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Properties of $s\tilde{\theta}$-Open Sets in Generalized Topological Spaces

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Abstract. A study of $\tilde{\theta}$-open sets in generalized topological spaces started in 2011 by Min [4]. In this paper, we introduce the concepts of $s\tilde{\theta}$-open sets, $s\tilde{\theta}$-closed sets, $s\tilde{\theta}$-continuous functions and $s\tilde{\theta}$-irresolute functions on generalized topological spaces. We also study some basic properties of such sets and functions.

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1. Introduction

In 2002, Császár [1] defined the concept of a generalized topological space which is an extension of the idea from a topological space. In addition, the concepts of the interior of sets and the closure of sets in topological spaces were introduced. If A is a subset of X, then the symbols $i_\mu(A)$ and $c_\mu(A)$ are used to represent the interior of A and the closure of A, respectively, in a generalized topological space (X, μ). In [1], the concept of (μ, μ')-continuous functions on generalized topological spaces was also introduced. In 2005, Császár [2] used the concepts of semi-open sets and semi-closed sets in topological spaces to define such sets in generalized topological spaces, and proved that if $A \subseteq X$, then $i_\mu(i_\mu(A)) = i_\mu(A)$ and $c_\mu(c_\mu(A)) = c_\mu(A)$. Moreover, if $A \subseteq B \subseteq X$, then $i_\mu(A) \subseteq i_\mu(B)$ and $c_\mu(A) \subseteq c_\mu(B)$. In 2008, Császár [3] defined the family $\theta(\mu) = \theta \subseteq P(X)$ in a generalized topological space (X, μ). A subset A of (X, μ) is an element of $\theta(\mu)$ if and only if there exists $M \in \mu$ such that $x \in M$ and $M \subseteq c_\mu(M) \subseteq A$ for all $x \in A$. The elements of $\theta(\mu)$ are called θ-open sets in (X, μ). The complements of θ-open sets are called θ-closed sets. In 2011, the concept of the collection $\theta(\mu)$ was developed into the collection $\tilde{\theta}(\mu) = \tilde{\theta} \subseteq P(X)$ by Min [4]. Additionally, he also concluded that $\theta \subseteq \tilde{\theta} \subseteq \mu$. In 2011, Roy [5] studied some properties of (μ, μ')-continuous functions on generalized topological spaces. In this paper, we use the concept of the above researches to define new types of sets in generalized topological spaces, along with studying some basic properties of such sets.

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2. Preliminaries

Definition 1. *[1] Let X be a non-empty set, $P(X)$ denotes the power set of X, we call a collection $\mu \subseteq P(X)$ a generalized topology on X if $\emptyset \in \mu$ and $\bigcup_{\alpha \in J} G_\alpha \in \mu$ for each $G_\alpha \in \mu$ and $\alpha \in J$. The elements in μ is called μ-open sets in X. The complement of each μ-open set is called μ-closed in X. The pair (X, μ) is called a generalized topological space. For $A \subseteq X$, the symbol $i_\mu(A)$ represents the interior of A which is the union of all μ-open sets contained in A, and the symbol $c_\mu(A)$ represents the closure of A, which means the intersection of all μ-closed sets containing A.*

Lemma 1. *[2] Let A be a subset of a generalized topological space (X, μ). Then:*
1. *$i_\mu(A)$ is the largest μ-open set contained in A,*
2. *$c_\mu(A)$ is the smallest μ-closed set containing A,*
3. *$c_\mu(X - A) = X - i_\mu(A)$.*

Definition 2. *[2] Let (X, μ) be a subset of a generalized topological space. Then, a subset A of X is called a μ-semi-open set if $A \subseteq c_\mu(i_\mu(A))$. The complement of μ-semi-open sets is called μ-semi-closed sets. The intersection of all μ-semi-closed sets containing A is denoted by $c_\sigma(A)$.*

