EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1804-1817
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Theoretical Foundations of h-Rough Sets

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Abstract. This paper introduces a pioneering advancement in rough set theory by presenting a new class of rough sets termed h-rough sets. Central to this novel approach are the concepts of h-lower and h-upper approximations, intricately tied to the notion of h-open sets. We delve into the fundamental properties of h-rough sets and establish the framework of h-approximation spaces, offering a comprehensive understanding of their theoretical underpinnings. Moreover, we introduce and rigorously analyze the concepts of h-rough equality and h-rough inclusion, providing formal definitions and insightful examinations of their implications in data approximation tasks. Through detailed examples and thorough exploration, this paper showcases how h-rough sets extend rough set theory, offering more flexible and precise techniques for data approximation. This study not only contributes to the theoretical development of rough set theory but also opens up exciting possibilities for practical applications across various domains.

2020 Mathematics Subject Classifications: 04A05,54A05,03E75, 54C08

Key Words and Phrases: Rough sets, upper and lower approximations, accuracy measure, h-open sets,h-rough sets, h-upper and h-lower approximations and h-accuracy measure

1. Introduction

Information technology is the most significant feature of the 21^{st} century, playing a vital role in information discovery through available knowledge. Rough set theory [12], a recent approach for reasoning about data, was created by Pawlak. This theory extends set theory by describing a subset of a universe with a pair of ordinary sets known as the lower and upper approximation. It depends on a specific topological structure and finds many applications across various real-life fields. The theory and applications of rough sets have impressively developed over time. Numerous papers have been written to generalize rough sets ([3],[2],[4],[5],[7],[14],[12],[15],[16],[18], [19],[20]). In [17] Wiweger introduced the concept of topological rough sets, one of the most important generalizations of rough sets. This generalization utilizes an approach starting with a topological space and defines the approximation via the interior and closure operators of topological spaces.

DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5253

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In [1] Abbas introduced h-open. In this paper, we introduce a new classification for the universe called h-approximation space. Additionally, we study the concepts of h-lower and h-upper approximations, investigate h-rough sets, compare this concept with rough sets, and provide some properties and examples.

2. Preliminaries

Rough set theory finds its roots in the necessity to represent subsets of a universe through equivalence classes within a partition of that universe, which defines a topological space, denoted as approximation space $A = (M, R)$. Here, M denotes the universe set and R stands for an equivalence relation ([8],[13]). The equivalence classes of R are referred to as granules, elementary sets, or blocks, denoted by $R_m \subseteq M$ for each $m \in M$. Within this approximation space, two operators are considered:

(i) $\overline{R}(K) = \{m \in M : R_m \cap K \neq \phi\}$ is called upper approximation of $K \subseteq M$.

(ii) $\underline{R}(K) = \{m \in M : R_m \subseteq K\}$ is called lower approximation of $K \subseteq M$.

Let $POS_R(K) = \underline{R}(K)$ denote the positive region of K, $NEG_R(K) = M - \overline{R}(K)$ denote the negative region of K, and $BN_R(K) = \overline{R}(K) - \underline{R}(K)$ denote the borderline region of M.

The degree of completeness can also be characterized by the accuracy measure, in which $\mid R \mid$ represents the cardinality of set R as follows:

$$\alpha_R(K) = \frac{\mid \underline{R}(K) \mid}{\mid \overline{R}(K) \mid}, where\ K \neq \phi$$

Accuracy measures aim to quantify the completeness of knowledge. $\alpha_R(K)$ helps gauge the size of the boundary region of datasets, yet it doesn't readily capture knowledge structure. One key advantage of rough set theory lies in its capability to manage categories that defy sharp definition within a knowledge base. The rough sets framework allows for the measurement of characteristics in potential datasets, facilitating the assessment of inexactness and expression of topological imprecision characterization.

(i) If $R(K) \neq \phi$ and $R(K) \neq M$, then K is roughly R-definable.

(ii) If $R(K) = \phi$ and $R(K) \neq M$, then K is internally R-undefinable.

(iii) If $R(K) \neq \phi$ and $R(K) = M$, then K is externally R-undefinable.

(iv) If $R(K) = \phi$ and $R(K) = M$, then K is totally R-undefinable.

We denote the set of all roughly R-definable (resp. internally R-undefinable, externally R-undefinable and totally R-undefinable) sets by $RD(M)$ (resp. $IUD(M)$, $EUD(M)$ and $TUD(M)$) ([8],[13]).

Using $\alpha_R(K)$ and these classifications, rough sets can be characterized by their boundary region size and structure. Rough sets are regarded as a specific subset of relative sets and are incorporated into the framework of Belnap's logic [10].

Definition 1 ([8],[13]). *If (M, R) be an approximation space and $K \subseteq M$. Then there are memberships which are defined by:*

(i) The strong membership is denoted by $\underline{\in}$, ($m\underline{\in}K \Leftrightarrow m \in \underline{R}(K)$).

(ii) The weak membership is denoted by $\overline{\in}$, ($m\overline{\in}K \Leftrightarrow m \in \overline{R}(K)$).

