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Two-Warehouse Inventory Model: Interval Valued Costs, Advanced Payment, Stock-Dependent Demand, and Partial Backlogging

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Abstract. A great deal of inventory costs are typically unpredictable because of the unpredictability of a competitive market. The fluctating demand from customer's and the unpredictability of the market economy make it difficult for researchers and operation research practitioners to appropriately replicate an inventory problem. In order to prevail this type of unpredictability circumstances, in this paper, we have represented the inventory parameters as interval. Using this concept, we progressed a two-warehouse inventory model for deteriorating items with a fixed Shelf life, partially backlogging shortages and interval-valued deterioration rate. In addition to this the parameters like ordering cost, purchase cost, shortage cost, deterioration cost for both the rented and owned warehouse except the backlogging parameter have been considered as interval-valued. The uncertainty in the inventory parameters motivated us to view them as interval-valued in the present research. Based on the assumptions, the cost function of this problem is a highly nonlinear constraint optimization problem. Mathematica is used to tackle this nonlinear optimization problem. A numerical example has been presented to demonstrate the computational result. Then, a sensitivity analysis has been performed to study the effect of changes of different parameters of the model on the optimal policy.

2020 Mathematics Subject Classifications: 90B05; 90C30; 93A30

Key Words and Phrases: Inventory, Two-warehouse, Partial backlogging, fixed-shelf life, Advance payment, Interval-valued inventory cost, Interval-valued Deterioration.

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1. Introduction

The growth of a competitive global market with significant swings has made inventory management much more difficult in recent years. Many academics and researchers have been working to develop various effective and efficient inventory models while accounting for the uncertainty of inventory parameters. The ultimate goal of these efforts is to improve cash flow, increase profits, and improve operational efficiency and efficiency through the business with the aid of appropriate purchasing and selling coordination. When it comes to competitive marketing, it is more sophisticated to assess the value of any inventory parameter in interval form rather than constant form while the behaviour of numerous important inventory measures fluctuates throughout the course of inventory storage. Consequently, the average profit, maximum shortage level, starting stock level, and market demand will all be interval valued. This analysis analyses the uncertainty of both the market demand for products and the inventory factors. By maximising operational efficiency and cash flow in an unpredictable environment, this activity supports industrial managers in increasing their earnings or decreasing their expenses.

A significant problem for any commercial organisation is inventory deterioration. A deterministic inventory model with trade credit finance and capacity restrictions for deteriorating products has been developed by Liao and Huang [6]. An inventory model with generalized-type demand and deterioration has been designed by Hung [3]. In their model, Widyadana and Wee[21] spoke about the economic production quantity model for degrading products. Eventually, an inventory model with non-instantaneous decaying objects with generalized-type degradation was developed by Shah et al. [11]. Using financial considerations and backordering into account, Taleizadeh and Nematollahi [14] suggested an inventory control problem for decaying products. Again applying the Stackelberg technique, Taleizadeh et al.[15] developed a vendor-managed inventory model for deteriorating items in supply chain systems. Tsao [20] designed an inventory model that aids in the decision-making process when determining a shared location, inventory, and preservation facility in the scenario of a payment delay. For deteriorating products with preservation technology investment, Shaikh et al.[13] introduced an economic order quantity model. A new inventory model with non-instantaneous deterioration with price and stock-dependent demand for completely backlogged shortages under inflation was investigated once again by Shaikh et al.[12]. After that, a stochastic production inventory model for deteriorating goods with finite life cycle was examined by Pal et al.[8].

Utilising adequate physical space to keep the items in order to maintain commercial operations is another aspect of inventory management. In this context, commercial organisations look at the two-warehouse facility. There are currently two warehouses in a two-warehouse system: an owned warehouse "OW" and a rented warehouse "RW". A commercial organisation may choose to store more items than they can handle for a wide range of reasons such as (i) in order to prevent stock-out scenarios, (ii) to hold enormous quantities of goods that are seasonal, (iii) in order to take benefit of the discount, (iv) in order to fulfil the product's high demand etc., In inventory management, a shortage occurs when demand for a product exceeds available inventory. The way this shortage is handled has a big influence on customer happiness and operational effectiveness. One solution to this problem is to partially backlog shortages. Partially backlog shortages are an inventory management approach that tries to balance immediate consumer needs with operational efficiency and cost-effectiveness. The decision of how much of the shortage to fill or backlog quickly is influenced by a number of factors, including "Cost of Backlogging," "Impact on Customer Service," "Inventory Holding Costs," and "Lead Time for Replenishment," among others. Partially backlogged shortages impact multiple areas of inventory management systems and supply chain models. For example: "Service Level," "Inventory Levels," "Operational Costs," and "Customer Satisfaction," etc.

