k \geq 2$. Suppose $S = \{v_1, v_2, v_5, v_6, \dots, v_{n-3}, v_{n-2}\}$. The same argument would have S to be a minimal geodetic hop dominating set so that S is a γ^+_{hg}-set. It follows that $|S| = \frac{4k}{2} = 2k = 2\lceil \frac{n}{4} \rceil$.

Let G be the Petersen graph shown in Figure 1.

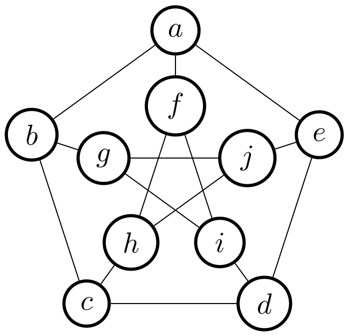

Figure 1: A petersen graph

Then $S = \{a, c, d, f, g, j\}$ is a γ^+_{hg}-set. Hence, $\gamma^+_{hg}(P) = 6$. □

Proposition 3. *Let $G = K_{m,n}$ with partite sets U and W with $|U| = m \geq 2$ and $|W| = n \geq 2$. Then $S \subseteq V(G)$ is a minimal geodetic hop dominating set of G if and only if S is one of the following:*

(i) $S = U \cup \{w\}$ *where* $w \in W$

(ii) $S = W \cup \{u\}$ *where* $u \in U$

(iii) $S = \{u, v, w, z\}$ *where* $u, v \in U$ *and* $w, z \in W$*, in case where* $m, n \geq 3$.

Proof. It is easy to verify that if S is any of the sets described in (i), (ii) and (iii), then S is a minimal geodetic hop dominating set of G.

Conversely, suppose S is a minimal geodetic hop dominating set of G. Being a hop dominating set, $S \cap U \neq \varnothing$ and $S \cap W \neq \varnothing$. Suppose that $U \subseteq S$. Note that if $w \in S \cap W$, then $U \cup \{w\}$ is a geodetic hop dominating set of G. By the minimality of S, $S = U \cup \{w\}$. Similarly, if $W \subseteq S$, then $S = W \cup \{u\}$, where $u \in U$. Now, suppose $U \setminus S \neq \varnothing$ and $W \setminus S \neq \varnothing$. Since S is geodetic, $|S \cap U| \geq 2$ and $|S \cap W| \geq 2$. Pick $u, v \in S \cap U$ and $w, z \in S \cap W$. Then, $\{u, v, w, z\}$ is a minimal geodetic hop dominating set of G. Thus, $S = \{u, v, w, z\}$.

Corollary 2. *Let* $G = K_{m,n}$ *where* $m, n \geq 2$. *Then* $\gamma^+_{hg}(G) = max\{m, n\} + 1$.

Proof. If $n = m = 2$, then $\gamma^+_{hg}(G) = 3 = 1+\max\{m, n\}$. If $n \geq 3$ or $m \geq 3$, then $\gamma^+_{hg}(G) \geq 4$. Thus, $\gamma^+_{hg}(G) = 1 + \max\{m, n\}$.

3. Realization problem

Theorem 3. *For every pair of positive integers* a *and* b *with* $2 \leq a \leq b$*, there exists a connected graph* G *such that* $\gamma_{hg}(G) = a$ *and* $\gamma^+_{hg}(G) = b$.

Proof. If $a = b$, then take $G = K_a$. For this graph G,

$$\gamma_{hg}(G) = a = b = \gamma^+_{hg}(G)$$

Assume $a < b$. Write $b = a + k$ for some $k \geq 1$. If $a = 2$, then we consider the graph $G = G_1$ in Figure 2 obtained by constructing $k + 2$ copies of P_4 with common end-vertices.