Definition 2 ([6]). *A topological space is defined as a pair (M, σ), where M is a set and σ is a family of subsets of M satisfying the following conditions:*

(i) $\phi, M \in \sigma$.

(ii) σ is closed under arbitrary union.

(iii) σ is closed under finite intersection.

The elements of M are referred to as the points of the space, while the subsets of M belonging to σ are termed open sets within the space. The complements of these subsets, belonging to the complement of σ, are known as closed sets within the space. Additionally, the family σ of open subsets of M is referred to as the topology for M.The closure of $K \subseteq M$ is the intersection of all closed sets containing K, denoted as ($\overline{K} = \cap\{F \subseteq M : F \textit{ is closed and } K \subseteq F\}$). Also, K is closed iff $K = \overline{K}$. The interior of K in M is the union of all open subsets of M contained in K denoted as ($K^\circ = \cup\{G \subseteq M : G \textit{ is open and } G \subseteq K\}$). Additionally, K is open iff $K = K^\circ$. The border of $K \subseteq M$ denoted as ($b(K) = K \setminus K^\circ$). A subset K is classified as exact if $b(K) = \phi$; otherwise, it's considered rough. It's evident that K is exact if and only if $\overline{K} = K^\circ$. In Pawlak space, a subset $K \subseteq M$ can either be rough or exact.

Definition 3 ([1]). *A subset K of the topological space M is termed h-open set if for every non-empty set $H \in M$ where $H \neq M$ and $H \in \sigma$, $K \subseteq (K \cup H)^\circ$ .The complement of the h-open set is referred to as h-closed. We denoted the collection of all h-open sets of a topological space (M, σ) as σ^h.*

Theorem 1 ([1]). *In any topological space (K, σ) every open set is h-open set.*

The converse of the Theorem 1 may not hold, as demonstrated in the following example.

Example 1. *Let $M = \{k, q, s\}$ with a topology $\sigma = \{\varphi, M, \{k\}, \{q\}, \{k, q\}\}$, then:*

$$\sigma^h = \{M, \varphi, \{k\}, \{q\}, \{s\}, \{k, q\}, \{k, s\}, \{q, s\}\}.$$

Definition 4 ([1]). *(i) The interior of K in M is the union of all h-open subsets of M contained in K denoted as ($Int_h(K) = \cup\{G \subseteq M : G \text{ is h-open and } G \subseteq K\}$).*

*(ii) The subset $b_h(K) = K \setminus Int_h(K)$ is said to be h-**border** of K.*

*(iii) The subset $Ext_h(K) = Int_h(M \setminus K)$ is called h-**exterior of** K.*

3. h-rough classification

In this section we introduce the h-approximation space which is a new class of approximation space. Additionally, we study the concepts of h-lower approximation and h-upper approximation and outline some properties of h-approximation.

Definition 5. *If M is a finite non-empty universe. the pair (M, R_h) is referred to as an h-approximation space, where R_h represents a general relation used to generate a subbase for a topology σ on M and a class of h-open sets σ^h.*

Example 2. *Let $M = \{k, q, s, t\}$ be a universe and a relation R defined by $R = \{(k, k), (k, s), (k, t), (q, q), (q, t), (s, k), (s, q), (s, t), (t, k)\}$, thus $kR = \{k, s, t\}$, $qR = \{q, t\}$, $sR = \{k, q, t\}$ and $tR = \{k\}$. Consequently, the topology associated with this relation is $\sigma = \{M, \phi, \{k\}, \{t\}, \{k, t\}, \{q, t\}, \{k, q, t\}, \{k, s, t\}\}$ and $\sigma^h = \{M, \phi, \{k\}, \{t\}, \{k, t\}, \{q, t\}, \{s, t\}, \{k, s, t\}, \{q, s, t\}, \{k, q, t\}\}$. So (M, R_h) is a h-approximation space.*

Example 3. *Let $M = \{k, q, s\}$ be a universe and a relation R defined by $kR = \{k, q\}$, $qR = \{q\}$, $sR = \{k, q\}$. Consequently, the topology associated with this relation is $\sigma = \{M, \phi, \{q\}, \{k, q\}\}$ and $\sigma^h = \{M, \phi, \{k\}, \{q\}, \{k, q\}, \{k, s\}\}$. So (M, R_h) is a h-approximation space.*

Definition 6. *If (M, R_h) is a h-approximation space and K is any non-empty subset of M. Then we defined,*

(i) The h-lower approximation, $\underline{R}_h(K) = \cup\{H \in \sigma^h : H \subseteq K\}$.

(ii) The h-upper approximation, $\overline{R}_h(K) = \cap\{F \in \sigma^{hc} : F \supseteq K\}$.

Definition 7. *If (M, R_h) is a h-approximation space and from the relation $Int(K) \subseteq Int_h(K) \subseteq K \subseteq Cl_h(K) \subseteq Cl(K)$, for any $K \subseteq M$. Then the universe M can be divided into 12 regions with respect to any $K \subseteq M$ as follows:*

(i) The internal edg of K ([12]) , $\underline{Edg}(K) = K - \underline{R}(K)$.

