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Modern Roman Dominating Functions in Graphs

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Abstract. Let $G = (V(G), E(G))$ be any connected graph. A function $f : V(G) \to \{0, 1, 2, 3\}$ is a modern Roman dominating function of G if for each $v \in V(G)$ with $f(v) = 0$, there exist $u, w \in N_G(v)$ such that $f(u) = 2$ and $f(w) = 3$; and for each $v \in V(G)$ with $f(v) = 1$, there exists $u \in N_G(v)$ such that $f(u) = 2$ or $f(w) = 3$. The weight of a modern Roman dominating function f of G is the sum $\omega_G^{mR}(f) = \sum_{v \in V(G)} f(v)$ and the minimum weight among all of the modern dominating functions on G is called the modern Roman domination number $\gamma_{mR}(G)$ of G. In this paper, we characterize graphs with smaller modern Roman domination number and obtain the $\gamma_{mR}(G)$ of some special graphs. Moreover, we investigate and characterize the modern Roman domination of the join and corona of graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Dominating set, Domination number, Modern Roman dominating function, and Modern Roman domination number.

1. Introduction

The concept of Roman domination is introduced in 2004 [6]. It is inspired by the strategies for defending the Roman Empire presented in the work of ReVelle and Rosing in [13] and Stewart, Cockayne, et al. in [15]. Since then, it has emerged as an active research field in graph theory (see [10],[8],[1],[4],[3],[12],[9],[7],[14],[11]). A new model of graph domination based on Roman domination is introduced in [8], called modern Roman domination. Studies and exploration on this variant can be found in [1, 11, 14]. Explicity, a function $f : V(G) \to \{0, 1, 2, 3\}$ is a modern Roman dominating function ($MRDF$) of G if for each $v \in V(G)$ with $f(v) = 0$, there exist $u, w \in N_G(v)$ such that $f(u) = 2$ and $f(w) = 3$; and for each $v \in V(G)$ with $f(v) = 1$, there exists $u \in N_G(v)$ such that $f(u) = 2$ or $f(u) = 3$. The minimum weight among all of the $MRDF$ is called the modern Roman

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domination number and is denoted by $\gamma_{mR}(G)$. In this model, the label of a vertex under the function f represents a type of weapon in a war zone. The four defensive weapon types are represented by the set of weights $\{0, 1, 2, 3\}$ under the function f. Weapon types are given ascending weights: light, medium, heavy, and air force. Light weapons are for pedestrians; heavy weapons can be tanks and rockets. The defense strategy of modern Roman domination relies on a support system of heavy weapons and air forces to back up the light and medium weapons. [1].

This study explores further the concept of modern Roman domination in graphs. It focuses on providing the modern Roman domination number of some specials graphs and some characterizations for the modern Roman dominating functions of the join and corona of graphs.

2. Terminology and Notation

The symbols $V(G)$ and $E(G)$ denote the *vertex set* and *edge set*, respectively, of a graph G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ and $|E(G)|$ are the *order* and *size*, respectively, of G. All graph terminologies that are not introduced but are being used here are adapted from [2].

The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u in G is the set $N_G[u] = N_G(u) \cup \{u\}$. If $S \subseteq V(G)$, the *open neighborhood* of S in G is the set $N_G(S) = \bigcup_{u \in S} N_G(u)$. The *closed neighborhood* of S in G is the set $N_G[S] = N_G(S) \cup S$. For $S \subseteq V(G)$ of a connected graph G, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$. A graph whose edge-set is empty is called an *empty graph* (also called *null graph* or *totally disconnected graph*). An empty graph of order n is denoted by $\overline{K}_n$. A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. Thus, S is a *dominating set* in G if and only if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. A dominating set S of G with $|S| = \gamma(G)$ is called a γ - set of G. Readers may refer to [5] for the introduction and more comprehensive discussion of the development of the concept of domination in graphs.

For a positive integer k, a set $D \subseteq V(G)$ is called a *k-dominating set* if each $x \in V(G) \setminus D$ is adjacent to at least k vertices in D. The *k-domination number* $\gamma_k(G)$ is then defined to be the smallest cardinality of a k-dominating set of G.

A *Roman dominating function* (RDF) on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $\omega_G(f) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight among all of the RDF on G. An RDF with $\omega_G(f) = \gamma_R(G)$ is referred to as a *γ_R-function* [3].

A function $f : V(G) \to \{0, 1, 2, 3\}$ is a *double Roman dominating function* of G, written $f \in DRD(G)$, if each of the following holds:

(1) for each $v \in V(G)$ with $f(v) = 0$ at least one of the following holds:

  (a) v has two adjacent vertices u and w for which $f(u) = f(w) = 2$; or

  (b) v has an adjacent vertex u for which $f(u) = 3$, and

(2) for each $v \in V(G)$ with $f(v) = 1, v$ is adjacent to a vertex u for which either $f(u) = 2$ or $f(u) = 3$.

The *double Roman domination number* of G denoted by $\gamma_{dR}(G)$, is the minimum weight $\omega_G(f) = \sum_{v\in V(G)} f(v)$ of all the double Roman dominating functions f of G. Any $f \in DRD(G)$ of weight equal to $\gamma_{dR}(G)$ is referred to as γ_{dR}-function of G [3].

A *modern Roman dominating function* $(MRDF)$ of G is a function $f : V(G) \to \{0, 1, 2, 3\}$ if

(P1) for each $v \in V(G)$ with $f(v) = 0$, there exist $u, w \in N_G(v)$ such that $f(u) = 2$ and $f(w) = 3$; and

(P2) for each $v \in V(G)$ with $f(v) = 1$, there exists $u \in N_G(v)$ such that $f(u) = 2$ or $f(u) = 3$.

The weight of a modern Roman dominating function f of G is the sum $\omega_G^{mR}(f) = \sum_{v\in V(G)} f(v)$ and its minimum weight among all of the modern Roman dominating function is called the *modern Roman domination number* $\gamma_{mR}(G)$ of G. A modern Roman dominating function of G with weight $\omega_G^{mR}(f) = \gamma_{mR}(G)$ is called a γ_{mR}-*function* of G [8].

For a function $f : V(G) \to \{0, 1, 2, 3\}$ on a graph G, let (V_0, V_1, V_2, V_3) be the ordered partition induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. Then we can write $f = (V_0, V_1, V_2, V_3)$. The weight of f is defined by $\omega_G(f) = |V_1| + 2|V_2| + 3|V_3|$.

Example 1. *Consider the given graph G with $V(G) = \{a, b, c, d, e, g, h\}$ in Figure 1. The function $f : V(G) \to \{0, 1, 2, 3\}$ given by*

$$f(v) = \begin{cases} 3, & \text{if } v = a. \\ 2, & \text{if } v = g. \\ 0, & \text{otherwise.} \end{cases}$$

is a modern Roman dominating function of G. It can be verified that the $\gamma_{mR}(G) = 5$.

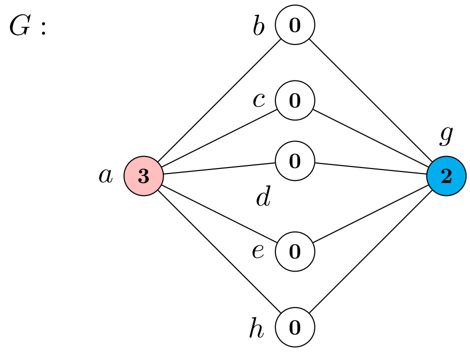

Figure 1: Graph G of order 7 with $\gamma_{mR}(G) = 5$.