Definition 3. *[3] Let A be a subset of a generalized topological space (X, μ). Then, A is an element of a collection $\theta(\mu) = \theta \subseteq P(X)$ if and only if there exists $M \in \mu$ such that $x \in M$ and $M \subseteq c_\mu(M) \subseteq A$ for all $x \in A$. The elements of $\theta(\mu)$ are called θ-open sets in X. The complements of θ-open sets are called θ-closed sets.*

Definition 4. *[4] Let (X, μ) be a generalized topological space and $\tilde{\theta}(\mu) = \tilde{\theta} \subseteq P(X)$. A subset A of X is an element of $\tilde{\theta}$ if and only if there exists $M \in \mu$ such that $x \in M \subseteq c_\mu(M) \cap M_\mu \subseteq A$ for all $x \in A$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. The elements of $\tilde{\theta}$ are called $\tilde{\theta}$-open sets in X. The complements of $\tilde{\theta}$-open sets are called $\tilde{\theta}$-closed sets. Additionally, in [4], the relationship between θ, $\tilde{\theta}$ and μ was concluded as follows: $\theta \subseteq \tilde{\theta} \subseteq \mu$.*

Definition 5. *[1] Let (X, μ) and (Y, μ') be generalized topological spaces. Then a function f from (X, μ) into (Y, μ') is called (μ, μ')-continuous if $f^{-1}(G)$ is μ-open in X for each μ'-open set G in Y.*

3. $s\tilde{\theta}$-open sets

In this section, we introduce the concept of $s\tilde{\theta}$-open sets. Furthermore, some properties of $s\tilde{\theta}$-open sets are studied.

Definition 6. *Let (X, μ) be a generalized topological space and $A \subseteq X$. Define the collection $s\tilde{\theta}(\mu) = s\tilde{\theta} \subseteq P(X)$ by $A \in s\tilde{\theta}$ if for each $x \in A$ there exists $M \in \mu$ such that $x \in M \subseteq c_\sigma(M) \cap M_\mu \subseteq A$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. The elements in $s\tilde{\theta}$ are called* $s\tilde{\theta}$-open sets *and the complements are called* $s\tilde{\theta}$-closed sets *in X.*

Theorem 1. *Let (X, μ) be a generalized topological space. Then, $s\tilde{\theta}$ is a generalized topology on X.*

Proof. By Definition 6, we can clearly see that $\emptyset \in s\tilde{\theta}$. Assume that $A_i \in s\tilde{\theta}$ for all $i \in J$. Let $x \in \bigcup_{i\in J} A_i$, we get $x \in A_i$ for some $i \in J$. By the assumption, there exists $M \in \mu$ such that $x \in M \subseteq c_\sigma(M) \cap M_\mu \subseteq A_i \subseteq \bigcup_{i\in J} A_i$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. Thus, $\bigcup_{i\in J} A_i \in s\tilde{\theta}$. Therefore, $s\tilde{\theta}$ is a generalized topology on X.

Theorem 2. *Let (X, μ) be a generalized topological space. Then $s\tilde{\theta} \subseteq \mu$.*

Proof. Let $A \subseteq X$ and $A \in s\tilde{\theta}$. If $A = \emptyset$, then $A \in \mu$. If $A \neq \emptyset$, let $x \in A$. Since $A \in s\tilde{\theta}$, there exists $M_x \in \mu$ such that $x \in M_x \subseteq c_\sigma(M_x) \cap M_\mu \subseteq A$ for each $x \in A$. Hence, $\bigcup_{x\in A}\{x\} \subseteq \bigcup_{x\in A} M_x \subseteq A$. As $A = \bigcup_{x\in A}\{x\}$, $A \subseteq \bigcup_{x\in A} M_x \subseteq A$. Consequently, $A = \bigcup_{x\in A} M_x \in \mu$. Therefore, $s\tilde{\theta} \subseteq \mu$.