(ii) The h-internal edg of K , $\underline{Edg}_h(K) = K - \underline{R}_h(K)$.

(iii) The external edg of K ([12]), $\overline{Edg}(K) = \overline{R}(K) - K$.

(iv) The h-external edg of K, $\overline{Edg}_h(K) = \overline{R}_h(K) - K$.

(v) The boundary of K ([12]), $b(K) = \overline{R}(K) - \underline{R}(K)$.

(vi) The h-boundary of K, $b_h(K) = \overline{R}_h(K) - \underline{R}_h(K)$.

(vii) The exterior of K ([12]), $Ext(K) = M - \overline{R}(K)$.

(viii) The h-exterior of K, $Ext_h(K) = M - \overline{R}_h(K)$.

(ix) $\overline{R}(K) - \underline{R}_h(K)$.

(x) $\overline{R}_h(K) - \underline{R}(K)$.

(xi) $\underline{R}_h(K) - \underline{R}(K)$.

(xii) $\overline{R}(K) - \overline{R}_h(K)$.

Remark 1. *In Figure 1, the study of h-approximation spaces is a generalization of the study of approximation spaces. This extension is evident as elements within the regions $[\underline{R}_h(K) - \underline{R}(K)]$ are well defined within K, contrasting with their undefined nature in Pawlak's approximation spaces. Furthermore, elements within the region $[\overline{R}(K) - \overline{R}_h(K)]$ lie outside of K, addressing a prior lack of clarity in Pawlak's spaces.*

Our paper involves redefining the boundary of K in Pawlak's approximation space as the h-boundary of K. Additionally, we expand the exterior of K, encompassing elements not belonging to K, termed as the h-exterior of K.

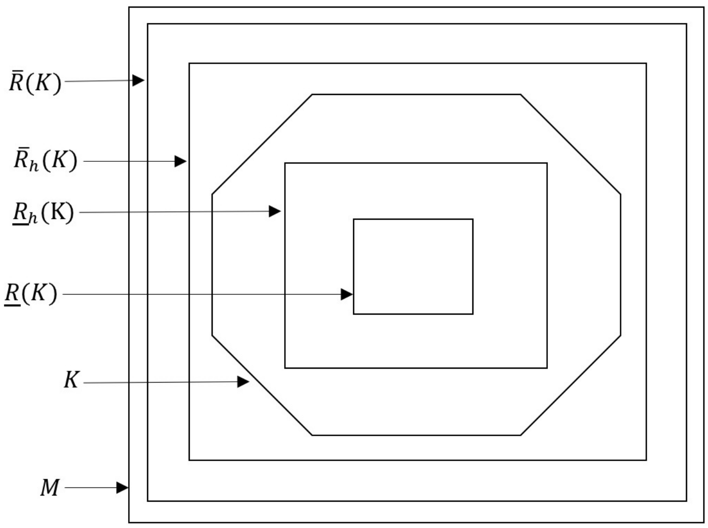

Figure 1: Representation of h-approximation spaces.

Proposition 1. *Let (M, R_h) be h-approximation spaces and $K \subseteq M$, then the following statements hold:*

(i) $b(K) = \underline{Edg}(K) \cup \overline{Edg}(K)$.

(ii) $b_h(K) = \underline{Edg}_h(K) \cup \overline{Edg}_h(K)$.

(iii) $\overline{R}(K) - \underline{R}_h(K) = \overline{Edg}(K) \cup \underline{Edg}_h(K)$.

(iv) $\overline{R}_h(K) - \underline{R}(K) = \overline{Edg}_h(K) \cup \underline{Edg}(K)$.

(v) $\underline{Edg}(K) = \underline{Edg}_h(K) \cup (\underline{R}_h(K) - \underline{R}(K))$.

(vi) $\overline{Edg}(K) = \overline{Edg}_h(K) \cup (\overline{R}(K) - \overline{R}_h(K))$.

Proof.

(i) Clear.

(ii) It follows from

$$\begin{aligned} b_h(K) &= \overline{R}_h(K) - \underline{R}_h(K) \\ &= (\overline{R}_h(K) - K) \cup (K - \underline{R}_h(K)) \\ &= \underline{Edg}_h(K) \cup \overline{Edg}_h(K). \end{aligned}$$

(iii) (iv), (v), and (vi) are obvious.

Definition 8. *If (M, R_h) be a h-approximation space and $K \subseteq M$. Then there are memberships which are defined by:*

(i) The h-strong membership is denoted by $\underline{\in}_h$, ($m \underline{\in}_h K \Leftrightarrow m \in \underline{R}_h(K)$).

(ii) The h-weak membership is denoted by $\overline{\in}_h$, ($m \overline{\in}_h K \Leftrightarrow m \in \overline{R}_h(K)$).

Remark 2. *Based on the Definition 8. we can be written h-lower and h-upper approximations of a set $K \subseteq M$ as*

(i) $\underline{R}_h(K) = \{m \in K : m \underline{\in}_h K\}$.