3. Known Results

We make use of the following known results from [8].

Proposition 1. *Let G be a graph of order n and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{mR}-function on G. Then each of the following statements holds:*

(i) *If $n \geq 4$, then $5 \leq \gamma_{mR}(G) \leq 2n$.*

(ii) *If there are two vertices that are adjacent to all other vertices in G, then $\gamma_{mR} = 5$.*

(iii) *If G is empty graph, then $2\gamma(G) = \gamma_{mR}(G)$.*

(iv) $V_2 \neq \varnothing$

(v) *$V_2 \cup V_3$ is a dominating set of G. Moreover, it is a 2-dominating set of $G[V_0]$*

(vi) *If v is a pendant vertex, then $f(v) \neq 0$.*

(vii) *If v is an isolated vertex, then $f(v) = 2$.*

Proposition 2. *For path $P_n, n \geq 1$,*

$$\gamma_{mR}(P_n) = n + \left\lceil \frac{n}{3} \right\rceil$$

Proposition 3. *For cycle $C_n, n \geq 3$, $\gamma_{mR}(C_n) = \begin{cases} 5, & \text{if } n = 4 \\ n + \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \neq 4 \end{cases}$*

4. Main Results

This section begins with the general and useful properties of modern Roman dominating functions. It also presents the characterizations of some graphs G with $\gamma_{mR}(G) \in \{2, 3, 4, 5\}$ and the modern Roman domination number of the n-barbell graph B_n, windmill graph $Wd(k, n)$, friendship graph G_3^n, butterfly graph G_3^2, complete bipartite graph

$K_{m,n}$, star graph S_n and fan graph F_n. For simplicity, we denote by $MRDF(G)$ the set of all modern Roman dominating functions on a graph G.

Remark 1. *If $f = (V_0, V_1, V_2, V_3)$ is a γ_{mR}-function of G and $v \in V_1$, then v need not be in $N_G(V_2) \cap N_G(V_3)$.*

Proposition 4. *Let G be any graph with no isolated vertex. If $f = (V_0, V_1, V_2, V_3)$ a γ_{mR}-function of G, then the following holds:*

(i) *$V_0 = \varnothing$ if and only if $V_3 = \varnothing$ and V_2 is a γ-set of G. Moreover, $\gamma_{mR}(G) = |V(G)| + \gamma(G)$.*

(ii) *$V_1 = \varnothing$ if and only if $V_2 \cup V_3$ is a 2-dominating set of G. Moreover, if $V_1 = \varnothing$, $\langle V_2 \cup V_3 \rangle$ is connected and V_3 is a γ-set of G, then $\gamma_{mR}(G) \geq \gamma(G) + 2\gamma_2(G)$.*

Proof. Clearly, $V_0 = \varnothing$ if and only if $V_3 = \varnothing$. Suppose $V_0 = \varnothing$. Since $V_2 \cup V_3$ is a dominating set of G and $V_3 = \varnothing$, it follows that V_2 is a dominating set of G. Suppose V_2 is not a γ-set of G. Let S be a γ-set of G and define $g = (V_0', V_1', V_2', V_3')$ where $V_0' = V_3' = \varnothing, V_1' = V(G)\backslash S$, and $V_2' = S$. Then there exists $V_2^* \subseteq V(G)$ such that V_2^* is a γ-set of G. Let $V_2' = V_2^*$, $V_0' = V_3' = \varnothing$ and $V_1' = V(G)\backslash V_2^*$. Thus, $g = (V_0', V_1', V_2', V_3') \in MRDF(G)$, and so, $\omega_G^{mR}(g) < \omega_G^{mR}(f)$, a contradiction. Hence, V_2 is a γ-set of G. Furthermore, $\gamma_{mR}(G) = |V_1| + 2|V_2| = |V(G)\backslash V_2| + 2|V_2| = |V(G)\backslash V_2| + 2\gamma(G) = |V(G)| - \gamma(G) + 2\gamma(G) = |V(G)| + \gamma(G)$. This proves (i).

Now we prove (ii). Suppose $V_1 = \varnothing$. Then by Proposition 1, $V_2 \cup V_3$ is a 2-dominating set of G. Conversely, suppose that $V_1 \neq \varnothing$ and take $\{v\} \in V_1$. Then by Remark 1, v need not be in $N_G(V_2) \cap N_G(V_3)$, which is a contradiction. Hence, the assertion follows. Moreover, assume that $\langle V_2 \cup V_3 \rangle$ is connected and let V_3 be a γ-set of G. Since $V_1 = \varnothing$, $\gamma_{mR}(G) = 2|V_2| + 3|V_3| = 2|V_2 \cup V_3| + |V_3| \geq 2\gamma_2(G) + \gamma(G)$. □

Proposition 5. *Let G be a connected graph. Then*

(i) *$\gamma_{mR}(G) = 2$ if and only if $G = K_1$.*

(ii) *$\gamma_{mR}(G) = 3$ if and only if $G = K_2$.*

(iii) *$\gamma_{mR}(G) = 4$ if and only if $G \in \{K_3, P_3\}$.*

(iv) *$\gamma_{mR}(G) = 5$ if and only if $|V(G)| = 4$ and $\gamma(G) = 1$ or $\gamma_2(G) = 2$ and $|V(G)| \geq 4$.*

Proof. (i) Suppose $\gamma_{mR}(G) = 2$, say $f = (V_0, V_1, V_2, V_3)$ is a γ_{mR}-function on G. By Proposition 1(iv), $V_2 = \{v\}$. Hence, $V_0 = V_1 = V_3 = \varnothing$. The converse is clear.

(ii) Suppose $\gamma_{mR}(G) = 3$, say $f = (V_0, V_1, V_2, V_3)$ is a γ_{mR}-function on G. By (i), $|V_2| \geq 2$. By Proposition 1(iv), and the assumption that $\gamma_{mR}(G) = 3, |V_2| = 1, |V_1| = 1$ and $V_0 = V_3 = \varnothing$. Therefore, $|V(G)| = 2$. Since G is connected, $G = K_2$. Conversely, suppose that $G = K_2$, say $V(G) = \{x, y\}$. Then $g = \{\varnothing, \{x\}, \{y\}, \varnothing\} \in MRDF(G)$ and

$\omega_G^{mR}(g) = 3$. Since $\gamma_{mR}(G) \geq 2$, it follows that $\gamma_{mR}(G) = 3$.

(iii) Note that if $\gamma_{mR}(G) = 4$, then $1 \leq |V_2| \leq 2$ by Proposition 1(iv). Hence, there are only two cases to consider, namely, $|V_2| = 1$ and $|V_2| = 2$. If $|V_2| = 2$, then $|V_1| = 0$. By (P2), this cases is not possible. So if $|V_2| = 1$, we have $|V_1| = 2$. By (P2), $\langle V_1 \cup V_2 \rangle$ must be connected. Thus, the result follows. The converse follows directly from Propositions 2 and 3.

