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Bipolar Fuzzy Commutative Ideals in BCK-algebras

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Abstract. This paper presents the notions of bipolar fuzzy commutative ideals of a BCK-algebra. We also study various properties regarding this concept. We prove various characterisations for bipolar fuzzy commutative ideals. Moreover, a relationship between bipolar fuzzy commutative ideals and bipolar fuzzy positive implicative ideals is presented. In addition, we present the notation of bipolar fuzzy characteristic commutative ideal. Then a relationship between a bipolar fuzzy characteristic commutative ideal and its level-cut commutative ideals is provided.

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Key Words and Phrases: Bipolar fuzzy ideal, commutative ideal, positive implicative ideal, characteristic ideal

1. Introduction

Zadeh introduced the notation of fuzzy sets [16]. In [6], BCK-algebra was provided as a logical class of algebras. This encouraged many researchers to apply fuzzy sets to BCK-algebras, see [8], [12], [13], [14], [15] and [10]

Then many concepts which related to fuzzy sets have been widely investigated. One of the most interesting concepts is a bipolar fuzzy set. The importance of it lies behind its properties and applications, for example see [3]. Many papers provide the study of bipolar fuzzy set theory in several algebraic structures see [5], [2], [11], [1], [9] and [4].

This paper provides bipolar fuzzy commutative ideals of a BCK-algebra. We also study various properties regarding this concept. We prove various characterisations for bipolar fuzzy commutative ideals. Moreover, we provide a relationship between bipolar fuzzy commutative ideals and bipolar fuzzy positive implicative ideals. In addition, we present the notation of bipolar fuzzy characteristic commutative ideal and give a relationship

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between a bipolar fuzzy characteristic commutative ideal and its level-cut commutative ideals.

Our motivation is to investigate bipolar fuzzy commutative ideals in BCK-algebras and their properties. This can lead to study this concept in soft case and apply the results in decision making algorithms.

2. Preliminaries

In [7], a BCK-algebra is defined as follows:

Consider the algebra $(X, *, 0)$. Suppose that, for all $a, g, w \in X$, we have
(BCK1) $((a * g) * (a * w) * (w * g) = 0)$,
(BCK2) $(a * (a * g)) * g = 0$,
(BCK3) $a * a = 0$,
(BCK4) $0 * a = 0$,
(BCK5) If $a * g = 0$ and $g * a = 0$, then we have $f = a$.
Then X is a BCK-algebra.

The subsequent properties are applicable to the operation $*$, for all $a, g, w \in X$:

$$(a * g) * w = (a * w) * g \tag{1}$$

$$(a * w) * (a * (a * w)) = (a * w) * w \tag{2}$$

We will use these properties in later sections.

A relation $\leq$ is defined on X as follows:

$$a \leq g \Longleftrightarrow a * g = 0$$

Definition 1. *[7] An ideal I of X is a non-empty subset satisfying two conditions:*

(1) $0 \in I$.

(2) If $a * g \in I$ and $g \in I$, then $a \in I$.

Definition 2. *[10] A bipolar fuzzy set $\varrho := (X, \varrho^+, \varrho^-)$ that satisfies the following conditions:*

(1) $\varrho^-(0) \leq \varrho^-(a)$, $\varrho^+(0) \geq \varrho^+(a)$ *for all* $a \in X$.

(2) $\varrho^-(a) \leq$ *max* $\{\varrho^-(a*g), \varrho^-(g)\}$ *and* $\varrho^+(a) \geq$ *min* $\{\varrho^+(a*g), \varrho^+(g)\}$ *for all* $a, g \in X$.

is called a bipolar fuzzy ideal.

Definition 3. *[10] A bipolar fuzzy subset $\varrho := (X, \varrho^+, \varrho^-)$ is called a bipolar fuzzy subalgebra if it meets two criteria for all $a, g \in X$:*

(1) $\varrho^+(a * g) \geq \min\{\varrho^+(a), \varrho^+(g)\}$.

(2) $\varrho^-(a * g) \leq \max\{\varrho^-(a), \varrho^-(g)\}$.