(ii) $\overline{R}_h(K) = \{m \in K : m \overline{\in}_h K\}$.

Proposition 2. *If (M, R_h) is an h-approximation space and $K \subseteq M$. Then*

(i) $m \underline{\in} K \Rightarrow m \underline{\in}_h K$.

(ii) $m \overline{\in}_h K \Rightarrow m \overline{\in} K$.

The converse of Proposition 2 may not be true in general as seen in the following example

Example 4. *Let $M = \{k, q, s, t\}$ be a universe and a relation R defined by $R = \{(k, k), (t, s), (t, t), (s, k), (s, t), (s, s)\}$, thus $kR = \{k\}$, $qR = \phi$, $sR = \{k, s, t\}$ and $tR = \{s, t\}$. Consequently, the topology associated with this relation is $\sigma = \{M, \phi, \{k\}, \{s, t\}, \{k, s, t\}\}$. So (M, R_h) is a h-approximation space. Let $K = \{q, s, t\}$, we have $q \underline{\in}_h K$ but $q \underline{\notin} K$. Also, let $R = \{k\}$. We have $q \overline{\in} R$ but $q \overline{\notin}_h K$.*

Definition 9. *If M is a finite none-empty universe , $K \subseteq M$ and $K \neq \phi$, then We can express the degree of completeness using a novel metric termed the h-accuracy measure, defined as follows:*

$$\alpha_{R_h}(K) = \frac{| \underline{R}_h(K) |}{| \overline{R}_h(K) |}$$

Example 5. *In Example 2, we can deduce the following table showing the degree of accuracy measure $\alpha_R(K)$ and h-accuracy measure $\alpha_{R_h}(K)$ for some sets.*

Table 1: The degree of accuracy measure and h-accuracy measure.

The set $K \subseteq M$	$\alpha_R(K)$	$\alpha_{R_h}(K)$
$\{k\}$	$\frac{1}{2}$	1
$\{t\}$	$\frac{1}{3}$	1
$\{k,q\}$	$\frac{1}{3}$	$\frac{1}{2}$
$\{k,s\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{k,t\}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{q,t\}$	$\frac{2}{3}$	$\frac{2}{3}$
$\{s,t\}$	$\frac{1}{3}$	$\frac{2}{3}$
$\{k,q,s\}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{k,q,t\}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{k,s,t\}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{q,s,t\}$	$\frac{1}{2}$	1

The degree of exactness of set $K = \{k\}$ is observed to be 50% using the accuracy measure and 100% using the h-accuracy measure. Thus, it follows that the h-accuracy measure outperforms the accuracy measure in this particular case.

4. h-rough equality and h-rough inclusion

In this section, the focus is on exploring h-rough equality and h-rough inclusion, drawing from the groundwork laid by Pawlak and Novotny ([11],[9]) in their introduction of rough equality and inclusion.

Definition 10. *If (M, R_h) is a h-approximation space and $K, Q \subseteq M$. Then K and Q are called:*

(i) h-roughly bottom equal ($K \eqsim_h Q$) if $\underline{R}_h(K) = \underline{R}_h(Q)$.

(ii) h-roughly top equal ($K \simeq_h Q$) if $\overline{R}_h(K) = \overline{R}_h(Q)$.

(iii) h-roughly equal ($K \approx_h Q$) if ($K \eqsim_h Q$) and ($K \simeq_h Q$).

Example 6. *In Example 2, we have the sets $\{k,s\}$, $\{k,q,s\}$ are h-roughly bottom equal and $\{s,t\}$, $\{q,s,t\}$ are h-roughly top equal.*

It's straightforward to demonstrate that $\approx_h$ forms an equivalence relation on $P(M)$, making the pair $(P(M), \approx_h)$ an approximation space. Additionally, this relation, $\approx_h$, is termed as the h-rough equality within the h-approximation space (M, R_h).

Definition 11. *Let (M, R_h) be a h-approximation space. The equivalence relation E_h on the set $P(M)$ is defined by the following condition:*

$$(K,Q) \in E_h \textit{ if } Int_h(K) = Int_h(Q) \textit{ and } Cl_h(K) = Cl_h(Q).$$

The equivalence relation E_h is identical to $\approx_h$, given that $\underline{R}_h(K) = Int_h(K)$ and $\overline{R}_h(K) = Cl_h(K)$

Remark 3. *Denoting the equivalence class of the relation ($\approx_h$ or E_h) containing any subset K of M as $[K]_{\approx_h}$ or $[K]_{E_h}$. We can conclude that:*

$$[K]_{\approx_h} = \{Q \subset M : \underline{R}_h(Q) = \underline{R}_h(K) \textit{ and } \overline{R}_h(Q) = \overline{R}_h(K)\}.$$

We denote by $R_h(M)$ the family of h-rough classes in a h-approximation space (M, R_h).

Definition 12. *If (M, R_h) be a h-approximation space and $K, Q \subseteq M$. Then:*

(i) K is h-roughly bottom included in Q ($K \underset{\sim}{\subset}_h Q$) if $\underline{R}_h(K) \subseteq \underline{R}_h(Q)$.