(iv) If $\gamma_{mR}(G) = 5$, then $|V_3| \leq 1$ and $1 \leq |V_2| \leq 2$. Also, by (iii), $|V(G)| \geq 4$. Now, if $|V_3| = 0$, then $|V_0| = 0$. Hence, there are only two cases to consider, namely, $|V_2| = 1$ and $|V_2| \leq 2$. If $|V_2| = 2$, then $|V_1| = 1$. Therefore, $|V(G)| = 3$ which is not possible by (iii). If $|V_2| = 1$, then $|V_1| = 3$. By (P2), $\langle V_1 \cup V_2 \rangle$ must be connected and V_2 is a dominating set in G, it follows that V_2 is a γ-set in G. Therefore, $|V(G)| = 4$ and $\gamma(G) = 1$. Now, suppose that $|V_3| = 1$. If $|V_2| = 0$, then $|V_1| = 2$. Consequently, $|V(G)| = 3$. Thus, $G \in \{K_3, P_3\}$, a contradiction by (iii). If $|V_2| = 1$, then $|V_1| = 0$. Since $V_2 \cup V_3$ is a 2-dominating set in G and $|V_2 \cup V_3| = 2$, it follows that $V_2 \cup V_3$ is a γ_2-set in G. Hence, $|V(G)| \geq 4$ and $\gamma_2(G) = 2$.
Conversely, suppose $|V(G)| = 4$ and $\gamma(G) = 1$. By (iii), $\gamma(G) \geq 5$. Let v be a dominating vertex of G and define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G)$ such that $V_0 = \varnothing = V_3, V_2 = \{v\},\ V_1 = V(G) \setminus \{v\}$. Then $f \in MRDF(G)$ and $\omega_G^{mR}(f) = 5$. This implies that $\gamma_{mR}(G) = 5$. Next, suppose that $\gamma_2(G) = 2$ and $|V(G)| \geq 4$. Let $D = \{u, v\}$ be the γ_2-set of G. Define a function $g = (V_0, V_1, V_2, V_3)$ such that $V_1 = \varnothing$ and

$$g(x) = \begin{cases} 3, & \text{if } x = u. \\ 2, & \text{if } x = v \\ 0, & \text{if } x \in V(G) \setminus D. \end{cases}$$

Then $g \in MRDF(G)$ and $\omega_G^{mR}(g) = 5$. Since $G \notin \{K_3, P_3\}$, we must have $\omega_G^{mR}(g) = 5$. Hence, $\gamma_{mR}(G) = 5$. □

Corollary 1. *For a connected graph G of order 4, $\gamma_{mR}(G) = 5$ if and only if $G \in \{K_1 + (K_1 \cup K_2), K_1 + K_3, K_1 + \overline{K_3}, K_1 + P_3\}$.*

Proof. The proof follows directly from Proposition 5 (iv). □

Remark 2. *Let G be a graph, then every γ_{mR}-function of G is a γ_{dR}-function of G if $V_0 = \varnothing$.*

Proposition 6. *For a complete graph K_n, $\gamma_{mR}(K_n) = 5$ for all $n \geq 4$.*

Proof. Pick any $x, y \in V(K_n)$ with $x \neq y$. Clearly, $g = (V(K_n) \setminus \{x, y\}, \varnothing, \{x\}, \{y\}) \in MRDF(K_n)$. It follows that $\gamma_{mR}(K_n) \leq 5$. On the other hand, suppose that $f = (V_0, V_1, V_2, V_3)$ is a γ_{mR}-function of K_n. If $V_0 = \varnothing$, then $V_3 = \varnothing$. Since f is a γ_{mR}-function of $K_n, |V_2| = 1$ and $|V_1| = n - 1$. Hence, $\gamma_{mR}(K_n) = \omega_{K_n}^{mR}(f) = n + 1 \geq 5$. If

$V_0 \neq \varnothing$, then $|V_2| \geq 1$ and $|V_3| \geq 1$. It follows that $\gamma_{mR}(K_n) = \omega^{mR}_{K_n}(f) = 2|V_2|+3|V_3| \geq 5$. Therefore, $\gamma_{mR}(K_n) = 5$. □

In what follows, we denote by $f|_G$ the restriction of f on the subgraph G of the graph H.

Proposition 7. *Let G be a disconnected graph with nontrivial components $G_1, G_2, \cdots, G_n$. Then*

$$\gamma_{mR}(G) = \textstyle\sum_{i=1}^{n} \gamma_{mR}(G_i).$$

Proof. Let $G_1, G_2, \cdots, G_n$ be the components of G. Let $f_1, f_2, \cdots, f_n$ be γ_{mR}-functions of $G_1, G_2, \cdots, G_n$ respectively. Define a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ given by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in V(G_1). \\ f_2(x), & \text{if } x \in V(G_2). \\ \vdots & \\ f_n(x), & \text{if } x \in V(G_n). \end{cases}$$

Then f is a γ_{mR}-function of G. Thus $\gamma_{mR}(G) \leq \sum_{i=1}^{n} \gamma_{mR}(G_i)$. Conversely, let f be a γ_{mR}-function of G. Then the restriction $f|_{G_i}$ of f to G_i, where $i = 1, 2, \cdots, n$ is a γ_{mR}-function of G_i. Thus, $\gamma_{mR}(G_i) \leq \omega^{mR}_{G}(f|_{G_i})$ for all $i = 1, 2, \cdots, n$. Hence, $\sum_{i=1}^{n} \gamma_{mR}(G_i) \leq \gamma_{mR}(G)$. Hence, the assertion follows by combining the results. □

Corollary 2. *Let G be a graph of order n. Then $\gamma_{mR}(G) = 2n$ if and only if $G = \overline{K_n}$.*

The *n-barbell* graph is the simple graph obtained by joining two copies of complete graph $K_{n\geq 3}$ by a bridge and is denoted by B_n. Figure 2 shows the n-barbell graphs B_3 and B_5, respectively.

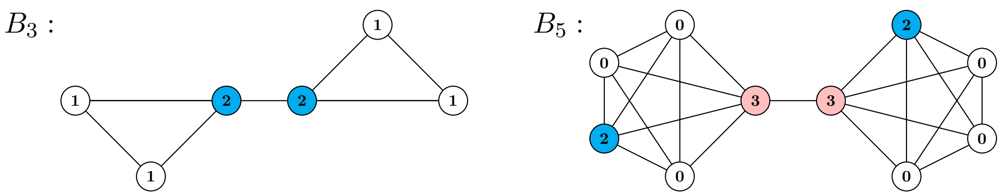

Figure 2: The graphs B_3 and B_5 with $\gamma_{mR}(B_3) = 8$ and $\gamma_{mR}(B_5) = 10$, respectively.