This paper interests in some types of bipolar fuzzy ideals of a BCK-algebra.

3. Bipolar fuzzy commutative ideal

Let $\varrho := (X, \varrho^+, \varrho^-)$ be a bipolar fuzzy set. A bipolar fuzzy set in X is said to be a bipolar fuzzy commutative ideal of X if

(1) $\varrho^+(0) \geq \varrho^+(a)$ and $\varrho^-(0) \leq \varrho^-(a)$.

(2) $\varrho^+(a * (g * (g * a))) \geq \min\{\varrho^+((a * g) * w), \varrho^+(w)\}$.

(3) $\varrho^-(a * (g * (g * a))) \leq \max\{\varrho^-((a * g) * w), \varrho^-(w)\}$.

for all $a, g, w \in X$:

Example 1. *Let $(X; *, 0)$ be a BCK-algebra given in Table 1 as follows:*

*	0	r	s	t
0	0	0	0	0
r	r	0	0	r
s	s	r	0	s
t	t	t	t	0

Table 1: : Tabular representation of a BCK-algebra X in example 1

Consider the bipolar fuzzy set $\varrho := (X, \varrho^+, \varrho^-)$ represented by:

*	0	r	s	t
ϱ^-	-0.5	-0.5	-0.5	-0.4
ϱ^+	0.8	0.8	0.8	0.6

Table 2: : Tabular representation of a bipolar fuzzy set in example 1

Direct calculations implies that $\varrho := (X, \varrho^+, \varrho^-)$ is a bipolar fuzzy commutative ideal of X.

The following theorem provides equivalent statements regarding a bipolar fuzzy commutative ideal.

Theorem 1. *Let X be a BCK-algebra and $\varrho := (X, \varrho^+, \varrho^-)$ a bipolar fuzzy ideal of X. Then ϱ is a bipolar fuzzy commutative ideal if and only if*

$$\begin{aligned} \varrho^+(a * (g * (g * a))) &\geq \varrho^+(a * g) \\ \varrho^-(a * (g * (g * a))) &\leq \varrho^-(a * g) \end{aligned} \tag{3}$$

Proof. Suppose that ϱ is a bipolar fuzzy commutative ideal. Thus, by definition, we have

$$\varrho^+(a * (g * (g * a))) \geq \min\{\varrho^+((a * g) * w), \varrho^+(w)\},$$

$$\varrho^-(a*(g*(g*a))) \leq \max\{\varrho^-((a*g)*w), \varrho^-(w)\}.$$

Set $w = 0$, use the fact that $a*0 = 0$ for all $a \in X$ and condition 1 from definition, the condition in (1) holds.
Conversely, assume that ϱ satisfies the conditions in (1). That ϱ is a bipolar fuzzy ideal implies that

$$\varrho^+(a*g) \geq \min\{\varrho^+((a*g)*w), \varrho^+(w)\},$$
$$\varrho^-(a*g) \leq \max\{\varrho^-((a*g)*w), \varrho^-(w)\}.$$

Applying (1),

$$\varrho^+(a*(g*(g*a))) \geq \min\{\varrho^+((a*g)*w), \varrho^+(w)\},$$
$$\varrho^-(a*(g*(g*a))) \leq \max\{\varrho^-((a*g)*w), \varrho^-(w)\},$$

and so ϱ is a bipolar fuzzy commutative ideal.

Theorem 2. *Every bipolar fuzzy commutative ideal of a BCK-algebra X is a bipolar fuzzy ideal of X.*

Proof. Let ϱ be a bipolar fuzzy commutative ideal of X and $a, w \in X$, then we have

$$\begin{aligned}\min\{\varrho^+(a*w), \varrho^+(w)\} &= \min\{\varrho^+((a*0)*w), \varrho^+(w)\}\\ &\leq \varrho^+(a*(0*(0*a)))\\ &= \varrho^+(a)\end{aligned}$$

and

$$\begin{aligned}\max\{\varrho^-(a*w), \varrho^-(w)\} &= \max\{\varrho^-((a*0)*w), \varrho^-(w)\}\\ &\geq \varrho^-(a*(0*(0*w)))\\ &= \varrho^-(a)\end{aligned}$$

Therefore, ϱ is a bipolar fuzzy ideal.