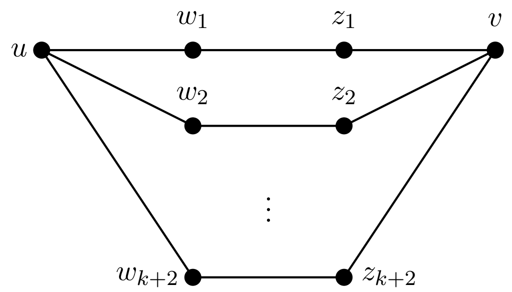

Figure 2: G_1: A connected graph complying with the specifications of Theorem 3 when $a = 2$

Let $S = \{u, v\}$ and

$$T = \begin{cases} \{w_{2i-1}, z_{2i} : i = 1, 2, 3, \ldots, \frac{k}{2} + 1\}, & k \text{ is even;} \\ \{w_1, w_{2i+1}, z_{2i} : i = 1, 2, 3, \ldots, \lfloor \frac{k+1}{2} \rfloor\}, & k \text{ is odd.} \end{cases}$$

Then S and T are γ_{hg}-set and γ^+_{hg}-set of G, respectively. Suppose that $a \geq 3$. Obtain the graph $G = G_2$ as in Figure 3 from G_1 by adding to G_1 $(a - 2)$ pendant edges vx_j, $j = 1, 2, \ldots, a - 2$.

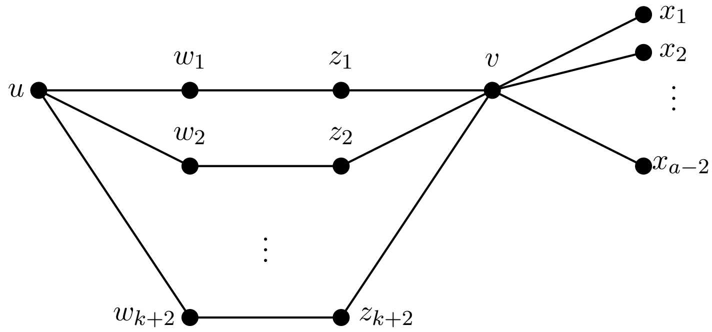

Figure 3: G_2: A connected graph complying with the specifications of Theorem 3 when $a \geq 3$

Put $S = \{u, v, x_i : i = 1, 2, \ldots, a-2\}$ and

$$T = \begin{cases} \{w_{2i-1}, z_{2i}, x_j : i = 1, 2, 3, \ldots, \frac{k}{2}+1; j = 1, 2, \ldots, a-2\}, & k \textit{ is even;} \\ \{w_1, w_{2i+1}, z_{2i}, x_j : i = 1, 2, 3, \ldots, \lfloor \frac{k+1}{2} \rfloor; j = 1, 2, \ldots, a-2\}, & k \textit{ is odd.} \end{cases}$$

Then S and T are γ_{hg}-set and γ_{hg}^+-set of G, respectively. In any case,

$$\gamma_{hg}(G) = a \quad \text{and} \quad \gamma_{hg}^+(G) = a + k = b$$

Corollary 3. *The difference between $\gamma_{hg}^+(G)$ and $\gamma_{hg}(G)$ can be made arbitrarily large.*

4. In the join of graphs

For the purposes of the remaining sections, we define the following variations of pointwise non-dominating sets.

A pointwise non-dominating set $S \subseteq V(G)$ is a ***minimal pointwise non-dominating set*** of G if S does not contain a proper subset which is itself a pointwise non-dominating set in G.

A set $S \subseteq V(G)$ is a 2***-path closure absorbing pointwise non-dominating set*** if S is both a 2-path closure absorbing set and a pointwise non-dominating set in G. A 2-path closure absorbing pointwise non-dominating set S is said to be a ***minimal*** 2***-path closure absorbing pointwise non-dominating set*** whenever S does not contain a proper subset which is itself a 2-path closure absorbing pointwise non-dominating set. We denote by $\rho_{2pnd}^+(G)$ the maximum cardinality of a minimal 2-path closure absorbing pointwise non-dominating set in G. A minimal 2-path closure absorbing pointwise non-dominating set S is called a ρ_{2pnd}^+-set if $|S| = \rho_{2pnd}^+(G)$.

Theorem 4. [26] *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a geodetic hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets of G and H, respectively, such that*

i) S_G is a 2-path closure absorbing set in G whenever $\langle S_H \rangle$ is a complete subgraph of H and

(ii) S_H is a 2-path closure absorbing set in H whenever $\langle S_G \rangle$ is a complete subgraph of G.