Corollary 1. *Let A be a subset of a generalized topological space (X, μ). If A is an $s\tilde{\theta}$-closed set, then A is μ-closed.*

Theorem 3. *Let (X, μ) be a generalized topological space. Then, $\tilde{\theta} \subseteq s\tilde{\theta}$.*

Proof. Let A be an arbitrary element in $\tilde{\theta}$. Assume that $x \in A$. Since $A \in \tilde{\theta}$, there exists $M \in \mu$ such that $x \in M \subseteq c_\mu(M) \cap M_\mu \subseteq A$. As $M \subseteq c_\sigma(M) \subseteq c_\mu(M)$ for all $x \in A$. Accordingly, $A \in s\tilde{\theta}$. Therefore, $\tilde{\theta} \subseteq s\tilde{\theta}$.

In a generalized topological space (X, μ), $s\tilde{\theta}$-open sets may not be $\tilde{\theta}$-open sets as the following example.

Example 1. *Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $A = \{b\}$. Then, $X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}$ are μ-closed and $M_\mu = \{a, b, c\}$.*
Moreover, only $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}$ and X are μ-semi-closed sets. Consider $b \in A$, then there exists $\{b\} \in \mu$ such that $b \in \{b\} \subseteq c_\sigma(\{b\}) \cap M_\mu = \{b\}$. Hence, $A \in s\tilde{\theta}$. Consider each μ-open set M containing b.

If $M = \{b\} \in \mu$, then $b \in \{b\} \subseteq c_\mu(\{b\}) \cap M_\mu = \{b, c\} \not\subseteq \{b\}$.

If $M = \{a, b\} \in \mu$, then $b \in \{a, b\} \subseteq c_\mu(\{a, b\}) \cap M_\mu = \{a, b, c\} \not\subseteq \{b\}$.

If $M = \{a, b, c\} \in \mu$, then $b \in \{a, b, c\} \subseteq c_\mu(\{a, b, c\}) \cap M_\mu = \{a, b, c\} \not\subseteq \{b\}$. From the above three cases, $A \notin \tilde{\theta}$. Therefore, $s\tilde{\theta} \not\subseteq \tilde{\theta}$.

By the generalized topological space (X, μ) in Example 1, we can A is $s\tilde{\theta}$-open but not $\tilde{\theta}$-open. Even though μ is a topology on X, $s\tilde{\theta}$-open sets may not be $\tilde{\theta}$-open as in the following example.

Example 2. *Let* $X = \{a,b,c,d\}$*,* $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\}$ *and* $A = \{b\}$*. Clearly,* $X, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{d\}, \emptyset$ *are* μ*-closed and* $M_\mu = X$*.*
Moreover, only $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, X$ *are* μ*-semi-closed sets. Consider* $b \in A$*, then there exists* $\{b\} \in \mu$ *such that* $b \in \{b\} \subseteq c_\sigma(\{b\}) \cap M_\mu = \{b\}$*. Thus,* $A \in s\tilde{\theta}$*. Since* $\{b\}, \{a,b\}, \{a,b,c\}, X$ *are* μ*-open sets containing* b*. Consider each* μ*-open set* M *containing* b*.*

If $M = \{b\} \in \mu$*, then* $b \in \{b\} \subseteq c_\mu(\{b\}) \cap M_\mu = \{b,c,d\} \not\subseteq \{b\}$*.*

If $M = \{a,b\} \in \mu$*, then* $b \in \{a,b\} \subseteq c_\mu(\{a,b\}) \cap M_\mu = X \not\subseteq \{b\}$*.*

If $M = \{a,b,c\} \in \mu$*, then* $b \in \{a,b,c\} \subseteq c_\mu(\{a,b,c\}) \cap M_\mu = X \not\subseteq \{b\}$*.*

If $M = X \in \mu$*, then* $b \in X \subseteq c_\mu(X) \cap M_\mu = X \not\subseteq \{b\}$*. From the above four cases,* $A \notin \tilde{\theta}$*. Therefore,* $s\tilde{\theta} \not\subseteq \tilde{\theta}$ *even though* μ *is a topology on* X*.*

Corollary 2. *Let* (X,μ) *be a generalized topological space. Then* $\theta \subseteq \tilde{\theta} \subseteq s\tilde{\theta} \subseteq \mu$*.*

In generalized topological spaces, μ-open sets need not be $s\tilde{\theta}$-open as can be seen from the following example.