The following corollary results from the aforementioned theorem:

Corollary 1. *Every bipolar fuzzy commutative ideal of a BCK-algebra X is a bipolar fuzzy subalgebra of X.*

Proof. Clear.

Remark 1. *A bipolar fuzzy ideal of a BCK-algebra need not be a bipolar fuzzy commutative ideal.*

A counter example is given as follows:

Consider the BCK-algebra X given in Table 3:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Table 3: : Tabular representation of a BCK-algebra X described above

*	0	1	2	3	4
ϱ^-	-0.6	-0.5	-0.5	-0.4	-0.4
ϱ^+	0.8	0.7	0.7	0.7	0.6

Table 4: : Tabular representation of a bipolar fuzzy set ϱ described above

Let $\varrho := (X, \varrho^+, \varrho^-)$ be a bipolar fuzzy set in X given by Table 4:

Thus ϱ is a bipolar fuzzy ideal. However, ϱ is not a bipolar fuzzy commutative ideal as

$$\varrho^+(2 * (3 * (3 * 2))) = 0.7 < 0.8 = \min\{\varrho^+((2 * 3) * 0), \varrho^+(0)\}.$$

Proposition 1. *Every bipolar fuzzy commutative ideal of a BCK-algebra is order preserving.*

Proof. Assume that $\varrho := (X, \varrho^+, \varrho^-)$ is a bipolar fuzzy commutative ideal and let $a, g, w \in X$ in which $a \leq w$. Then $a * w = 0$. That

$$\varrho^+(a * (g * (g * a))) \geq \min\{\varrho^+((a * g) * w), \varrho^+(w)\},$$

$$\varrho^-(a * (g * (g * a))) \leq \max\{\varrho^-((a * g) * w), \varrho^-(w)\},$$

and setting $g = 0$ implies that

$$\begin{aligned} \varrho^+(a) &= \varrho^+(a * (0 * (0 * a))) \\ &\geq \min\{\varrho^+((a * 0) * w), \varrho^+(w)\} \\ &= \min\{\varrho^+(a * w), \varrho^+(w)\} \\ &= \min\{\varrho^+(0), \varrho^+(w)\} \\ &= \varrho^+(w), \end{aligned}$$

and

$$\begin{aligned} \varrho^-(a) &= \varrho^-(a * (0 * (0 * a))) \\ &\leq \max\{\varrho^-((a * 0) * w), \varrho^-(w)\} \\ &= \max\{\varrho^-(a * w), \varrho^-(w)\} \\ &= \max\{\varrho^-(0), \varrho^-(w)\} \end{aligned}$$

$$= \varrho^{-}(w)$$

Therefore, $\varrho := (X, \varrho^{+}, \varrho^{-})$ is order preserving as required.

Lemma 1. *Suppose that $\varrho := (X, \varrho^{+}, \varrho^{-})$ is a bipolar fuzzy ideal. If $a * g \leq w$ holds for some $a, g, w \in X$, then*

$$\varrho^{+}(a) \geq \{\varrho^{+}(g), \varrho^{+}(w)\},$$
$$\varrho^{-}(a) \leq \ max\ \{\varrho^{-}(g), \varrho^{-}(w)\}.$$

Recall that a BCK-algebra X is commutative if it satisfies the condition:

$$f * (f * g) = g * (g * f).$$

Theorem 3. *Let X be a commutative BCK-algebra. Then every bipolar fuzzy ideal of X is a bipolar fuzzy commutative ideal.*