Proposition 4. *Let G and H be connected graphs, and let $S \subseteq G+H$. If S is a minimal geodetic hop dominating set of $G+H$, then $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are pointwise non-dominating sets of G and H, respectively, such that*

(i) S_G is a minimal 2-path closure absorbing pointwise non-dominating set in G whenever $\langle S_H \rangle$ is a complete subgraph of H;

(ii) *S_H is a minimal 2-path closure absorbing pointwise non-dominating set in H whenever $\langle S_G\rangle$ is a complete subgraph of G.*

Proof. Let S be a minimal geodetic hop dominating set of $G+H$. By Theorem 4, $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets in G and H, respectively. Suppose that $\langle S_H\rangle$ is a complete subgraph of H. By Theorem 4, S_G is a 2-path closure absorbing set in G. Let $S^*_G \subseteq V(G)$ be a 2-path closure absorbing set in G with $S^*_G \subseteq S_G$. By Theorem 4, $S^*_G \cup S_H$ is a geodetic hop dominating set of $G+H$. Since $S^*_G \cup S_H \subseteq S$, the minimality of S implies that $S_G = S^*_G$. Thus, S_G is a minimal 2-path closure absorbing pointwise non-dominating set in G and (i) holds. Similarly, if $\langle S_G\rangle$ is complete, then (ii) holds. □

Lemma 1. [1] *Let G be a connected noncomplete graph and $S \subseteq V(G)$. If S is a 2-path closure absorbing set in G, then $\langle S\rangle$ is not complete.*

Proposition 5. *Let G and H be connected graphs, and let $S \subseteq G+H$. Suppose that $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are pointwise non-dominating sets of G and H, respectively, such that one of the following holds:*

(i) *$\langle S_H\rangle$ is a complete subgraph of H and S_G is a minimal 2-path closure absorbing pointwise non-dominating set in G.*

(ii) *$\langle S_G\rangle$ is a complete subgraph of G and S_H is a minimal 2-path closure absorbing pointwise non-dominating set in H.*

Then S is a minimal geodetic hop dominating set of $G+H$.

Proof. Assume that S_G and S_H are pointwise non-dominating sets of G and H, respectively. Then S is a hop dominating set in $G+H$. Assume further that (i) holds for S. Let $x \in V(G+H) \setminus S$. Suppose that $x \in V(G) \setminus S$. Since S_G is a 2-path closure absorbing set in G, there exists $u, v \in S_G$ for which $d_G(u,v) = 2$ and $x \in I_G(u,v)$. Observe that $d_{G+H}(u,v) = d_G(u,v) = 2$ and $I_G(u,v) \subseteq I_{G+H}(u,v)$. Suppose that $x \in V(H) \setminus S$. By Lemma 1, $\langle S_G\rangle$ is noncomplete. Thus, there exists $u, v \in S_G$ such that $d_G(u,v) = 2$. Then $d_{G+H}(u,v) = 2$ and $x \in I_{G+H}(u,v)$. Since x is arbitrary, S is a geodetic set in $G+H$. Therefore, S is a geodetic hop dominating set in $G+H$.

Now, let $T \subseteq V(G+H)$ be a geodetic hop dominating set in $G+H$ with $T \subseteq S$. Write $T = T_G \cup T_H$, where $T_G = T \cap V(G)$ and $T_H = T \cap V(H)$. Then T_G and T_H are pointwise non-dominating sets in G and H, respectively, by Theorem 4. Note that $T_G \subseteq S_G$ and $T_H \subseteq S_H$. Since $\langle S_H\rangle$ is complete, $\langle T_H\rangle$ is a complete subgraph of H. If $S_H \setminus T_H \neq \varnothing$ and $x \in S_H \setminus T_H$, then $xy \in E(G+H)$ for all $y \in T_H$, a contradiction since T_H is a pointwise non-dominating set in H. Thus, $S_H = T_H$. By Theorem 4, T_G is a 2-path closure absorbing set in G. Since $T_G \subseteq S_G$, the minimality of S_G implies that $S_G = T_G$. Therefore, $S = T$ and S is a minimal geodetic hop dominating set in $G+H$.