Example 3. *By the generalized topological space* (X,μ) *in Example 2. Let* $A = \{a,b,c\}$*, then* $A \in \mu$*. Consider* $c \in A$*. Since only* $\{a,b,c\}$ *and* X *are* μ*-open sets containing* c*. Consider each* μ*-open set* M *containing* c*.*

If $M = \{a,b,c\} \in \mu$*, then* $c \in \{a,b,c\} \subseteq c_\mu(\{a,b,c\}) \cap M_\mu = X \not\subseteq \{a,b,c\}$*.*

If $M = X \in \mu$*, then* $c \in X \subseteq c_\mu(X) \cap M_\mu = X \not\subseteq \{a,b,c\}$*. From the above two cases,* $A \notin s\tilde{\theta}$*.*

In [4], Min defined the set $\gamma_{\tilde{\theta}}(A)$, where A is a subset of a generalized topological space (X,μ) as follows: $x \in \gamma_{\tilde{\theta}}(A)$ if $c_\mu(G) \cap M_\mu \cap A \neq \emptyset$ for each $G \in \mu$ such that $x \in G$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. In this paper, the aforementioned concepts are used to define the set $\gamma_{s\tilde{\theta}}(A)$ as follows.

Definition 7. *Let* A *be a subset of a generalized topological space* (X,μ)*. The set* $\gamma_{s\tilde{\theta}}(A)$ *is a subset of* X *defined by* $x \in \gamma_{s\tilde{\theta}}(A)$ *if* $c_\sigma(G) \cap M_\mu \cap A \neq \emptyset$ *for each* $G \in \mu$ *such that* $x \in G$*, where* $M_\mu = \cup\{M \subseteq X : M \in \mu\}$*.*

Theorem 4. *Let* A *and* B *be subsets of a generalized topological space* (X,μ)*. If* $A \subseteq B \subseteq X$*, then* $\gamma_{s\tilde{\theta}}(A) \subseteq \gamma_{s\tilde{\theta}}(B)$*.*

Proof. Let $A \subseteq B \subseteq X$ and $x \in \gamma_{s\tilde{\theta}}(A)$. Thus, $c_\sigma(G) \cap M_\mu \cap A \neq \emptyset$ for all $G \in \mu$ such that $x \in G$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. Let H be an arbitrary μ-open set such that $x \in H$. By the assumption, $c_\sigma(H) \cap M_\mu \cap B \neq \emptyset$. Consequently, $x \in \gamma_{s\tilde{\theta}}(B)$. Therefore, $\gamma_{s\tilde{\theta}}(A) \subseteq \gamma_{s\tilde{\theta}}(B)$.

Theorem 5. *Let* (X,μ) *be a generalized topological space and* $A \subseteq X$*. Then,* $A \subseteq \gamma_{s\tilde{\theta}}(A) \subseteq \gamma_{\tilde{\theta}}(A)$*.*

Proof. Assume that $x \in A$. If $x \notin M_\mu$, then $x \in X - M_\mu$. It follows that $x \in X$ and $x \notin M_\mu$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. Thus, there is no $G \in \mu$ such that $x \in G$.

By Definition 7, $x \in \gamma_{s\tilde{\theta}}(A)$. If $x \in M_\mu$, then $c_\sigma(G) \cap M_\mu \cap A \neq \emptyset$ for each $G \in \mu$ such that $x \in G$. Hence, $x \in \gamma_{s\tilde{\theta}}(A)$. Thus, $A \subseteq \gamma_{s\tilde{\theta}}(A)$. By Definition 7, we can see that $\gamma_{s\tilde{\theta}}(A) \subseteq \gamma_{\tilde{\theta}}(A)$, where A is a subset of a generalized topological space (X, μ). Therefore, $A \subseteq \gamma_{s\tilde{\theta}}(A) \subseteq \gamma_{\tilde{\theta}}(A)$.