Proof. Let $\varrho := (X, \varrho^{+}, \varrho^{-})$ be a bipolar fuzzy ideal and $a, g, w \in X$. Then

$$\begin{aligned}((a * (g * (g * a))) * ((a * g) * w)) * w &= ((a * (g * (g * a))) * w) * ((a * g) * w) \\ &\leq (a * (g * (g * a))) * (a * g) \\ &= (a * (a * g)) * (g * (g * a)) \\ &= 0,\end{aligned}$$

and so

$$(a * (g * (g * a))) * ((a * g) * w) \leq w.$$

Applying lemma 1,

$$\varrho^{+}(a * (g * (g * a))) \geq \ \min\{\varrho^{+}((a * g) * w), \varrho^{+}(w)\},$$

$$\varrho^{-}(a * (g * (g * a))) \leq \ \max\ \{\varrho^{-}((a * g) * w), \varrho^{-}(w)\}.$$

Therefore, ϱ is a bipolar fuzzy commutative ideal.

Suppose that $\varrho := (X, \varrho^{+}, \varrho^{-})$ is a bipolar fuzzy set. Let $(\alpha, \beta) \in [-1, 0) \times (0, 1]$. Recall that

$$N(\varrho; \alpha) = \{a \in X : \varrho^{-}(a) \leq \alpha\}$$

is said to be the negative α-cut of ϱ, and

$$P(\varrho; \beta) = \{r \in X : \varrho^{+}(r) \geq \beta\}$$

is called the positive β-cut of ϱ.

Theorem 4. *Let $\varrho := (X, \varrho^{+}, \varrho^{-})$ be a bipolar fuzzy set. Then ϱ is a bipolar fuzzy commutative ideal of X if and only if both the non-empty negative α-cut and the non-empty positive β-cut of ϱ are commutative ideals of X for all $(\alpha, \beta) \in [-1, 0) \times (0, 1]$.*

Proof. Let ϱ be a bipolar fuzzy commutative ideal of X. For any fixed $\alpha \in [-1,0)$ and $\beta \in (0,1]$, if $\varrho^-(0) \leq \alpha$ and $\varrho^+(0) \geq \beta$, it implies that $0 \in \varrho^-(\alpha)$ and $0 \in \varrho^+(\beta)$. Thus the first condition holds. Let $((a*g)*w) \in \varrho^-(\alpha) \cap \varrho^+(\beta)$ and $t \in \varrho^-(\alpha) \cap \varrho^+(\beta)$. It follows that

$$\varrho^-((a*g)*w) \leq \alpha, \quad \varrho^+((a*g)*w) \geq \beta$$

and

$$\varrho^-(w) \leq \alpha, \quad \varrho^+(w) \geq \beta.$$

By definition, we obtain

$$\varrho^+(a*(g*(g*a))) \geq \min\{\varrho^+((a*g)*w), \varrho^+(w)\} \geq \beta,$$
$$\varrho^-(a*(g*(g*a))) \leq \max\{\varrho^-((a*g)*w), \varrho^-(w)\} \leq \alpha.$$

Therefore, ϱ is a commutative ideal of X.

Now, assume that $\varrho := (X, \varrho^+, \varrho^-)$ is a commutative ideal of X for all $(\alpha,\beta) \in [-1,0)(0,1]$, $\varrho^-(0) \leq \alpha$ and $\varrho^+(0) \geq \beta$. Suppose that $\varrho^-((a*g)*w) = \alpha_1$ and $\varrho^-(w) = \alpha_2$ for some $a, g, w \in X$. This implies that $((a*g)*w) \in \varrho^-(\alpha_1)$ and $w \in \varrho^-(\alpha_2)$. Without loss of generality, let $\alpha_1 \leq \alpha_2$. Let $\varrho^+((a*g)*w) = \beta_1$, $\varrho^+(w) = \beta_2$ and $\beta_1 \geq \beta_2$. Then $\varrho^-(\alpha_2) \leq \varrho^-(\alpha_1)$ which means that $w \in \varrho^-(\alpha_1)$. Now, $\varrho^+(\beta_1) \geq \varrho^+(\beta_2)$ and hence $w \in \varrho^+(\beta_1)$. That $\varrho^-(\alpha_1)$ and $\varrho^+(\beta_1)$ are commutative ideals of X, implies that

$$a*(g*(g*a)) \in (\varrho^-(\alpha_1) \cap \varrho^+(\beta_1)).$$

Hence

$$\varrho^-(a*(g*(g*a))) \leq \alpha_1 = \min\{\varrho^-((a*g)*w), \varrho^-(h)\},$$
$$\varrho^+(a*(g*(g*a))) \geq \beta_1 = \max\{\varrho^+((a*g)*w), \varrho^+(w)\}.$$

Thus $\varrho^-(0) \leq \varrho^-(a)$ and $\varrho^+(0) \geq \varrho^+(a)$. Hence, ϱ is a bipolar fuzzy commutative ideal as required.