(ii) K is h-roughly top included in Q ($K \tilde{\subset}_h Q$) if $\overline{R}_h(K) \subseteq \overline{R}_h(Q)$.

(iii) K is h-roughly included in Q ($K \underset{\sim}{\tilde{\subset}}_h Q$) if ($K \underset{\sim}{\subset}_h Q$) and ($K \tilde{\subset}_h Q$).

Example 7. *In Example 2, we have $\{k, s\}$ is h-roughly bottom included in $\{k, q, s\}$. Also, $\{s, t\}$ is h-roughly top included in $\{q, s, t\}$.*

5. h-rough sets

In this section, we introduce a new concept known as the h-rough set, and we illustrate its properties and provide examples.

Definition 13. *Let (M, R_h) be h-approximation space and the set $K \subseteq M$ is called:*

(i) R_h-definable (h-exact) if $\overline{R}_h(K) = \underline{R}_h(K)$ or $b_h(K) = \phi$.

(ii) h-rough if $\overline{R}_h(K) \neq \underline{R}_h(K)$ or $b_h(K) \neq \phi$.

Example 8. *In Example 4, consider the h-approximation space (M, R_h). Here, the set $\{q, s, t\}$ is h-exact, whereas $\{q\}$ is h-rough set.*

Proposition 3. *Let (M, R_h) be a h-approximation space. Then:*

(i) Every exact set in M is h-exact.

(ii) Every h-rough set in M is rough.

Proof. Clear.

The converse of all parts of Proposition 3 may not hold in general as demonstrated in the following example.

Example 9. *In Example 4, If we consider the h-approximation space (M, R_h). Then the set $\{q, s, t\}$ is h-exact but not exact and the set $\{k\}$ is rough but not h-rough.*

Remark 4. *The intersection of two h-exact sets may not necessarily result in a h-exact set.*

Example 10. *In Example 4, consider the h-approximation space* (M, R_h). *We have* $\{q, s, t\}$ *and* $\{k\}$ *are two h-exact sets but* $\{q, s, t\} \cap \{k\} = \phi$ *does not h-exact.*

Definition 14. *If* (M, R_h) *is a h-approximation space, then the set* $K \subseteq M$ *is called:*

(i) Roughly R_h*-definable, if* $\underline{R}_h(K) \neq \phi$ *and* $\overline{R}_h(K) \neq M$.

(ii) Internally R_h*-undefinable, if* $\underline{R}_h(K) = \phi$ *and* $\overline{R}_h(K) \neq M$.

(iii) Externally R_h*-undefinable, if* $\underline{R}_h(K) \neq \phi$ *and* $\overline{R}_h(K) = M$.

(iv) Totally R_h*-undefinable, if* $\underline{R}_h(K) = \phi$ *and* $\overline{R}_h(K) = M$.

The set of all roughly R_h*-definable (resp. internally* R_h*-undefinable, externally* R_h*-undefinable and totally* R_h*-undefinable) sets is denoted by* $RD_h(M)$ *(resp.* $IUD_h(M)$, $EUD_h(M)$ *and* $TUD_h(M)$*).*

Remark 5. *Let* (M, R_h) *be any h-approximation space. Then the following are hold:*

(i) $RD_h(M) \supseteq RD(M)$.

(ii) $IUD_h(M) \subseteq IUD(M)$.

(iii) $EUD_h(M) \subseteq EUD(M)$.

(iv) $TUD_h(M) \subseteq TUD(M)$.

Example 11. *In Example 3, we have the set* $\{k, s\} \in RD_h(M)$ *but* $\{k, s\} \notin RD(M)$. *The set* $\{s\} \in IUD(M)$ *but* $\{s\} \notin IUD_h(M)$. *Also, the set* $\{q, s\} \in EUD(M)$ *but* $\{q, s\} \in EUD_h(M)$.

Proposition 4. *Let* (M, R_h) *be any h-approximation space and for all* $m, n \in M$, *if* $m \in \overline{R}_h(\{n\})$ *and* $n \in \overline{R}_h(\{m\})$, *then it implies that* $\overline{R}_h(\{m\}) = \overline{R}_h(\{n\})$.

Proof. According to the definition, the h-upper approximation of a set is the h-closure of that set. Given that $Cl_h(\{n\})$ is a h-closed set containing m (based on the condition) and $Cl_h(\{m\})$ is the smallest h-closed set containing m, it follows that $Cl_h(\{m\}) \subseteq Cl_h(\{n\})$. Consequently, $\overline{R}_h(\{m\}) \subseteq \overline{R}_h(\{n\})$. Symmetrically, The reverse inclusion holds: $Cl_h(\{n\}) \subseteq Cl_h(\{m\})$. Thus, $\overline{R}_h(\{n\}) \subseteq \overline{R}_h(\{m\})$, completing the proof.

Proposition 5. *Let* (M, R_h) *be a h-approximation space, where every h-open subset K of M is h-closed. If* $n \in \overline{R}_h(\{m\})$ *,then it implies that* $m \in \overline{R}_h(\{n\})$ *for all* $m, n \in M$.