Proposition 8. *For any n-barbell graph B_n where $n \geq 3$,*

$$\gamma_{mR}(B_n) = \begin{cases} 8, & \text{if } n = 3. \\ 10, & \text{if } n \geq 4. \end{cases}$$

Proof. Let B_n be an n-barbell graph and $uv \in E(B_n)$ be the bridge that joins the two copies of K_n. If $n = 3$, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 2, & x \in \{u, v\}. \\ 1, & \text{otherwise.} \end{cases}$$

Then $f \in MRDF$ of B_3. It follows that $\gamma_{mR}(B_3) \leq 8$. Now, suppose that $f' = (V_0', V_1', V_2', V_3')$ is a γ_{mR}-function of B_3. If $V_0' = \varnothing$, then $V_3' = \varnothing$. Since f' is a γ_{mR}-function of B_3, $|V_2'| = 2$ and $|V_1'| = V(B_3) \setminus |V_2'|$. Hence, $\gamma_{mR}(B_3) = \omega_{B_3}^{mR}(f') \geq 8$. If $|V_0'| \neq 0$, then $|V_2'| \geq 2$ and $|V_3'| \geq 1$. It follows that $\gamma_{mR}(B_3) = \omega_{B_3}^{mR}(g) \geq 8$. Therefore, $\gamma_{mR}(B_3) = 8$. If $n \geq 4$. Pick any $v', u' \in V(B_n)$ such that $v' \neq u, u' \neq v$, and $v'v, u'u \in E(B_n)$. Now, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 0, & x \in V(B_n) \setminus \{u, v, u', v'\}. \\ 3, & x \in \{u, v\}. \\ 2, & x \in \{u', v'\}. \end{cases}$$

Then $f \in MRDF$ of $B_n, n \geq 4$. It follows that $\gamma_{mR}(B_n) \leq 10$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of B_n. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{mR}-function of B_n, $|W_2| = 2$ and $|W_1| = V(B_n) \setminus |W_2|$. Hence, $\gamma_{mR}(B_n) = \omega_{B_n}^{mR}(g) \geq 10$. If $|W_0| \neq 0$, then $|W_2| \geq 2$ and $|W_3| \geq 2$. It follows that $\gamma_{mR}(B_n) = \omega_{B_n}^{mR}(g) = 2|W_2| + 3|W_3| \geq 10$. Therefore, $\gamma_{mR}(B_3) = 10$. □

The *windmill graph* $Wd(k, n) = G = K_1 + nK_{k-1}$ is constructed for $k \geq 2$ and $n \geq 2$ by joining n copies of the complete graph K_k at a shared vertex. It has $n(k-1)+1$ vertices and $\frac{1}{2}nk(k-1)$ edges. The case $k = 3$ corresponds to the *dutch windmill graph* (also called friendship graph) $G_3^n = K_1 + nK_2$ and the case $n = 2$ corresponds to the butterfly graph $G_3^2 = K_1 + 2K_2$. The graphs in Figures 3, 4, and 5 are the windmill graph $Wd(4, 2)$, friendship graph G_3^4 and butterfly graphs G_3^2, respectively.

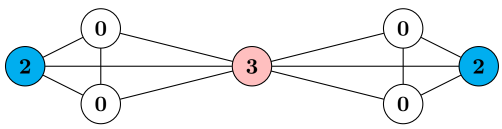

Figure 3: A windmill graph $Wd(4, 2)$ with $\gamma_{mR}(Wd(4, 2)) = 7$

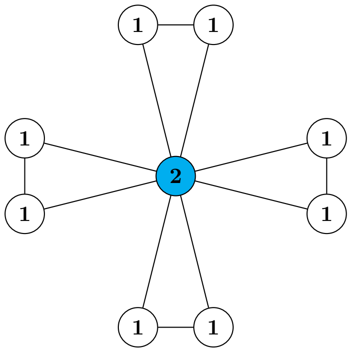

Figure 4: A friendship graph G_3^4 with $\gamma_{mR}(G_3^4) = 10$

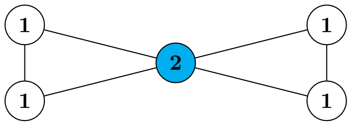

Figure 5: A butterfly graph G_3^2 with $\gamma_{mR}(G_3^2) = 6$

Proposition 9. *For any windmill graph $G = K_1 + nK_{k-1}$, where $k \geq 4$ and $n \geq 2$, $\gamma_{mR}(G) = 2n + 3$.*

Proof. Let $G = K_1 + nK_{k-1}$, where $k \geq 4$ and $n \geq 2$. Suppose $V(K_1) = \{u\}$ be the central vertex in G, then pick a vertex v in each n copies of the complete graph K_{k-1} and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 3, & x = u. \\ 2, & x = v. \\ 0, & \textit{otherwise.} \end{cases}$$

Then $f \in MRDF(G)$. It follows that $\gamma_{mR}(G) \leq 2n + 3$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of G. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{mR}-function of G, $|W_2| = \{u\} = 1$ and $|W_1| = V(G) \setminus \{u\}$. Hence, $\gamma_{mR}(G) = \omega_G^{mR}(g) \geq 2n+3$. If $|W_0| \neq 0$, then $|W_2| \geq 2$ and $|W_3| \geq 1$. It follows that $\gamma_{mR}(G) = \omega_G^{mR}(g) = 2|W_2| + 3|W_3| \geq 2n + 3$. Therefore, $\gamma_{mR}(G) = 2n + 3$. □

Proposition 10. *For any friendship graph G, $\gamma_{mR}(G) = 2n + 2$.*

Proof. Let $G = K_1 + nK_2$, $n \geq 2$. Let $V(K_1) = \{u\}$ be the central vertex in G. Define a function $f = (\varnothing, V(G) \setminus \{u\}, \{u\}, \varnothing)$. Then for all $v_i \in V_1, 1 \leq i \leq n, f(N_G[v_i]) = 2n + 2$. Thus, $f \in MRDF(G)$. It follows that $\gamma_{mR}(G) \leq 2n + 2$. Now, suppose that $f' = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of G. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since f' is a γ_{mR}-function of G, by Proposition 4 (i), $\gamma_{mR}(G) = 2n + 2$. □

Corollary 3. *For a butterfly graph G, $\gamma_{mR}(G) = 6$.*

Proof. The result follows from Proposition 10. □

A graph G is called *bipartite* if the vertex set $V(G)$ of G can be partitioned into two subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. If G is bipartite such that G contains every edge incident with any pair of vertices in V_1 and V_2, then G is a *complete bipartite graph*; in this case, $G = K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$. Figure 6 shows the complete bipartite graph $K_{7,5}$.

$K_{7,5}$:

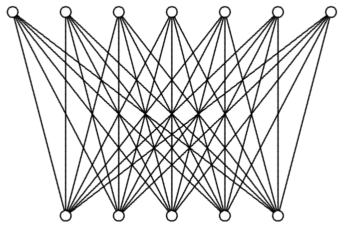

Figure 6: A complete bipartite $G = K_{7,5}$

Proposition 11. *For a complete bipartite graph* $K_{m,n}$*, let* $p = min\{m,n\}, m,n \geq 2$*. Then*

$$\gamma_{mR}(K_{m,n}) = \begin{cases} 5, & \text{if } p = 2. \\ 7, & \text{if } p = 3 \\ 9, & \text{if } p = 4 \\ 10, & \text{if } p \geq 5. \end{cases}$$