Similarly, if condition (ii) holds, then S is a minimal geodetic hop dominating set in $G+H$. □

Corollary 4. *Let G be a nontrivial connected graphs and $p \geq 1$, and let $S \subseteq V(G+K_p)$. Then S is a minimal geodetic hop dominating set of $G+K_p$ if and only if $S = V(K_p) \cup S_G$ where $S_G \subseteq V(G)$ is a minimal 2-path closure absorbing pointwise non-dominating set of G. More precisely,*

$$\gamma^+_{hg}(G+K_p) = p + \rho^+_{2pnd}(G).$$

Proof. If G is complete, then $G+H$ is complete, and the assertion is obvious. Suppose that G is noncomplete. Put $H=K_p$, and let $S_G=S\cap V(G)$ and $S_H=S\cap V(H)=V(K_p)$. Assume S is a minimal geodetic hop dominating set in $G+K_p$. Note that $\langle S_H\rangle$ is complete and $S_H=V(K_p)$ is a pointwise non-dominating set in H. Thus, S_G is a minimal 2-path closure absorbing pointwise non-dominating set in G by Proposition 4.

Conversely, suppose that S_G is a minimal 2-path closure absorbing pointwise non-dominating set in G. By Proposition 5, $S=S_G\cup S_H$ is a minimal geodetic hop dominating set in $G+H$. $\square$

Proposition 6. *Let G and H be connected noncomplete graphs. Then*

$$\gamma^+_{hg}(G+H)\geq \max\{\rho^+_{2pnd}(G)+\omega(H),\rho^+_{2pnd}(H)+\omega(G)\}.$$

Proof. Let $S_G\subseteq V(G)$ be a ρ^+_{2pnd}-set of G and $S_H\subseteq V(H)$ be ω-set of H. Because H is noncomplete, $V(H)\neq S_H$. Let $x\in V(H)\setminus S_H$. Since $\langle S_H\cup\{x\}\rangle$ is a not complete, there exists $y\in S_H$ for which $xy\notin E(H)$. Since x is arbitrary, S_H is a pointwise non-dominating set in H. By Proposition 4, $S=S_G\cup S_H$ is a minimal geodetic hop dominating set in $G+H$. Thus,

$$\gamma^+_{hg}(G+H)\geq |S|=\rho^+_{hg}(G)+\omega(H).$$

Similarly, $\gamma^+_{hg}(G+H)\geq |S|=\rho^+_{hg}(H)+\omega(G)$. $\square$

Since $\gamma^+_{hg}(P_3+C_3)=5=\rho^+_{2pnd}(P_3)+\omega(P_3)$, the lower bound in Proposition 6 is sharp.

5. In the corona of graphs

Theorem 5. [27] *Let G and H be any two graphs. A set $S\subseteq V(G\circ H)$ is a geodetic hop dominating set of $G\circ H$ if and only if*

$$S=A\cup\left(\cup_{v\in V(G)}S_v\right),$$

where $A\subseteq V(G)$ and $S_v\subseteq V(H^v)$ for each $v\in V(G)$, and satisfies the following conditions:

(i) S_v is a pointwise non-dominating set in H^v for each $v\in V(G)\setminus N_G(A)$;

(ii) For each $w\in V(G)\setminus A$, one of the following holds:

- *(a) $\exists a,b\in S_w$ with $d_{H^w}(a,b)\neq 1$;*
- *(b) $\exists x,y\in V(G)$ with $w\in I_G(x,y)$;*
- *(c) $\exists s\in S_w$ and $t\in A$.*

(iii) S_v is a 2-path closure absorbing set in H^v for all $v\in V(G)$.

Observe that if G is a nontrivial connected graph, then condition (ii) in Theorem 5 may be removed.

Corollary 5. *Let G and H be two graphs, where G is connected and nontrivial. A set $S\subseteq V(G\circ H)$ is a geodetic hop dominating set of $G\circ H$ if and only if*

$$S=A\cup\left(\cup_{v\in V(G)}S_v\right),$$

where $A\subseteq V(G)$ and $S_v\subseteq V(H^v)$ for each $v\in V(G)$, and satisfies the following conditions:

(*i*) *S_v is a pointwise non-dominating set in H^v for each $v \in V(G) \setminus N_G(A)$;*

(*ii*) *S_v is a 2-path closure absorbing set in H^v for all $v \in V(G)$.*

Proof. The necessity part follows from Theorem 5. Suppose that (*i*) and (*ii*) hold for S. Following the same proof as one given in [9] for Theorem 5, S is a hop dominating set in $G \circ H$ and for every $w \in V(H^w) \setminus S_w$, there exist $u, v \in S$ such that $w \in I_{G+H}(u, v)$. Now, suppose that $w \in V(G) \setminus A$. Since G is nontrivial and connected, $N_G(w) \neq \varnothing$, say $z \in N_G(w)$. By (*ii*), $S_z \neq \varnothing$ and $S_w \neq \varnothing$. Pick $u \in S_z$ and $v \in S_w$. Then $w \in I_{G\circ H}(u, v)$. □

Proposition 7. *Let G and H be two graphs, where G is connected of order $n \geq 2$. Then*

$$\gamma^+_{hg}(G \circ H) \geq n \cdot \rho^+_{2pnd}(H),$$

and this bound is sharp.