We use some properties of the operation $\gamma_{s\tilde{\theta}}$ to characterize $s\tilde{\theta}$-closed sets as the following theorem.

Theorem 6. *Let (X, μ) be a generalized topological space and $A \subseteq X$. Then A is $s\tilde{\theta}$-closed if and only if $\gamma_{s\tilde{\theta}}(A) = A$.*

Proof. $(\rightarrow)$ Let A be an $s\tilde{\theta}$-closed. Then, $X - A$ is $s\tilde{\theta}$-open. Assume that $x \in X - A$. Hence, there exists $M \in \mu$ such that $x \in M \subseteq c_\sigma(M) \cap M_\mu \subseteq X - A$, where $M_\mu = \cup\{M \subseteq X : M \in \mu\}$. Thus, $c_\sigma(M) \cap M_\mu \cap A = \emptyset$, for some $M \in \mu$ and $x \in M$. It follows that $x \notin \gamma_{s\tilde{\theta}}(A)$. Consequently, $x \in X - \gamma_{s\tilde{\theta}}(A)$. Accordingly, $X - A \subseteq X - \gamma_{s\tilde{\theta}}(A)$. Therefore, $\gamma_{s\tilde{\theta}}(A) \subseteq A$. By Theorem 5, $\gamma_{s\tilde{\theta}}(A) = A$.
$(\leftarrow)$ Suppose that $\gamma_{s\tilde{\theta}}(A) = A$. Then, $X - A = X - \gamma_{s\tilde{\theta}}(A)$. Assume that $x \in X - A$. It follows that $x \notin \gamma_{s\tilde{\theta}}(A)$. Hence, there exists $M \in \mu$ such that $x \in M$ and $c_\sigma(M) \cap M_\mu \cap A = \emptyset$. Thus, $x \in M \subseteq c_\sigma(M) \cap M_\mu \subseteq X - A$. Accordingly, $X - A$ is $s\tilde{\theta}$-open. Therefore, A is $s\tilde{\theta}$-closed.

Corollary 3. $\gamma_{s\tilde{\theta}}(X) = X$, *where (X, μ) is a generalized topological space.*

Proof. By Theorem 1, $\emptyset$ is $s\tilde{\theta}$-open. It follows that X is $s\tilde{\theta}$-closed. By Theorem 6, $\gamma_{s\tilde{\theta}}(X) = X$.

Definition 8. *Let (X, μ) and (Y, μ') be generalized topological spaces. Then, a function f from (X, μ) into (Y, μ') is called $\tilde{\theta}$-continuous if $f^{-1}(G)$ is $\tilde{\theta}$-open in X for each μ'-open set G in Y.*

Example 4. *Let $X = \{a, b, c, d\}$, $Y = \{t, v, w\}$. $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, \{t\}, \{t, v\}\}$. Let f be a function from (X, μ) into (Y, μ') defined by $f(a) = t$, $f(b) = t$, $f(c) = t$, $f(d) = w$. Then, only $\emptyset$ and $\{a, b, c\}$ are $\tilde{\theta}$-open sets in X. Consider $\emptyset, \{t\}$ and $\{t, v\}$ are μ'-open sets in Y. Then $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{t\}) = \{a, b, c\}$ and $f^{-1}(\{t, v\}) = \{a, b, c\}$. Thus, $f^{-1}(G)$ is $\tilde{\theta}$-open in X for each μ'-open set G in Y. Therefore, f is a $\tilde{\theta}$-continuous function.*

Theorem 7. *Let (X, μ) and (Y, μ') be generalized topological spaces. Then, $f : (X, \mu) \rightarrow (Y, \mu')$ is $\tilde{\theta}$-continuous if and only if $f^{-1}(V)$ is $\tilde{\theta}$-closed in X for each μ'-closed set V in Y.*

Proof. $(\rightarrow)$ Assume that f is a $\tilde{\theta}$-continuous function. Let V be a μ'-closed set in Y. Then, $Y - V$ is μ'-open in Y. Hence, $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\tilde{\theta}$-open in X. Therefore, $f^{-1}(V)$ is $\tilde{\theta}$-closed in X.
$(\leftarrow)$ Assume that $f^{-1}(V)$ is $\tilde{\theta}$-closed in X for each μ'-closed set V in Y. Let H be μ'-open in Y. Then $Y - H$ is μ'-closed in Y. By the assumption, $f^{-1}(Y - H)$ is $\tilde{\theta}$-closed in X. It follows that $f^{-1}(H)$ is $\tilde{\theta}$-open in X. Hence, f is $\tilde{\theta}$-continuous.