Proof. Let G be a complete bipartite graph $K_{m,n}$ and X and Y be partite sets of $K_{m,n}$, where $|X| = m$ and $|Y| = n$. Let $p = min\{m,n\}$. For $p = 2$, since $|V_5| = 5$ and $\gamma_2(G) = 2$, it follows from 5(iv) that $\gamma_{mR}(G) = 5$. For $p = 3$, let X and Y be partite sets of G and assume that $|X| = 3$, say $X = \{x_1, x_2, x_3\}$. Let $V_0 = Y, V_1 = \varnothing, V_2 = \{x_1, x_2\}$ and $V_3 = \{x_3\}$. Then $f = (V_0, V_1, V_2, V_3) \in MRDF(G)$ and its weight is 7. Hence, $\gamma_{mR}(G) \leq 7$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function. If $W_0 = \varnothing$ then $W_3 = \varnothing$. Since g is a γ_{mR}-function, then $\gamma_{mR}(G) \geq 7$. If $W_0 \neq \varnothing$, let $W_0 = Y, W_1 = \varnothing, W_2 = \{x_1, x_2\}$ and $W_3 = \{x_3\}$. Since g is a γ_{mR}-function, then $\gamma_{mR}(G) \geq 7$. Thus, $\gamma_{mR}(G) = 7$. If $p = 4$ and WLOG, let $X = \{x_1, x_2, x_3, x_4\}$ such that $f(x_1) = 2$, $f(x_2) = 3$ and $f(x_3) = 2 = f(x_4)$. Then $f(y_i) = 0$, $i = 1, 2, \cdots, n$, and for every $y_i \in Y$, $y_i \in N_G(X)$ by (P2). Define $f = (V_0, V_1, V_2, V_3)$ such that $V_0 \neq \varnothing, V_1 = \varnothing, V_2 = \{x_1, x_3, x_4\}, V_3 = \{x_2\}$. Then $f \in MRDF(G)$. Thus, $\gamma_{mR}(G) \leq 9$. Suppose to the contrary that $\gamma_{mR}(G) < 9$ and $p = 4$. So, $\gamma_{mR}(G) = 8$. Now, if $X = \{x_1, x_2, x_3, x_4\}$ then $\{x_1, x_2, x_3\}$ is a γ_3-set of $G \setminus \{x_4\}$. Clearly, $7 = \gamma_{mR}(G \setminus \{x_4\}) \leq \gamma_{mR}(G) = 8$. This means that $f(x_4) = 1$, a contradiction. Therefore, $\gamma_{mR}(G) = 9$. If $p \geq 5$, let $X = \overline{K_m}$ and $Y = \overline{K_n}$. WLOG, let

$\{u_1, u_2\} \in V(\overline{K_m}), \{v_1, v_2\} \in V(\overline{K_n})$ and define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(z) = \begin{cases} 3, & z \in \{u_1, v_1\}. \\ 2, & z \in \{u_2, v_2\}. \\ 0, & otherwise. \end{cases}$$

Then $f \in MRDF(G)$. Since $|V_1| = 0$ and $u_i v_i \in E(G)$, for all $i = 1, 2$, $\{u_1, v_1\} = V_3$ is a dominating set of G. Thus, $\gamma(G) = 2$. Since $\langle V_2 \cup V_3 \rangle$ is connected, by Proposition 4 (ii), $\gamma_{mR}(G) = 10$. □

The *fan* F_n of order $n+1$ is the graph $P_n + K_1$ and the *star* S_n of order $n+1$ is the graph $\overline{K_n} + K_1$. The graphs in Figures 7 and 8 are the star graph S_6 and fan graph F_6, respectively.

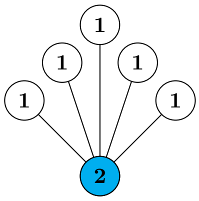

Figure 7: A star graph S_6 with $\gamma_{mR}(S_6) = 7$

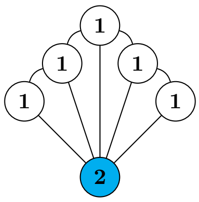

Figure 8: A fan graph F_6 with $\gamma_{mR}(F_6) = 7$

Proposition 12. *If* $G \in \{F_n, S_n\}$*,* $n \geq 1$*, then* $\gamma_{mR}(G) = n + 2$*.*

Proof. WLOG, let $G = F_n$ where $V(G) = V(K_1 + P_n)$ and $V(K_1) = \{u\}$ is a central vertex of G. Now, define a function $f = (V_0, V_1, V_2, V_3)$ given by

$$f(x) = \begin{cases} 2, & x = \{u\}. \\ 1, & \text{otherwise}. \end{cases}$$

Then $f \in MRDF(G)$. It follows that $\gamma_{mR}(G) \leq n + 2$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of G. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{mR}-function of G, $|W_2| = |V(K_1)| = 1$ and $|W_1| = |V(P_n)| = n$. Hence, $\gamma_{mR}(G) = \omega_G^{mR}(g) \geq n + 2$. If $|W_0| \neq 0$, then $|W_2| \geq 1$ and $|W_3| \geq 1$. It follows that $\gamma_{mR}(G) = \omega_G^{mR}(g) = 2|W_2| + 3|W_3| \geq n + 2$. Therefore, $\gamma_{mR}(G) = n + 2$. □

5. On the join of graphs

Given two graphs G and H with disjoint vertex sets, the *join* $G+H$ of graphs G and H, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$ [3].

In this section, the following proposition characterizes all $MRDF$ on the join of graphs.

Proposition 13. *Let G and H be any graphs and let $f \in (V_0, V_1, V_2, V_3)$ be a function on $V(G+H)$ with $V_2 \neq \varnothing$ and $V_3 \neq \varnothing$. Then $f \in MRDF(G+H)$ if and only if one of the following holds:*

(i) *$f|_G \in MRDF(G)$ and one of the following holds:*

- (a) $|V_2 \cap V(G)| \geq 1$ *and* $|V_3 \cap V(G)| \geq 1$
- (b) $V_2 \cap V(G) = \varnothing$ *and each of the following holds:*
  - (b1) V_3 *is a dominating set of* G.
  - (b2) $V_2 \cap V(H)$ *is a dominating set of* $H[V_0]$.
- (c) $V_3 \cap V(G) = \varnothing$ *and each of the following holds:*
  - (c1) V_2 *is a dominating set of* G.
  - (c2) $V_3 \cap V(H)$ *is a dominating set of* $H[V_0]$.

(ii) *$f|_H \in MRDF(H)$ and one of the following holds:*

- (a) $|V_2 \cap V(H)| \geq 1$ *and* $|V_3 \cap V(H)| \geq 1$
- (b) $V_2 \cap V(H) = \varnothing$ *and each of the following holds:*
  - (b1) V_3 *is a dominating set of* H.
  - (b2) $V_2 \cap V(G)$ *is a dominating set of* $G[V_0]$.
- (c) $V_3 \cap V(H) = \varnothing$ *and each of the following holds:*
  - (c1) V_2 *is a dominating set of* H.
  - (c2) $V_3 \cap V(G)$ *is a dominating set of* $G[V_0]$.

(iii) *$f|_G \notin MRDF(G)$, $f|_H \notin MRDF(H)$ and each of the following holds:*

- (a) $V_2 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_2 = \varnothing$ *for some* $x \in V_0$.
- (b) $V_3 \cap V(H) \neq \varnothing$ *whenever* $N_G(x) \cap V_3 = \varnothing$ *for some* $x \in V_0$.
- (c) $V_2 \cap V(H) \neq \varnothing$ *or* $V_3 \cap V(H) \neq \varnothing$ *whenever* $\exists x \in V_1$ *with* $N_G(x) \cap V_2 = \varnothing$ *and* $N_G(x) \cap V_3 = \varnothing$
- (d) $V_2 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_2 = \varnothing$ *for some* $x \in V_0$.
- (e) $V_3 \cap V(G) \neq \varnothing$ *whenever* $N_H(x) \cap V_3 = \varnothing$ *for some* $x \in V_0$.