Proof. If $m \notin \overline{R}_h(\{n\})$, then there exists a h-open set H containing m such that $H \cap \{m\} = \phi$, implying that $\{n\} \subseteq (M \setminus H)$. However, $(M \setminus H)$ is both a h-closed set and also is a h-open set that does not contain m. Therefore, $(M \setminus H) \cap \{m\} = \phi$, which means $n \neq \overline{R}_h(\{m\})$.

Proposition 6. *Let (M, R_h) be a h-approximation space, where every h-open subset K of M is h-closed. Then the family of sets $\{\overline{R}_h(\{m\}) : m \in K\}$ is a partition of the set M.*

Proof. If $m, n, p \in K$ and $p \in \overline{R}_h(\{m\}) \cap \overline{R}_h(\{n\})$, then $p \in \overline{R}_h(\{m\})$ and $p \in \overline{R}_h(\{n\})$. Consequently, by Proposition 5, $m \in \overline{R}_h(\{p\})$ and $n \in \overline{R}_h(\{p\})$. By Proposition 4, it follows that $\overline{R}_h(\{m\}) = \overline{R}_h(\{p\})$ and $\overline{R}_h(\{n\}) = \overline{R}_h(\{p\})$. Therefore $\overline{R}_h(\{m\}) = \overline{R}_h(\{n\}) = \overline{R}_h(\{p\})$. Hence either $\overline{R}_h(\{m\}) = \overline{R}_h(\{n\})$ or $\overline{R}_h(\{m\}) \cap \overline{R}_h(\{n\}) = \phi$.

6. Properties of h-approximation spaces

In this section, we introduce some properties of h-approximation spaces and provide counterexamples.

Proposition 7. *Let (M, R_h) be h-approximation space and $K, Q \subseteq M$. Then*

(i) $\underline{R}_h(K) \subseteq K \subseteq \underline{R}_h(K)$.

(ii) $\underline{R}_h(\phi) = \overline{R}_h(\phi) = \phi$, $\underline{R}_h(M) = \overline{R}_h(M) = M$.

(iii) *If $K \subseteq Q$ then $\underline{R}_h(K) \subseteq \underline{R}_h(Q)$ and $\overline{R}_h(K) \subseteq \overline{R}_h(Q)$.*

Proof.

(i) Let $m \in \underline{R}_h(K)$ which mean that $m \in \cup\{H \in \sigma^h, H \subseteq K\}$. Then there exists $H_0 \in \sigma^h$ such that $m \in H_0 \subseteq K$. Thus $m \in K$. Hence $\underline{R}_h(K) \subseteq K$. Also, let $m \in M$ and by definition of $\overline{R}_h(K) = \cap\{F \in \sigma^{hc}, K \subseteq F\}$, then $m \in F$ for all $F \in \sigma^{hc}$. Hence $K \subseteq \overline{R}_h(K)$.

(ii) It directly follows.

(iii) Let $m \in \underline{R}_h(K)$, by definition of h-lower approximation of K, we have $m \in \cup\{H \in \sigma^h, H \subseteq K\}$ but $K \subseteq Q$, thus $H \subseteq Q$ and $m \in H$, then $m \in \underline{R}_h(Q)$. Also, let $m \neq \overline{R}_h(Q)$ this means that $m \neq \cap\{F \in \sigma^{hc}, Q \subseteq F\}$ then, there exists $F \in \sigma^{hc}$,$Q \subseteq F$ and $m \notin F$ which means that, there exists $F \in \sigma^{hc}$, $K \subseteq Q \subseteq F$ and $m \notin F$ which implies $m \notin \cap\{F \in \sigma^{hc}, K \subseteq F\}$, thus $m \notin \overline{R}_h(K)$. Therefore $\overline{R}_h(K) \subseteq \overline{R}_h(Q)$.

Proposition 8. *Let (M, R_h) be a h-approximation space and $K, Q \subseteq M$. Then*

(i) $\underline{R}_h(M \setminus K) = M \setminus \overline{R}_h(K)$.

(ii) $\overline{R}_h(X \setminus K) = M \setminus \underline{R}_h(K)$.

(iii) $\underline{R}_h(\underline{R}_h(K)) = \underline{R}_h(K)$.

(iv) $\overline{R}_h(\overline{R}_h(K)) = \overline{R}_h(K)$.

(v) $\underline{R}_h(\underline{R}_h(K)) \subseteq \overline{R}_h(\underline{R}_h(K))$.

(vi) $\underline{R}_h(\overline{R}_h(K)) \subseteq \overline{R}_h(\overline{R}_h(K))$.

Proof.

(i) Let $m \in \underline{R}_h(M \setminus K)$ which is equivalent to $m \in \cup\{H \in \sigma^h, H \subseteq M \setminus K\}$. So there exists $H_0 \in \sigma^h$ such that $m \in H_0 \subseteq M \setminus K$. Then there exists H_0^c such that $K \subset H_0^c$ and $m \notin H_0^c$, $H_0^c \in \sigma^{hc}$. Thus, $m \notin R_h(K)$. So $m \in M \setminus \overline{R}_h(K)$. Therefore $\underline{R}_h(M \setminus K) = M \setminus \overline{R}_h(K)$.