Definition 9. *Let (X, μ) and (Y, μ') be generalized topological spaces. Then, a function f from (X, μ) into (Y, μ') is called $s\tilde{\theta}$-continuous if $f^{-1}(G)$ is $s\tilde{\theta}$-open in X for each μ'-open set G in Y.*

It is easy to see that all $\tilde{\theta}$-continuous functions are $s\tilde{\theta}$-continuous. But the converse need not to be true as the following example.

Example 5. *By the generalized topological spaces (X, μ) and (Y, μ') in Example 4, if f is a function from (X, μ) into (Y, μ') defined by $f(a) = t$, $f(b) = t$, $f(c) = v$, $f(d) = w$. Then, only $\emptyset, \{a\}, \{b\}, \{a, b\}$ and $\{a, b, c\}$ are $s\tilde{\theta}$-open in X. It can be checked that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{t\}) = \{a, b\}$ and $f^{-1}(\{t, v\}) = \{a, b, c\}$. Thus, $f^{-1}(G)$ is $s\tilde{\theta}$-open in X for each $s\tilde{\theta}$-open set G in Y. Consequently, f is $s\tilde{\theta}$-continuous. Since there exists a μ'-open set $\{t\}$ in Y such that $f^{-1}(\{t\}) = \{a, b\}$ is not $\tilde{\theta}$-open in X. Therefore, f is $s\tilde{\theta}$-continuous but f is not $\tilde{\theta}$-continuous.*

Theorem 8. *Let (X, μ) and (Y, μ') be generalized topological spaces. Then, for a function f from (X, μ) into (Y, μ') the followings are equivalent:*

a) f is $s\tilde{\theta}$-continuous;

b) $f^{-1}(F)$ is $s\tilde{\theta}$-closed in X for each μ'-closed set F in Y;

c) for all $x \in X$, there exists an $s\tilde{\theta}$-open set H in X such that $x \in H$ and $f(H) \subseteq G$, for all $G \in \mu'$ such that $f(x) \in G$

Proof.

a) $\to$ *b)* Let f be an $s\tilde{\theta}$-continuous function. Then, $f^{-1}(G)$ is $s\tilde{\theta}$-open in X for all μ'-open set G in Y. Assume that F is μ'-closed in Y. We get $Y - F$ is μ'-open in Y. Hence $f^{-1}(Y - F)$ is $s\tilde{\theta}$-open in X. Thus, $X - f^{-1}(F)$ is $s\tilde{\theta}$-open in X. Therefore, $f^{-1}(F)$ is $s\tilde{\theta}$-closed in X.

b) $\to$ *c)* Assume that $f^{-1}(F)$ is $s\tilde{\theta}$-closed in X for each μ'-closed set F in Y. Let $x \in X$ and $G \in \mu'$ such that $f(x) \in G$. Then, there exists a μ'-closed set F in Y such that $G = X - F$. Since $f(x) \in G$, $x \in f^{-1}(G) = f^{-1}(X - F) = Y - f^{-1}(F)$. By the assumption, $f^{-1}(F)$ is $s\tilde{\theta}$-closed in Y. It follows that $f^{-1}(G)$ is $s\tilde{\theta}$-open in X and $f(f^{-1}(G)) \subseteq G$.

c) $\to$ *a)* Assume that for all $x \in X$, there exists an $s\tilde{\theta}$-open set H in X such that $x \in H$ and $f(H) \subseteq G$, for all $G \in \mu'$ such that $f(x) \in G$. Let K be μ'-open in Y.