(f) *$V_2 \cap V(G) \neq \varnothing$ or $V_3 \cap V(G) \neq \varnothing$ whenever $\exists x \in V_1$ with $N_H(x) \cap V_2 = \varnothing$ and $N_H(x) \cap V_3 = \varnothing$*

Proof. Suppose $f|_G \in MRDF(G)$. Assume that (i)(a) holds. Let $v \in V_0$. If $v \in V(G)$, then there exist $u, w \in V(G)$ such that $\{u, w\} \subseteq N_G(v)$ and $f(u) = 2$ and $f(w) = 3$, by (P1). This implies that $\{u, w\} \subseteq N_{G+H}(v)$. Now, assume that $v \in V(H)$. Note that $|V_2 \cap V(G)| \geq 1$ and $|V_3 \cap V(G)| \geq 1$. Now, take $u \in V_2 \cap V(G)$ and $w \in V_3 \cap V(G)$ such that $vu, vw \in E(G + H)$. Thus, $\{u, v\} \subseteq N_{G+H}(v)$. Moreover, let $v \in V_1$. Assume $v \in V(G)$. Then there exists $z \in V_2 \cap V(G)$ or $z \in V_3 \cap V(G)$ such that $z \in N_G(v)$ by (P2). This means that $z \in N_{G+H}(v)$. Now, assume $v \in V(H)$. Since $|V_2 \cap V(G)| \geq 1$ and $|V_3 \cap V(G)| \geq 1$, there exists $z \in V_2 \cap V(G)$ or $z \in V_3 \cap V(G)$ such that $z \in N_{G+H}(v)$. Thus, $f \in MRDF(G + H)$. Similarly, if $f|_H \in MRDF(H)$ with $|V_2 \cap V(H)| \geq 1$ and $|V_3 \cap V(H)| \geq 1$, then $f \in MRDF(G + H)$. Assume (i)(b) holds. Since $V_0 \cap V(G) = \varnothing$, $V_0 \subseteq V(H)$. Let $v \in V_0$. By (b2), there exists $u \in V_2 \cap V(H)$ such that $uv \in E(H) \subseteq E(G + H)$. Also, since V_3 is a dominating set of G, $V_3 \cap V(G) \neq \varnothing$. Pick $u \in V_3 \cap V(G)$. Then $uv \in E(G + H)$. Let $v \in V_1 \cap V(G)$. By (b1), there exists $u \in V_3 \cap V(G)$ such that $uv \in E(G) \subseteq E(G + H)$. Now, let $v \in V_1 \cap V(H)$. By (b1), there exists $u \in V_3 \cap V(G)$. Then $uv \in E(G + H)$. Therefore, $f \in MDRF(G + H)$. Similarly, if (ii)(b) holds, then $f \in MRDF(G + H)$. Assume (i)(c) holds. Since $V_0 \cap V(G) = \varnothing$, then $V_0 \subseteq V(H)$. Let $v \in V_0$. By (c2), there exists $u \in V_3 \cap V(H)$ such that $uv \in E(H) \subseteq E(G+H)$. Also, since V_2 is a dominating set of G, $V_2 \cap V(G) \neq \varnothing$. Pick $u \in V_2 \cap V(G)$. Then $uv \in E(G + H)$. Let $v \in V_1 \cap V(G)$. By (c1), there exists $u \in V_2 \cap V(G)$ such that $uv \in E(G) \subseteq E(G+H)$. Now, let $v \in V_1 \cap V(H)$. By (c1), there exists $u \in V_2 \cap V(G)$. Then $uv \in E(G + H)$. Therefore, $f \in MDRF(G + H)$. Similarly, if (ii)(c) holds, then $f \in MRDF(G + H)$.
Suppose (iii) holds, that is $f|_G \notin MRDF(G)$ and $f|_H \notin MRDF(G)$. Let $v \in V_0 \cap V(G)$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 \neq \varnothing$. Take $u \in V_3 \cap V(G)$ such that $uv \in E(G) \subseteq E(G + H)$. Since $N_G(v) \cap V_2 = \varnothing$, by assumption there exists $w \in V_2 \cap V(H)$ such that $vw \in E(G + H)$. If $N_G(v) \cap V_2 \neq \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Pick $u \in V_2 \cap V(G)$ such that $uv \in E(G) \subseteq E(G + H)$. Since $N_G(v) \cap V_3 = \varnothing$, by assumption there exists $w \in V_3 \cap V(H)$ such that $vw \in E(G + H)$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Then by assumption, $V_2 \cap V(H) \neq \varnothing$ and $V_3 \cap V(H) \neq \varnothing$ and so, there exist $u \in V_2 \cap V(H)$ and $w \in V_3 \cap V(H)$ such that $vu, vw \in E(G + H)$. Now, suppose $f(v) = 1$. If $N_G(v) \cap V_2 = \varnothing$ and $N_G(v) \cap V_3 = \varnothing$. Then by assumption, there exist $z \in V_2 \cap V(H)$ or $z \in V_3 \cap V(H)$ such that $vz \in E(G + H)$ satisfying (P2). Therefore, $f \in MRDF(G + H)$. Similarly, for $v \in V(H)$ such that $f(v) \in \{0, 1\}$, $f \in MRDF(G + H)$.

Conversely, suppose $f \in MRDF(G + H)$. Consider the following cases:
Case 1: Suppose $f|_G \in MRDF(G)$. If (i)(a) holds, we are done. Suppose (i)(a) does not hold. Thus, either $V_2 \cap V(G) = \varnothing$ or $V_3 \cap V(G) = \varnothing$. Suppose $V_2 \cap V(G) = \varnothing$. Necessarily, $V_0 \cap V(G) = \varnothing$. Let $v \in V_1 \cap V(G)$. Since $f|_G \in MRDF(G)$, there exists $u \in V_3$ such that $uv \in E(G)$. Thus, V_3 is a dominating set of G, and so, (b1) holds. Also, since $V_2 \cap V(G) = \varnothing$, we have $V_2 \subseteq V(H)$. This means that $V_2 \cap V(H) \neq \varnothing$, say $w \in V_2 \cap V(H)$. Suppose $v \in V_0 \cap V(H)$. Then since $f \in MRDF(G + H)$, $vw \in E(H) \subseteq E(G + H)$. And so, (b2) holds. Also, since V_3 is a dominating set of G, there exists $u \in V_3 \cap V(G)$ where