(ii) Comparable to (i)

(iii) Since $\underline{R}_h(K) = \cup\{H \in \sigma^h, H \subseteq K\}$. This implies that $\underline{R}_h(\underline{R}_h(K)) = \cup\{\cup\{H \in \sigma^h, H \subseteq K\}\} = \cup\{H \in \sigma^h, H \subseteq K\} = \underline{R}_h(K)$.

(iv) $\overline{R}_h(\overline{R}_h(K)) = \overline{R}_h(M \setminus \underline{R}_h(M \setminus K)) = M \setminus \underline{R}_h(M \setminus \underline{R}_h(M \setminus K))$. From (i), (ii) and (iii), we get $\overline{R}_h(\overline{R}_h(K)) = M \setminus \underline{R}_h(M \setminus K) = M \setminus (M \setminus \overline{R}_h(K)) = \overline{R}_h(K)$.

(v) Since $\underline{R}_h(K) \subseteq \overline{R}_h(\underline{R}_h(K))$ and by (iii) we have $\underline{R}_h(\underline{R}_h(K)) = \underline{R}_h(K)$, then $\underline{R}_h(\underline{R}_h(K)) \subseteq \overline{R}_h(\underline{R}_h(K))$.

(vi) Since $\underline{R}_h(\overline{R}_h(K)) \subseteq \overline{R}_h(K)$ and by (iv), we have $\overline{R}_h(\overline{R}_h(K)) = \overline{R}_h(K)$, then $\underline{R}_h(\overline{R}_h(K)) \subseteq \overline{R}_h(\overline{R}_h(K))$.

Proposition 9. *Let (M, R_h) be a h-approximation space and $K, Q \subseteq M$. Then*

(i) $\underline{R}_h(K \cup Q) \supseteq \underline{R}_h(K) \cup \underline{R}_h(Q)$.

(ii) $\overline{R}_h(K \cup Q) \supseteq \overline{R}_h(K) \cup \overline{R}_h(Q)$.

(iii) $\underline{R}_h(K \cap Q) \subseteq \underline{R}_h(K) \cap \underline{R}_h(Q)$.

(iv) $\overline{R}_h(K \cap Q) \subseteq \overline{R}_h(K) \cap \overline{R}_h(Q)$.

Proof.

(i) Since we have $K \subseteq K \cup Q$ and $Q \subseteq K \cup Q$. Then $\underline{R}_h(K) \subseteq \underline{R}_h(K \cup Q)$ and $\underline{R}_h(Q) \subseteq \underline{R}_h(K \cup Q)$ by (iii) in Proposition 7, then $\underline{R}_h(K \cup Q) \supseteq \underline{R}_h(K) \cup \underline{R}_h(Q)$.

(ii) (iii) and (iv) Similar to (i).

The equality of all parts in Proposition 9 does not hold, as demonstrated in the following example.

Example 12. *In Example 2:*

(i) If $K = \{t\}$, $Q = \{k, q\}$, then we have $\underline{R}_h(K \cup Q) = \{k, q, t\}$, $\underline{R}_h(K) = \{t\}$, $\underline{R}_h(Q) = \{k\}$. Therefore $\underline{R}_h(K \cup Q) \neq \underline{R}_h(K) \cup \underline{R}_h(Q)$.

(ii) *If* $K = \{t\}$, $Q = \{k, q\}$, *then we have* $\overline{R}_h(K \cup Q) = M$, $\overline{R}_h(K) = \{q, s, t\}$, $\overline{R}_h(Q) = \{k, q\}$.*Therefore* $\overline{R}_h(K) \cup \overline{R}_h(Q) \neq \overline{R}_h(K \cup Q)$.

(iii) *If* $K = \{k, q, s\}$, $Q = \{q, s, t\}$, *then we have* $\underline{R}_h(K \cap Q) = \phi$, $\underline{R}_h(K) = \{k\}$ *and* $\underline{R}_h(Q) = \{q, s, t\}$. *Therefore* $\underline{R}_h(K \cap Q) \neq \underline{R}_h(K) \cap \underline{R}_h(Q)$.

(iv) *If* $K = \{k\}$, $Q = \{q, t\}$, *then we have* $\overline{R}_h(K \cap Q) = \phi$, $\overline{R}_h(K) = \{k\}$, $\overline{R}_h(Q) = \{q, s, t\}$.*Therefore* $\overline{R}_h(K) \cap \overline{R}_h(Q) \neq \overline{R}_h(K \cap Q)$.

The following theorems are generalization of Proposition 9.

Proposition 10. *Let* (M, R_h) *be a h-approximation space and* $K, Q \subseteq M$. *If* K *is* R_h-*definable. Then the following are hold.*

(i) $\underline{R}_h(K \cup Q) = \underline{R}_h(K) \cup \underline{R}_h(Q)$.