4. Characteristic Commutative Ideals

Recall that a commutative ideal A of a BCK-algebra X that satisfies:
if for every $\varphi \in \mathrm{Aut}(X)$, the set of all automorphisms of X, we have $\varphi(A) = A$, is said to be a characteristic commutative ideal of X

A bipolar fuzzy commutative ideal A of a BCK-algebra X is said to be a bipolar fuzzy characteristic commutative ideal of X if for every $\varphi \in \mathrm{Aut}(X)$ we have $\varrho^+(\varphi(a)) = \varrho^+(a)$ and $\varrho^-(\varphi(a)) = \varrho^-(a)$.

Theorem 5. *Let X be a BCK-algebra and let $\varrho := (X, \varrho^+, \varrho^-)$ be a bipolar fuzzy characteristic commutative ideal of X. Then any negative α-cut and positive β-cut commutative ideals of ϱ are characteristic commutative ideals of X.*

Proof. Consider a negative α-cut, $N(\varrho;\alpha) = \varrho_\alpha$ and a positive β-cut, $P(\varrho;\beta) = \varrho_\beta$ of ϱ. Let $\alpha,\beta \in Im(f)$, the image of ϱ, $\varphi \in \mathrm{Aut}(X)$, $r \in \varrho_\beta$ and $s \in \varrho_\alpha$. That ϱ is a bipolar fuzzy characteristic commutative ideal of X, implies that

$$\varrho^+(\varphi(r)) = \varrho^+(r) \geq \beta,$$

$$\varrho^-(\varphi(s)) = \varrho^-(s) \leq \alpha.$$

Then $\varphi(r) \in \varrho_\beta$ and $\varphi(s) \in \varrho_\alpha$ which means that $\varphi(\varrho_\beta) \subseteq \varrho_\beta$ and $\varphi(\varrho_\alpha) \subseteq \varrho_\alpha$.
On the other hand, let $r \in \varrho_\beta, s \in \varrho_\alpha$ and $a,b \in X$ in which $\varphi(a) = r$ and $\varphi(b) = s$. It follows that

$$\varrho^+(a) = \varrho^+(\varphi(a)) = \varrho^+(r) \geq \beta,$$

$$\varrho^-(b) = \varrho^-(\varphi(b)) = \varrho^-(s) \leq \alpha,$$

and thus $a \in \varrho_\beta$ and $b \in \varrho_\alpha$. Now,

$$r = \varphi(a) \in \varphi(\varrho_\beta),$$

$$s = \varphi(b) \in \varphi(\varrho_\alpha),$$

and hence $\varrho_\beta \subseteq \varphi(\varrho_\beta)$ and $\varrho_\alpha \subseteq \varphi(\varrho_\alpha)$. Therefore, ϱ_β and ϱ_α are bipolar characteristic commutative ideals of X as required.

Theorem 6. *Let X be a BCK-algebra and let $\varrho := (X,\varrho^+,\varrho^-)$ be a bipolar fuzzy commutative ideal of X. If each negative α-cut and each positive β-cut commutative ideal of ϱ is a characteristic commutative ideal of X, then ϱ is a bipolar fuzzy characteristic commutative ideal of X.*