$uv \in E(G+H)$. Suppose $V_3 \cap V(G) = \varnothing$, then similarly, (i)(c1) and (i)(c2) hold.
Case 2: Suppose $f|_H \in MRDF(H)$. This case can be proven similarly with Case 1.
Case 3: Suppose $f|_G \notin MRDF(G)$ and $f|_H \notin MRDF(H)$. If $f|_G \notin MRDF(G)$, then there exists $x \in V_0 \cap V(G)$ such that $N_G(x) \cap V_2 = \varnothing$ or $N_G(x) \cap V_3 = \varnothing$. Moreover, there exists $y \in V_1 \cap V(G)$ such that $N_G(y) \cap V_2 = \varnothing$ and $N_G(y) \cap V_3 = \varnothing$. If $N_G(x) \cap V_2 = \varnothing$ and $N_G(x) \cap V_3 \neq \varnothing$. Note that $f \in MRDF(G+H)$. Then, there exists $u \in (V_2 \cap V(G+H))$ such that $u \in N_{G+H}(x)$ for some $x \in V_0 \cap V(G)$. Since $N_G(x) \cap V_2 = \varnothing, u \in N_H(x)$. Consequently, $u \in V_2 \cap V(H)$ for some $x \in V_0$. Thus, (iii)(a) holds. If $N_G(x) \cap V_3 = \varnothing$ and $N_G(x) \cap V_2 \neq \varnothing$. Since $f \in MRDF(G+H)$, then there exists $w \in (V_3 \cap V(G+H))$ such that $w \in N_{G+H}(x)$ for some $x \in V_0 \cap V(G)$. Consequently, by assumption, (iii)(b) holds. If $N_G(x) \cap V_2 = \varnothing$ and $N_G(x) \cap V_3 = \varnothing$. Since $f \in MRDF(G+H)$, then there exist $u \in (V_2 \cap V(G+H))$ and $w \in (V_3 \cap V(G+H))$ such that $u, w \in N_{G+H}(x)$ for some $x \in V_0 \cap V(G)$. Thus, by assumption, $u, w \in N_H(x)$ and consequently, $V_2 \cap V(H) \neq \varnothing$ and $V_3 \cap V(H) \neq \varnothing$. Hence, (iii)(a) and (iii)(b) hold. Furthermore, suppose $N_G(y) \cap V_2 = \varnothing$ and $N_G(y) \cap V_3 = \varnothing$ for some $y \in V_1 \cap V(G)$. Since $f \in MRDF(G+H)$, there exist $u \in V_2 \cap V(G+H)$ or $w \in V_3 \cap V(G+H)$ such that $u, w \in N_{G+H}(y)$. By assumption, $u, w \in N_H(y)$ and consequently, (iii)(c) holds. Similarly, if $f|_H \notin MRDF(H)$, then (iii)(d), (iii)(e), and (iii)(f) hold. □

Proposition 14. *Let G and H be any graphs. Then*

$$3 \leq \gamma_{mR}(G+H) \leq 10$$

Proof. Suppose G and H are trivial graphs. Then by Proposition 5 (ii), $\gamma_{mR}(G+H) = \gamma_{mR}(K_2) = 3$. Suppose G and H are not trivial graphs, then $\gamma_{mR}(G+H) > 2$. That is, $\gamma_{mR}(G+H) \geq 3$. On the other hand, let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(H) = \{u_1, u_2, \cdots, u_n\}$. Now, define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G+H)$ given by

$$f(x) = \begin{cases} 2, & \text{if } x \in \{v_1, u_1\}. \\ 3, & \text{if } x \in \{v_2, u_2\}. \\ 0, & \text{if } x \in V(G+H) \setminus \{v_1, v_2, u_1, u_2\}. \end{cases}$$

for every $x \in V(G+H)$. Then $f \in MRDF(G+H)$. Thus, $\gamma_{mR}(G+H) \leq \omega^{mR}_{G+H}(f) = 10$. Hence, $3 \leq \gamma_{mR}(G+H) \leq 10$. □

Proposition 15. *Let G and H be any graphs. Then*

(i) $\gamma_{mR}(G+H) = 3$ *if and only if* $G = K_1$ *and* $H = K_1$

(ii) $\gamma_{mR}(G+H) = 4$ *if and only if* $G = K_1$ *and* $H \in \{K_2, \overline{K_2}\}$

(iii) $\gamma_{mR}(G+H) = 5$ *if and only if one of the following holds:*

  (a) $G = K_1$ *and* $H \in \{P_3, K_3, \overline{K_3}, K_1 \cup K_2\}$ *or* $H = K_1$ *and* $G \in \{P_3, K_3, \overline{K_3}, K_1 \cup K_2\}$.

(b) *If* $|V(G+H)| \geq 4$*, then* $\gamma_2(G) = 2$ *or* $\gamma_2(H) = 2$.

(c) *If* $|V(G+H)| \geq 4$*, then* $\gamma(G) = 1$ *and* $\gamma(H) = 1$.

Proof. The proof follows immediately from Proposition 5. □

Corollary 4. *Let* m *and* n *be positive integers.*
(i) *If* $G = K_n$ *and* $H = K_m$ *with* $n, m \geq 2, \gamma_{mR}(G+H) = 5$.
(ii) *If* $G = \overline{K_n}$ *and* $H = \overline{K_m}$ *with* $n, m \geq 5, \gamma_{mR}(G+H) = 10$.

6. On the corona of graphs

Let G and H be graphs with disjoint vertex sets. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex of the i^{th} copy of H. For convenience, we adapt the notation $H^v + v$ used in [3] to denote the subgraph of $G \circ H$ corresponding to the join $H^v + \langle\{v\}\rangle, v \in V(G)$. Moreover, for convenience, we define for $i = 0, 1, 2, 3$ and $v \in V(G)$, $V_i^v = \{u \in V(H^v) | f(u) = i\}$.

Proposition 16. *Let* G *be any nontrivial connected graph and* H *be any graph. Let* $f = (V_0, V_1, V_2, V_3)$ *be any function on* $V(G \circ H)$*. Then* $f \in MRDF(G)$ *if and only if each of the following holds:*

(i) *For every* $v \in (V_0 \cup V_1) \cap V(G), f|_{H^v} \in MRDF(H^v)$*. Moreover, if* $v \in V_0 \cap V(G)$*, then the following holds:*

(a) *If* $|V_2^v| \neq 0$ *and* $|V_3^v| = 0$*, then* $|N_G(v) \cap V_3| \geq 1$*; and*

(b) *If* $|V_2^v| = 0$ *and* $|V_3^v| \neq 0$*, then* $|N_G(v) \cap V_2| \geq 1$.

(ii) *For every* $v \in V_2 \cap V(G), V_3^v$ *dominates* V_0^v.

(iii) *For every* $v \in V_3 \cap V(G), V_2^v$ *dominates* V_0^v.

Proof. Suppose $f \in MRDF(G \circ H)$ and let $v \in (V_0 \cup V_1) \cap V(G)$. Let $u \in V_0^v$. Then $u \in V_0$ and by definition, there exist $w, z \in N_{G \circ H}(u)$ such that $w \in V_2$ and $z \in V_3$. But $N_{G \circ H}(u) = \{v\} \cup N_{H^v}(u)$ and $v \in V_0 \cup V_1$, and thus, $w, z \in N_{H^v}(u)$. Moreover, let $u \in V_1^v$. Then $u \in V_1$ and by definition, there exists $x \in V_2 \cup V_3$ such that $x \in N_{G \circ H}(u)$, so that $x \in (V_2^v \cup V_3^v)$ and so, $x \in N_{H^v}(u)$. Hence, $f|_{H^v} \in MRDF(G \circ H)$. Now, let $v \in V_0$. Suppose that $|V_2^v| \neq 0$ and $|V_3^v| = 0$. Then if $u \in N_{G \circ H}(v)$ and $u \in V_3$, we have $u \in N_G(v) \cap V_3$. Thus, $|N_G(v) \cap V_3| \geq 1$. Moreover, suppose that $|V_3^v| \neq 0$ and $|V_2^v| = 0$. Similarly, if $w \in N_{G \circ H}(v)$ and $w \in V_2$, then $u \in N_G(v) \cap V_2$. Thus, $|N_G(v) \cap V_2| \geq 1$. This proves (i).