(ii) $\overline{R}_h(K \cap Q) = \overline{R}_h(K) \cap \overline{R}_h(Q)$.

Proof.

(i) It is evident that $\underline{R}_h(K) \cup \underline{R}_h(Q) \subseteq \underline{R}_h(K \cup Q)$. For the converse inclusion, let $m \in \underline{R}_h(K \cup Q)$, that means, $m \in \cup\{H \in \sigma^h, H \subseteq K \cup Q\}$. Then there exists $H_0 \in \sigma^h$ such that $m \in H_0 \subset K \cup Q$. We distinguish three cases:
Case (1) If $H_0 \subset K$, $m \in H_0$ and H_0 is a h-open set, then $m \in \underline{R}_h(K)$.
Case (2) If $H_0 \cap K = \phi$, then $H_0 \subseteq Q$ and $m \in H_0$, thus $m \in \underline{R}_h(Q)$.
Case (3) If $H_0 \cap K \neq \phi$. Since $m \in H_0$ and H_0 is an h-open set, then $m \in Cl_h(K)$, for every H_0 Which satisfies the aforementioned condition, thus $m \in \overline{R}_h(K)$, then $m \in \underline{R}_h(K)$, because K is R_h- definable. Therefore, in three cases $m \in \underline{R}_h(K) \cup \underline{R}_h(Q)$.

(ii) It is evident that $\overline{R}_h(K \cap Q) \subseteq \overline{R}_h(K) \cap \overline{R}_h(Q)$. We prove the converse inclusion. Let $m \in \overline{R}_h(K) \cap \overline{R}_h(Q)$, then $m \in \overline{R}_h(K)$ implies $m \in \underline{R}_h(K)$ and $m \in H \subseteq M$, where H is an h-open set and $m \in \overline{R}_h(Q)$ implies for all $H \in \sigma^h$, $H \cap Q \neq \phi$. Therefore $H \cap (K \cap Q) = (H \cap K) \cap Q = H \cap N \neq \phi$. Hence $m \in \overline{R}_h(K \cap Q)$.

Proposition 11. *If* (M, R_h) *is a h-approximation space and* $K, Q \subseteq M$. *Then the following are hold.*

(i) $\overline{R}_h(Cl(K) \cup Q) = Cl(K) \cup \overline{R}_h(Q)$.

(ii) $\underline{R}_h(Int(K) \cap Q) = Int(K) \cap \underline{R}_h(Q)$.

Proof.

(i) Based on Proposition 7 (i) and Proposition 9 (ii), we have $Cl(K) \subset \overline{R}_h(Cl(K))$.Then $Cl(K) \cup \overline{R}_h(Q) \subset \overline{R}_h(Cl(K)) \cup \overline{R}_h(Q) \subset \overline{R}_h(Cl(K) \cup Q)$. Conversely, since $Cl(K) \cup Q \subset Cl(K) \cup \overline{R}_h(Q)$ and the union of an h-open set and a closed set is h-closed,then $\overline{R}_h(Cl(K) \cup Q) \subset \overline{R}_h(Cl(K) \cup \overline{R}_h(Q)) = Cl(K) \cup \overline{R}_h(Q)$. Therefore, $\overline{R}_h(Cl(K) \cup Q) = Cl(K) \cup \overline{R}_h(Q)$.

(ii) Considering the intersection of an open set $Int(K)$ and an h-open set $\underline{R}_h(Q)$ is h-open,$Int(K) \cap \underline{R}_h(Q) = \underline{R}_h(Int(K) \cap \underline{R}_h(Q)) \subset \underline{R}_h(Int(K) \cap Q)$. Conversely, by using Proposition 9 (iii), $\underline{R}_h(Int(K) \cap Q) \subset \underline{R}_h(Int(K)) \cap \underline{R}_h(Q) \subset Int(K) \cap \underline{R}_h(Q)$. Therefore $\underline{R}_h(Int(K) \cap Q) = Int(K) \cap \underline{R}_h(Q)$.

7. Conclusions

This paper introduced h-rough sets, an extension of rough set theory, by incorporating h-open sets to define h-lower and h-upper approximations. Alongside these new approximations, we explored h-rough equality and h-rough inclusion, thoroughly examining the properties of h-approximation spaces. Our findings illustrate that h-rough sets provide improved precision and flexibility in data approximation and analysis. Future research should focus on integrating h-rough sets with fuzzy set theory to better manage uncertainty, developing efficient algorithms for processing large-scale data, and combining h-rough sets with neural networks and decision trees for enhanced decision-making processes. Additionally, applying h-rough sets to machine learning tasks such as feature selection, clustering, and classification holds significant potential. Further theoretical advancements, particularly in exploring the topological properties of h-rough sets, will continue to expand and deepen the utility of rough set theory, ensuring its ongoing relevance and effectiveness in modern data analysis.

Acknowledgements

Gratitude is extended to my late parents, whose unwavering support and encouragement continue to inspire and drive my pursuit of scientific inquiry, forever embedded in the foundation of this research.

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