Liang and Zhou[5] developed a two-warehouse inventory model recently for items that were deteriorating with payment delayed. A two warehouse inventory model with time-varying deterioration and higher demand has been established by Sett et al.[10]. Using an exclusive real-coded genetic algorithm, Bhunia et al.[1] have also proposed a two-warehouse deterministic inventory model. Furthermore, utilising a genetic algorithm with shifting population size method, Das et al.[2] have provided a two-warehouse production inventory model with time-varying demand. With an ideal credit duration and partial backlog under inflation, Palanivel and Uthayakumar[9] have developed a two-warehouse inventory model. An inventory model with two warehouses and deteriorating goods of unsatisfactory quality under payment delay has been introduced by Jaggi et al.[4].

Nowadays, the prepayment or advance payment strategy is frequently preferred by companies. Whenever suppliers pay in advance for the supply of products, they are only covering a portion of the total amount due; the remaining amount is only added to the invoice upon delivery. In other words, suppliers and retailers may operate their businesses and purchase the entire inventory by paying a portion of the total advance. The balance that remains can then be paid in intervals. Only a few of the researchers have created inventory models and worked on advance payment. An inventory model with a stochastic lead time and price-dependent demand that includes an advance payment has been developed by Maiti et al.[7]. Thangam [18] has presented a strategy on the best course of action for dominating retailers in a supply chain with trade credit and advance payment systems. Thangam[19] has expanded the model once more by including the most effective price discounting and lot-sizing rules under the advance payment plans and two-echelon trade credit. With an expiration date and prior payment, Teng et al.[17] have developed lot-size rules for deteriorating products. In order to create an EOQ model, Tavakoli and Taleizadeh[16] included conditional discounts for deteriorating products that require full prior payment.

Several studies that have been performed incorporated fixed inventory costs into account when evaluating in the collection of available literature. But in actuality, because of the unpredictability of competing marketing scenarios, inventory prices aren't always fixed. The uncertainty in the inventory parameters motivated us to view them as interval valued in the present research. This idea was implemented in a two-warehouse inventory model with fixed shelf life, partial backlog shortage, and advanced payment Method. The two-warehouse inventory model, which allows for partial backlogging of shortages, is highly adaptable and may be applied to a wide range of inventory problems. This model is flexible in its ability to handle varying rates of deterioration, patterns of demand, and strategies for backlogging. The effectiveness of this solution is derived from its capacity to enhance inventory management procedures, while simultaneously guaranteeing customer satisfaction and operational efficiency in dynamic and diversified corporate environments.

Finally we discuss a numerical example with the help of Mathematica Software to validate our proposed model and performed a sensitivity analysis to study the effect of changes of different parameters. Below is a summary of the major contributions:

- Inventory costs like ordering cost, holding cost, purchasing cost, shortage cost, deterioration cost and capital cost are interval valued.
- n equal installment before received the product.
- The demand of the item is linearly dependent on price i.e. $D(p) = a - bp$
- Advance Payment, Partial backlogging and fixed Shelf life have been considered.
- Inventory parameters with values for intervals have been taken into account.
- Partially backlogged shortages at a steady rate.
- The product's demand is stock-dependent.
- Rate of deterioration is constant.

The remaining sections are arranged as follows: Different symbols and Notations used in this study were covered in Section 2 of the paper. In Section 3, the model's mathematical formulation is described. In Section 4, we examine the Optimal solution of the model by applying different parameters. Finally in Section 5, the Conclusion of the paper is provided.