Proof. For each $v \in V(G)$, let $S_v \subseteq V(H^v)$ be a ρ^+_{2pnd}-set of H^v. By Corollary 5, $\cup_{v\in V(G)} S_v$ is a geodetic hop dominating set in $G \circ H$. Let $T \subseteq V(G \circ H)$ be a geodetic hop dominating set in $G \circ H$ with $T \subseteq S$. By Corollary 5 and since $T \subseteq S$, $T = \cup_{v\in V(G)} T_v$, where $T_v \subseteq S_v$ is a 2-path closure absorbing pointwise non-dominating set in H^v for each $v \in V(G)$. By the minimality of S_v, $T_v = S_v$ for each $v \in V(G)$. Therefore, $S = T$ and S is a minimal geodetic hop dominating set in $G \circ H$. Thus,

$$\gamma^+_{hg}(G \circ H) \geq |S| = n \cdot \rho^+_{2pnd}(H).$$

Further, since $\gamma^+_{hg}(P_2 \circ P_2) = 4 = 2\rho^+_{2pnd}(P_2)$, the given bound is sharp. □

Proposition 8. *Let G and H be any two graphs, where G is connected and nontrivial. If $S \subseteq V(G \circ H)$ is a minimal geodetic hop dominating set in $G \circ H$, then*

$$S = A \cup \left(\cup_{v\in V(G)} S_v\right),$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:

(*i*) *S_v is a minimal 2-path closure absorbing pointwise non-dominating set in H^v for each $v \in V(G) \setminus N_G(A)$;*

(*ii*) *S_v is a minimal 2-path closure absorbing set in H^v for all $v \in V(G) \cap N_G(A)$.*

Proof. In view of Corollary 5, we are left to work only on the minimality part. Let S be a minimal geodetic hop dominating set in $G \circ H$. Let $v \in V(G) \setminus N_G(A)$. Let $D \subseteq V(H^v)$ be a 2-path closure absorbing pointwise non-dominating set in H^v with $D \subseteq S_v$. Then $T = A \cup \left(\cup_{u\in V(G)\setminus\{v\}} S_u\right) \cup D$ is a geodetic hop dominating set of $G \circ H$. Since $T \subseteq S$, the minimality of S implies that $T = S$. Necessarily, $S_v = D$, showing that S_v is a minimal 2-path closure absorbing pointwise non-dominating set in H^v, and (*i*) holds. Similarly, (*ii*) holds. □

Corollary 6. *Let G and H be two graphs, where G is connected of order $n \geq 2$ and $\rho^+_{2pnd}(H) = \rho^+_2(H)$. Then*

$$\gamma^+_{hg}(G \circ H) = n \cdot \rho^+_{2pnd}(H).$$

In particular, for $p \geq 1$,

$$\gamma^+_{hg}(G \circ K_p) = np.$$

Proof. Let S be a γ_{hg}^{+}-set of $G \circ H$. By Proposition 8, $S = A \cup \left(\cup_{v \in V(G)} S_v\right)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$such that S_v is a minimal 2-path closure absorbing pointwise non-dominating set in H^v for each $v \in V(G) \setminus N_G(A)$ and S_v is a minimal 2-path closure absorbing set in H^v for all $v \in V(G) \cap N_G(A)$. Since S is a γ_{hg}^{+}-set, $|S_v| = \rho_{2pnd}^{+}(H)$ for all $v \in V(G)$. Consequently, the minimality of S implies that $A = \varnothing$. Thus,

$$\gamma_{hg}^{+}(G \circ H) = |S| = |V(G)| \cdot \rho_{2pnd}^{+}(H).$$

□

6. In the lexicographic product of graphs

Theorem 6. *[27] Let G and H be connected nontrivial graphs. A subset $C = \cup_{x \in S} (\{x\} \times S_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $S_x \subseteq V(H)$, is a geodetic hop dominating set in $G[H]$ if and only if the following conditions hold:*

(i) *S is a geodetic hop dominating set in G;*

(ii) *S_x is a pointwise non-dominating set in H for each $x \in S \setminus N_G^2(S)$;*

(iii) *S_x is a 2-path closure absorbing set in H for each $x \in S \setminus I_G(S)$.*

The following follows from Theorem 6.