Proof. Let $r \in X$, $\varphi \in \mathrm{Aut}(X)$ and $f(r) = \beta$. It follows that $r \in \varrho_\beta$ and $r \in \varrho_\gamma$ for all $\gamma > \beta$. That $\varphi(\varrho_\beta) = \varrho_\beta$, implies that $\varphi(r) \in \varrho_\beta$. Thus $\varrho^+(\varphi(r)) \geq \beta$. Let $\gamma = \varrho^+(\varphi(r))$. Then $\varphi(r) \in \varrho_\gamma = \varphi(\varrho_\gamma)$ by hypothesis. That $\varphi \in \mathrm{Aut}(X)$, implies that φ is injective and so $r \in \varrho_\gamma$. This means that γ cannot be greater than β and hence $\varrho^+(\varphi(r)) = \beta = \varrho^+(r)$. Similarly, we can show that $\varrho^-(\varphi(r)) = \alpha = \varrho^-(r)$ for every $r \in X$, $\varphi \in \mathrm{Aut}(X)$ and $f(r) = \alpha$.
Therefore, ϱ is a bipolar fuzzy characteristic commutative ideal of X.

5. Bipolar fuzzy commutative positive ideal

Recall that a bipolar fuzzy ideal of a BCK-algebra X is said to be a bipolar fuzzy positive implicative ideal of X if

$$\varrho^+(a*w) \geq \min\{\varrho^+((a*g)*w), \varrho^+(g*w)\},$$

$$\varrho^-(a*w) \leq \max\{\varrho^-((a*g)*w)), \varrho^-(g*w)\}.$$

If X is positive implicative, then we have

$$(a*w)*(g*w) = ((a*g)*w) \tag{4}$$

Theorem 7. *Let X be positive implicative. A bipolar fuzzy commutative ideal of X is a bipolar fuzzy positive implicative ideal of X if and only if:*

$$\begin{aligned}\varrho^+(a*g) &\geq \varrho^+((a*g)*w) \\ \varrho^-(a*g) &\leq \varrho^-((a*g)*w)\end{aligned} \tag{5}$$

Proof. Let $\varrho := (X, \varrho^+, \varrho^-)$ be a bipolar fuzzy commutative ideal of X. Assume that ϱ is a bipolar fuzzy positive implicative ideal. Then

$$\varrho^+(m*p) \geq \min\{\varrho^+((a*g)*w), \varrho^+(n*p)\},$$

$$\varrho^-(m*p) \leq \max\{\varrho^-((a*g)*w), \varrho^-(n*p)\}.$$

Put $p = n$, we obtain

$$\varrho^+(m*n) \geq \min\{\varrho^+((a*g)*w), \varrho^+(0)\},$$

$$\varrho^-(m*n) \leq \max\{\varrho^-((a*g)*w), \varrho^-(0)\}.$$

Thus we have

$$\begin{aligned}\varrho^+(m*n) &\geq \varrho^+((a*g)*w) \\ \varrho^-(m*n) &\leq \varrho^-((a*g)*w)\end{aligned}$$

Conversely, suppose that (5) holds. We have

$$\begin{aligned}
\varrho^+(a*w) &\geq \varrho^+((a*g)*w) && by(5)\\
&= \varrho^+((a*w)*(a*(a*w))) && by(2)\\
&\geq \varrho^+((a*w)*(a*(a*(a*w)))) && X \text{ is positive implicative}\\
&\geq \min\{\varrho^+(((a*w)*a)*g), \varrho^+(g)\} && \varrho \text{ is a bipolar fuzzy commutative ideal}\\
&= \min\{\varrho^+(((a*w)*g)*a), \varrho^+(g)\} && by(1)\\
&= \min\{\varrho^+((a*g)*w)*a), \varrho^+(g)\} && by(1)\\
&\geq \min\{\varrho^+((a*g)*w), \varrho^+(g)\} && by \text{ Proposition 1}\\
&= \min\{\varrho^+((a*(g*w))*w), \varrho^+(g*w)\} && (\text{Set } g = g*w)\\
&= \min\{\varrho^+((a*w)*(g*w)), \varrho^+(g*w)\} && by(1)\\
&= \min\{\varrho^+((a*g)*w), \varrho^+(g*w)\} && by(4)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varrho^-(a*w) &\leq \varrho^-((a*w)*w) && by(5)\\
&= \varrho^-((a*w)*(a*(a*w))) && by(2)\\
&\leq \varrho^-((a*w)*(a*(a*(a*w)))) && X \text{ is positive implicative}\\
&\leq \max\{\varrho^-(((a*w)*a)*g), \varrho^-(g)\} && \varrho \text{ is a bipolar fuzzy commutative ideal}
\end{aligned}$$