Case 1. $f^{-1}(K) = \emptyset$. Since $s\tilde{\theta}$ is a generalized topology on X, $\emptyset \in s\tilde{\theta}$. Hence, $f^{-1}(K) \in s\tilde{\theta}$. Thus, $f^{-1}(K)$ is $s\tilde{\theta}$-open in X.

Case 2. $f^{-1}(K) \neq \emptyset$. Let $x \in f^{-1}(K)$, then $f(x) \in K$. Hence, $x \in X$ such that $f(x) \in K$ and K is μ'-open in Y. By the assumption, there exists an $s\tilde{\theta}$-open set H_x in X such that $x \in H_x$ and $f(H_x) \subseteq K$. It follows that $H_x \subseteq f^{-1}(f(H_x)) \subseteq f^{-1}(K)$. Thus, $x \in H_x \subseteq f^{-1}(K)$. As x is an arbitrary element in $f^{-1}(K)$, $\{x\} \subseteq H_x \subseteq f^{-1}(K)$ for all $x \in f^{-1}(K)$. Hence, $\bigcup_{x \in f^{-1}(K)} \{x\} \subseteq \bigcup_{x \in f^{-1}(K)} H_x \subseteq f^{-1}(K)$. Thus $f^{-1}(K) = \bigcup_{x \in f^{-1}(K)} H_x$. As $s\tilde{\theta}$ is a generalized topology on X and H_x is $s\tilde{\theta}$-open for each $x \in f^{-1}(K)$, $\bigcup_{x \in f^{-1}(K)} H_x$ is $s\tilde{\theta}$-open in X. Consequently, $f^{-1}(K)$ is $s\tilde{\theta}$-open in X. Therefore, f is a $s\tilde{\theta}$-continuous function.

Theorem 9. *Let* (X, μ)*,* (Y, μ') *and* (Z, μ'') *be generalized topological spaces. If* $f : X \to Y$ *is* $s\tilde{\theta}$*-continuous and* $g : Y \to Z$ *is* (μ, μ')*-continuous, then* $g \circ f : X \to Z$ *is* $s\tilde{\theta}$*-continuous.*

Proof. Assume that $V \in \mu''$. Since g is (μ, μ')-continuous, $g^{-1}(V)$ is μ'-open in Y. As f is $s\tilde{\theta}$-continuous, $f^{-1}(g^{-1}(V))$ is $s\tilde{\theta}$-open in X. It follows that $(g \circ f)^{-1}(V)$ is $s\tilde{\theta}$-open in X for all $V \in \mu''$. Therefore, $g \circ f : X \to Z$ is $s\tilde{\theta}$-continuous.

Theorem 10. *Let* (X, μ) *and* (Y, μ') *be generalized topological spaces. If* $f : X \to X$ *is an identity function and* $g : X \to Y$ *is* $s\tilde{\theta}$*-continuous, then* $g \circ f : X \to Y$ *is* $s\tilde{\theta}$*-continuous.*

Proof. Assume that V is μ'-open. Since g is $s\tilde{\theta}$-continuous, $g^{-1}(V)$ is $s\tilde{\theta}$-open in X. As f is an identity function, $g^{-1}(V) = f^{-1}(g^{-1}(V))$. Consequently, $(g \circ f)^{-1}(V) = g^{-1}(V)$ is $s\tilde{\theta}$-open in X for all $V \in \mu'$. Thus, $g \circ f : X \to Y$ is $s\tilde{\theta}$-continuous.