Corollary 7. *Let G and H be connected nontrivial graphs. Let $C = \cup_{x \in S} (\{x\} \times S_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $S_x \subseteq V(H)$, such that the following conditions hold:*

(i) *S is a minimal geodetic hop dominating set in G;*

(ii) *S_x is a minimal pointwise non-dominating set in H for each $x \in S \setminus N_G^2(S)$;*

(iii) *S_x is a minimal 2-path closure absorbing set in H for each $x \in S \setminus I_G(S)$.*

Then C is minimal geodetic hop dominating set in $G[H]$.

Proof. By Theorem 6, C is a geodetic hop dominating set in $G[H]$. Let $C^* \subseteq C$ be a geodetic hop dominating set in $G[H]$. By Theorem 6, there exist $T \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in T$ such that

$$C^* = \cup_{x \in T} (\{x\} \times T_x).$$

Moreover, T is a geodetic hop dominating set in G, T_x is a pointwise non-dominating set in H for each $x \in T \setminus N_G^2(T)$, and T_x is a 2-path closure absorbing set in H for each $x \in T \setminus I_G(T)$. Since $C^* \subseteq C$, $T \subseteq S$ and $T_x \subseteq S_x$ for each $x \in T$. The minimality of S implies that $T = S$. Consequently, the minimality of S_x implies that $S_x = T_x$ for all $x \in T \setminus N_G^2(T)$ and $S_x = T_x$ for all $x \in T \setminus I_G(T)$. Hence, $C = C^*$, showing that C is a minimal geodetic hop dominating set in $G[H]$.

Corollary 8. *For nontrivial connected graphs G and $p \geq 2$,*

(i) *$\gamma_{hg}^{+}(G[K_p]) = p \cdot \gamma_{hg}^{+}(G)$.*

(ii) *$\gamma_{hg}^{+}(K_p[G]) = p \cdot \rho_{2pnd}^{+}(H)$.*

Proof. Let $S \subseteq V(G)$ be a γ^+_{hg}-set of G, and let $D = V(K_p)$. Then $C = \cup_{x\in S}(\{x\} \times D)$ is a minimal geodetic hop dominating set in $G[K_p]$ by Corollary 7. Thus,

$$\gamma^+_{hg}(G[K_p]) \geq |C| = |S| \cdot |D| = p \cdot \gamma^+_{hg}(G).$$

To get the other inequality, let $C = \cup_{x\in S}(\{x\} \times S_x) \subseteq V(G[H])$ be a γ^+_{hg}-set of $G[K_p]$. By Theorem 6 and the minimality of C, S is a minimal geodetic hop dominating set in G and $S_x = V(K_p)$ for all $x \in S \setminus N^2_G(S)$ and for all $x \in S \setminus I_G(S)$. Thus,

$$\gamma^+_{hg}(G[K_p]) = |C| = \sum_{x\in S} |S_x| \leq \sum_{x\in S} |V(K_p)| = p \cdot |S| \leq p \cdot \gamma^+_{hg}(G).$$

This proves (i).

To prove (ii), let $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(K_p[G])$ be a γ^+_{hg}-set of $K_p[G]$. By Theorem 6, $S = V(K_p)$ and S_x is a 2-path closure absorbing pointwise non-dominating set in H. Moreover, by the minimality of C, S_x is a minimal 2-path closure absorbing pointwise non-dominating set in H. Thus,

$$\gamma^+_{hg}(K_p[G]) = |C| = \sum_{x\in V(G)} |S_x| \leq p \cdot \rho^+_{2pnd}(H).$$

To get the other inequality, $D \subseteq V(H)$ be a ρ^+_{2pnd}-set of H. Then

$$C = \cup_{v\in V(K_p)}(\{x\} \times D) = V(K_p) \times D$$

is a minimal geodetic hop dominating set in $K_p[G]$. Therefore,

$$\gamma^+_{hg}(K_p[G]) \geq |C| = p \cdot \rho^+_{2pnd}(H).$$

□

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References

[1] I.S. Aniversario and F.P. Jamil. The minimal closed geodetic numbers of graphs. *Utilitas Mathematica*, 79, 2009.