Suppose that $v \in V_2 \cap V(G)$ and let $u \in V_0^v$. By definition, there exists $\{x, y\} \subseteq N_{G \circ H}(u) = \{v\} + N_{H^v}(u)$ such that $x \in V_2$ and $y \in V_3$. If $v \in V_2$ and take $x = v$, then

$y \in V_3^v$ and $y \in N_{H^v}(u)$. Thus, V_3^v dominates V_0^v. This proves (ii). Similarly, if $v \in V_3$ and taking $y = v$, then $x \in V_2^v$ and $x \in N_{H^v}(u)$. This means that V_2^v dominates V_0^v. This proves (iii).

Conversely, let $u \in V_0$ and let $v \in V(G)$ for which $u \in V(H^v + v)$. If $u = v$, by (i), $f|_{H^v} \in MRDF(G \circ H)$. Thus, there exist $w \in V_2^v$ and $z \in V_3^v$ such that $w, z \in N_{H^v}(u)$ and so, $w, z \in N_{G\circ H}(u)$. Now, if $|V_3^v| = 0$, by (i) $|N_G(v) \cap V_3| \geq 1$. Thus, there exists $w \in V_2^v$ and $z \in N_G(v) \cap V_3$ such that $w, z \in N_{G\circ H}(u)$. Similarly, if $|V_2^v| = 0$, by (i), $|N_G(v) \cap V_2| \geq 1$. Thus, there exist $w \in N_G(v) \cap V_2$ and $z \in V_3^v$ such that $w, z \in N_{G\circ H}(u)$. If $u \neq v$, then $u \in V_0^v$. Suppose $v \in V_1 \cap V(G)$. By (i), $f|_{H^v} \in MRDF(G \circ H)$. Thus, there exist $x, y \in N_{H^v}(u)$ such that $x \in V_2^v$ and $y \in V_3^v$. If $v \in V_2 \cap V(G)$. By (ii), V_3^v dominates V_0^v. Thus, there exist $w, v \in N_{H^v}(u)$ such that $w \in V_3^v$. Also, if $v \in V_3 \cap V(G)$. By (iii), V_2^v dominates V_0^v. Thus, there exist $w, v \in N_{H^v}(u)$ such that $w \in V_2^v$. Now, let $u \in V_1$. If $u \in V(G)$, then there exist $x \in V_2^u \cup V_3^u$ such that $x \in N_{H^u}(u)$ since $f|_{H^u} \in MRDF(H^u)$. This implies that $x \in V_2 \cup V_3$ and $x \in N_{G\circ H}(u)$. Suppose $u \in V(H^v)$ for some $v \in V(G)$. If $v \in (V_2 \cup V_3) \cap V(G)$, then $v \in (V_2 \cup V_3) \cap N_{G\circ H}(u)$. If $v \in V_0 \cup V_1$, then there exist $w \in V_2^v \cup V_3^v$ such that $w \in N_{H^v}(u)$ since $f|_{H^v} \in MRDF(H^v)$ by (i). It implies that $w \in V_2 \cup V_3$ and $w \in N_{G\circ H}(u)$. Therefore, $f \in MRDF(G \circ H)$. □

Proposition 17. *Let G and H be any graph with $|V(G)| = n$ and $|V(H)| = m$ and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{mR}-function of $G \circ H$. Then $3n \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$, for each $v \in V(G)$.*

Proof. Let $v \in V(G)$. If $v \in V_2 \cup V_3$, then $3n \leq \sum_{p\in V(H^v)} f(p) \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. Suppose that $v \in V_0$. By Proposition 16, $f|_{H^v} \in MRDF(H^v)$. Thus, $3n \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. If $v \in V_1$, then by Proposition 16, $f|_{H^v} \in MRDF(H^v)$. Thus, $3n \leq \sum_{p\in V(H^v)} f(p) \leq \sum_{a\in V(v+H^v)} f(a) \leq 2n + nm$. Moreover, the bounds are sharp if $H = K_1$ and $G \in \{P_n, C_n, \overline{K_n}\}$. □

Proposition 18. *Let G be a connected graph of order $n \geq 1$ and K_m be the complete graph of order $m \geq 2$, then*

$$\gamma_{mR}(G \circ K_m) = \begin{cases} 4n, & \text{if } m = 2. \\ 5n, & \text{if } m \geq 3. \end{cases}$$

Proof. If $n = 1$, then $G \circ K_m = K_{m+1}$. Hence, if $m = 2, \gamma_{mR}(G \circ K_2) = \gamma_{mR}(K_3) = 4$ by Proposition 5 (iii). If $m \geq 4$, then $\gamma_{mR}(K_{m+1}) = 5$ by Proposition 6. Now, If $n > 1$, then for $m = 2$, let $V(K_2) = \{x, y\}$ and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G \circ K_2)$ where $V_0 = \varnothing = V_3, V_1 = \bigcup_{v\in V(G)} V(H^v), V_2 = V(G)$. Then $f \in MRDF(G \circ K_2)$. It follows that $\gamma_{mR}(G \circ K_2) \leq 4n$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of $G \circ K_2$. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{mR}-function of $G \circ K_2$, by Proposition 4, $\gamma_{mR}(G \circ K_2) = 4n$.

For $m \geq 3$, let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and WLOG, pick a vertex $u \in V(K_m)$. Define a function $f = (V_0, V_1, V_2, V_3)$ on $V(G \circ K_m)$ by

$$f(x) = \begin{cases} 3, & \text{if } x \in V(G). \\ 2, & \text{if } x \in \bigcup_{v \in V(G)} V(u^v). \\ 0, & \text{if } x \in \bigcup_{v \in V(G)} V((H \setminus u)^v). \end{cases}$$

Then $f \in MRDF(G \circ K_m)$. It follows that $\gamma_{mR}(G \circ K_m) \leq 5n$. Now, suppose that $g = (W_0, W_1, W_2, W_3)$ is a γ_{mR}-function of $G \circ K_m$. If $W_0 = \varnothing$, then $W_3 = \varnothing$. Since g is a γ_{mR}-function of $G \circ K_m$, $|W_2| = V(G)$ and $|W_1| = V(H^v)$. Hence, $\gamma_{mR}(G \circ K_m) = \omega^{mR}_{G \circ K_m}(g) \geq 5n$. If $|W_0| \neq 0$, then $|W_2| \geq 1$ and $|W_3| \geq 1$. It follows that $\gamma_{mR}(G \circ K_m) = \omega^{mR}_{G \circ K_m}(g) = 2|W_2| + 3|W_3| \geq 5n$. Therefore, $\gamma_{mR}(G \circ K_m) = 5n$. □

Corollary 5. *If K_n is a complete graph of order $n \geq 1$, then*

(i) $\gamma_{mR}(K_1 \circ \overline{K_n}) = n + 2$.

(ii) $\gamma_{mR}(\overline{K_n} \circ K_1) = 3n$.

Proof. Statement (i) follows from the fact that $K_1 \circ \overline{K_n} = S_n$ and by Proposition 12 (ii), $\gamma_{mR}(K_1 \circ \overline{K_n}) = \gamma_{mR}(S_n) = n + 2$. For (ii), note that $\overline{K_n} \circ K_1$ is the disjoint union n copies of K_2. Using proposition 5 (ii) and Proposition 7, we have $\gamma_{mR}(\overline{K_n} \circ K_1) = 3n$. □

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