Definition 10. *Let* (X, μ) *and* (Y, μ') *be generalized topological spaces. Then, a function* f *from* (X, μ) *into* (Y, μ') *is called* $s\tilde{\theta}$*-irresolute if* $f^{-1}(G)$ *is* $s\tilde{\theta}$*-open in* X *for each* $s\tilde{\theta}$*-open set* G *in* Y*.*

Theorem 11. *Let* (X, μ) *and* (Y, μ') *be generalized topological spaces. Then,* $f : (X, \mu) \to (Y, \mu')$ *is* $s\tilde{\theta}$*-irresolute if and only if* $f^{-1}(F)$ *is* $s\tilde{\theta}$*-closed in* X *for each* $s\tilde{\theta}$*-closed set* F *in* Y*.*

Proof. $(\to)$ Let f be a $s\tilde{\theta}$-irresolute function. Suppose that F is a $s\tilde{\theta}$-closed set in Y. Then, $Y - F$ is $s\tilde{\theta}$-open in Y. Consequently, $X - f^{-1}(F)$ is $s\tilde{\theta}$-open in X. It follows that $f^{-1}(F)$ is $s\tilde{\theta}$-closed in X.
$(\leftarrow)$ Assume that $f^{-1}(F)$ is $s\tilde{\theta}$-closed in X for each $s\tilde{\theta}$-closed set F in Y. Let K be $s\tilde{\theta}$-open in Y. Then, $Y - K$ is $s\tilde{\theta}$-closed in Y. By the assumption, $f^{-1}(Y - K)$ is $s\tilde{\theta}$-closed in X. It follows that $f^{-1}(K)$ is $s\tilde{\theta}$-open in X. Hence, f is $s\tilde{\theta}$-irresolute.

Theorem 12. *Each* $s\tilde{\theta}$*-irresolute function is* $s\tilde{\theta}$*-continuous.*

Proof. Assume that V is μ'-open in Y. By Theorem 3, V is $s\tilde{\theta}$-open. Since f is $s\tilde{\theta}$-irresolute, $f^{-1}(V)$ is $s\tilde{\theta}$-open in X. Thus, f is $s\tilde{\theta}$-continuous.

Theorem 13. *Let* (X, μ)*,* (Y, μ') *and* (Z, μ'') *be generalized topological spaces. Then,* $g \circ f : X \to Z$ *is* $s\tilde{\theta}$*-continuous if* $f : X \to Y$ *is* $s\tilde{\theta}$*-irresolute and* $g : X \to Z$ *is* $s\tilde{\theta}$*-continuous.*

Proof. Assume that H is μ''-open. Since g is $s\tilde{\theta}$-continuous, $g^{-1}(H)$ is $s\tilde{\theta}$-open in Y. As f is $s\tilde{\theta}$-irresolute, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is $s\tilde{\theta}$-open in X for all $H \in \mu''$. Thus, $g \circ f$ is $s\tilde{\theta}$-continuous.

Theorem 14. *Let* (X, μ) *and* (Y, μ') *be generalized topological spaces. Then,* $g \circ f : X \to Y$ *is* $s\tilde{\theta}$*-irresolute if* $f : X \to X$ *is an identity function and* $g : X \to Y$ *is* $s\tilde{\theta}$*-irresolute.*

Proof. Suppose that M is $s\tilde{\theta}$-open in Y. As g is $s\tilde{\theta}$-irresolute, $g^{-1}(M)$ is $s\tilde{\theta}$-open in X. Since f is an identity function, $f^{-1}(g^{-1}(M)) = g^{-1}(M)$ is $s\tilde{\theta}$-open in X for each M is $s\tilde{\theta}$-open in Y. Thus, $g \circ f : X \to Y$ is $s\tilde{\theta}$-irresolute.

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References

[1] Á. Császár. Generalized topology, generalized continuity. *Acta Mathematica Hungarica*, 96(4):351–357, 2002.

[2] Á. Császár. Generalized open sets in generalized topologies. *Acta Mathematica Hungarica*, 106(1-2):53–66, 2005.

[3] Á. Császár. δ-and θ- modifications of generalized topologies. *Acta Mathematica Hungarica*, 120(3):275–279, 2008.

[4] W. K. Min. Remarks on θ-open sets in generalized topological spaces. *Applied Mathematics Letters*, 24(2):165–168, 2011.

[5] B. Roy. On a type of generalized open sets. *Applied Generalized Topology*, 12(2):163–173, 2011.