[2] I.S. Aniversario, F.P. Jamil, and S.R. Canoy. The closed geodetic numbers of graphs. *Utilitas Mathematica*, 74, 2007.

[3] D. Anusha. Geodetic hop domination number in join and corona of graphs. *Advances and Applications in Mathematical Sciences*, 21(3):1117–1127.

[4] D. Anusha, J. John, and S. Joseph Robin. Further results on the hop domination number of a graph. *Boletim Da Sociedade Paranaense de Matemática*, 42:1–12, 2024.

[5] S Ayyaswamy, Natarajan Chidambaram, and Venkatakrishnan Yanamandram. Hop domination in graphs. 2022.

[6] C. Berge. *Theory of graphs and applications.* Methuen Co. Ltd. London, John wiley & Sons Inc, New York, 1962.

[7] Boštjan Brešar, Matjaž Kovše, and Aleksandra Tepeh. *Geodetic Sets in Graphs*, pages 197–218. 1 1970.

[8] G. Cagaanan and S.R. Canoy Jr. On the geodetic covers and geodetic bases of the composition g[km]. *Ars Comb.*, 79, 2006.

[9] S.R. Canoy and G.P. Salasalan. A variant of hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 15(2):342–353, 2022.

[10] G. Chartrand, F. Harary, and P. Zhang. The geodetic number of a graph. *Networks: An International Journal*, 39(1):1–6, 2002.

[11] F. Harary. *Graph theory.* Addison-Wesley Publishing Company, 1991.

[12] F. Harary. The geodetic number of a graph. *Mathl. Comput. Modelling*, 17(11):89–95, 1993.

[13] M. Henning, S. Pal, and D. Pradhan. Algorithm and hardness results on hop domination in graphs. *Information Processing Letters*, 2019.

[14] M. Henning and N. Jafari Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics*, 33:1–15, 2017.

[15] F.P. Jamil, I.S. Aniversario, and S.R. Canoy. The closed geodetic numbers of the corona and composition of graphs. *Utilitas Mathematica*, 82, 2010.

[16] F.P. Jamil and H. Nuenay-Maglanque. On minimal geodetic domination in graphs. *Discussiones Mathematicae Graph Theory*, 2015.

[17] J. John. The geodetic hop domination number of complementary prisms. *Discrete Mathematics Algorithms and Applications*, 13(12), 2020.

[18] S.R. Canoy Jr and G. Cagaanan. On the geodesic and hull numbers of the sum of graphs. *Congressus Numerantium*, 2003.

[19] S.R. Canoy Jr., G. Cagaanan, and S. Gervacio. Convexity, geodetic, and hull numbers of the join of graphs. *Utilitas Mathematica*, 71, 2006.

[20] S.R. Canoy Jr, R. Mollejon, and J. Canoy. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 11, 2019.

[21] C. Natajaran and S.K. Ayyaswamy. Hop domination in graphs-ii. *Analele Universitatii "Ovidius" Constanta - Seria Matematica*, 2015.

[22] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-European Journal of Mathematics*, 11, 2017.

[23] G. Salasalan and S.R. Canoy Jr. Global hop domination numbers of graphs. *European Journal of Pure and Applied Mathematics*, 14:112–125, 2021.

[24] G. Salasalan and S.R. Canoy Jr. Revisiting domination, hop domination, and global hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 14:1415–1428, 2021.

[25] C.J. Saromines and S.R. Canoy. Outer-connected hop dominating sets in graphs. *European Journal of Pure and Applied Mathematics*, 15:1966–1981, 2022.

[26] C.J. Saromines and S.R. Canoy. Another look at geodetic hop domination in a graph. *European Journal of Pure and Applied Mathematics*, 2023.

[27] C.J. Saromines and S.R. Canoy. Geodetic hop dominating sets in a graph. *European Journal of Pure and Applied Mathematics*, 2023.

[28] T. Tacbobo, F.P. Jamil, and S.R. Canoy Jr. Monophonic and geodetic dominations in the join, corona and composition of graphs. *Ars Combinatoria*, 112:13–31, 2013.