$$
\begin{aligned}
&= \max\{\varrho^-(((a*w)*g)*a), \varrho^-(g)\} && by(1)\\
&= \max\{\varrho^-(((a*g)*w)*a), \varrho^-(g)\} && by(1)\\
&\leq \max\{\varrho^-((a*g)*w), \varrho^-(g)\} && by \text{ Proposition 1}\\
&= \max\{\varrho^-(((a*(g*w))*w), \varrho^-(g*w)\} && (\text{Set } g = g*w)\\
&= \max\{\varrho^-((a*w)*(g*w)), \varrho^-(g*w)\} && by(1)\\
&= \max\{\varrho^-((a*g)*w), \varrho^-(g*w)\} && by(4)
\end{aligned}
$$

This completes the proof.

References

[1] D Al-Kadi and G Muhiuddin. Bipolar fuzzy bci-implicative ideals of bci-algebras. *Ann. Commun. Math*, 3(1):88–96, 2020.

[2] ANAS Al-Masarwah and Abd Ghafur Ahmad. Doubt bipolar fuzzy subalgebras and ideals in bck/bci-algebras. *J. Math. Anal*, 9(3):9–27, 2018.

[3] Ghous Ali, G Muhiuddin, Arooj Adeel, and Muhammad Zain Ul Abidin. Ranking effectiveness of covid-19 tests using fuzzy bipolar soft expert sets. *Mathematical Problems in Engineering*, 2021, 2021.

[4] H Alshehri and A Almuhaimeed. On bipolar fuzzy d-subalgebras and d-ideals. *JP Journal of Algebra, Number Theory and Applications*, 45(1):73–83, 2020.

[5] Isabelle Bloch. Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. *Information Sciences*, 181(10):2002–2015, 2011.

[6] Y Imai and K Iseki. On axioms of proportional calculi xiv proc. *Japan Acad*, 42:19–22, 1966.

[7] K. Is6ki and S. Tanaka. An introduction to the theory of bck-algebra. *Math*, 23:1–26, 1978.

[8] YB Jun, HS Kim, and DS Yoo. Fuzzy bck-algebras. *Scientiae Mathematicae Japonicae*, 36:935–942, 1991.

[9] Young Bae Jun, Tahsin Oner, Duygu Selin Turan, and Burak Ordin. Bipolar-valued fuzzy filters of sheffer stroke bl-algebras. *New Mathematics and Natural Computation*, pages 1–17, 2023.

[10] Kyoung Ja Lee. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of bck/bci-algebras. *Bull. Malays. Math. Sci. Soc*, 32(3):361–373, 2009.

[11] Kyoung-Ja Lee and Young-Bae Jun. Bipolar fuzzy a-ideals of bci-algebras. *Communications of the Korean Mathematical Society*, 26(4):531–542, 2011.

[12] Jie Meng and Xiu-e Guo. On fuzzy ideals in bck/bci-algebras. *Fuzzy sets and Systems*, 149(3):509–525, 2005.

[13] G Muhiuddin, M Mohseni Takallo, RA Borzooei, and YB Jun. m-polar fuzzy q-ideals in bci-algebras. *Journal of King Saud University-Science*, 32(6):2803–2809, 2020.

[14] Tahsin Oner, T Kalkan, and Arsham Borumand Saeid. (anti) fuzzy ideals of sheffer stroke bck-algebras. *Journal of Algebraic Systems*, 11(1):105–135, 2023.

[15] Tahsin Oner, Tugce Kalkan, and Arsham Borumand Saeid. Class of sheffer stroke bck-algebras. *Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică*, 30(1):247–269, 2022.

[16] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.