2. Notations

The following notations have been taken into consideration while we developed the inventory model:

Notations	Units	Description
$A = [A_L, A_U]$	\$/Order	Interval-valued Ordering Cost
η	Units	Backlogging unit $(0 < \eta < 1)$
S	Units	Total Inventory level
a	Constant	Demand rate's coefficient part $(a > 0)$

Notations	Units	Description
M	yr	Enterprise's lead time for paying prepayments
b	Constant	Demand rate Constant for price ($b > 0$)
α	Constant	Rate of Deterioration at OW
β	Constant	Rate of Deterioration at RW
n	Constant	Prepayments evenly spaced throughout the Lead period
W_1	Units	Level of Inventory at OW
$C_p = [C_{pL}, C_{pU}]$	Units	Interval-valued purchasing Cost per unit
k	Constant	An amount that needs to be paid in multiple installments ($0 < k < 1$)
R	Units	Backlogged units
f_1	Constant	Fixed Shelf-life
t_a	yr	The time where the inventory level in RW falls to zero
$c = [c_L, c_U]$	Unit	Interval valued shortage cost per unit
t_b	yr	The time where the inventory level in OW falls to zero
q	Unit	Rupees per unit time spent in OW
γ	Constant	In OW, the deterioration rate's value exists between $(0, 1)$
θ	Constant	In RW, the deterioration rate's value exists between $(0, 1)$
$D_1 = [D_{1L}, D_{1U}]$	Unit	Interval valued Cost of deterioration, Rupee per unit time in RW
$D_2 = [D_{2L}, D_{2U}]$	Unit	Interval valued Cost of deterioration, Rupee per unit time in OW
p	Unit	Rupees per unit time spent in RW
$h = [h_L, h_U]$	Constant	Interval valued Holding Cost Constant that does not depend on Time
T	yr	Total length of an inventory cycle,hence $T = t_a + t_b$
$I_r(t)$	Unit	Inventory level at any time t in RW
$I_o(t)$	Unit	Inventory level at any time t in OW

3. PROBLEM DEFINITION

Suppose a scenario where a company place a request for $(S + R)$ units of a given product and pays for a number "k" of the purchase price by making "n" uniform payments at uniform periods over the lead time "M" before paying the balance at time $t = 0$ to receive the lot. The on-hand inventory level changes to "S" shortly after "R" are used

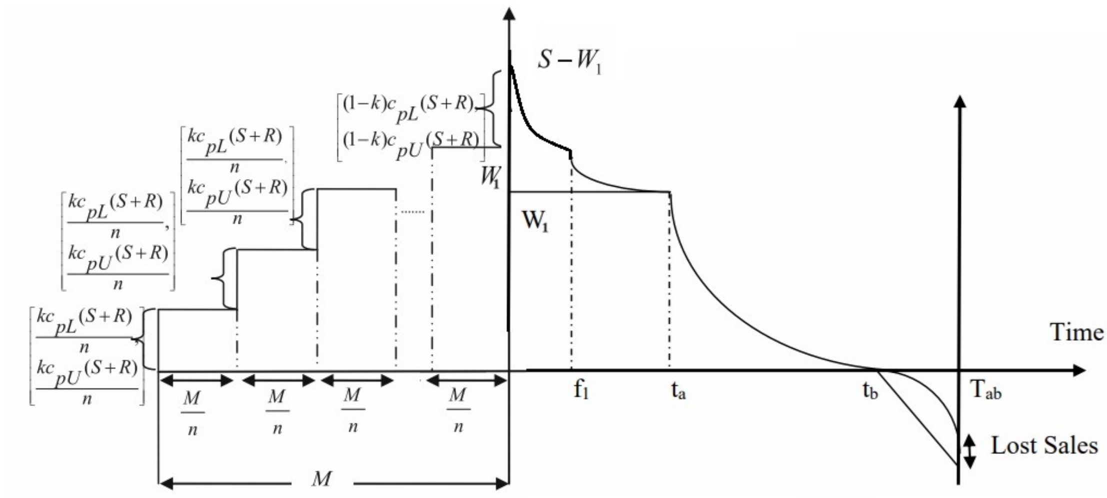


Figure 1: Graphical representation of two-warehouse inventory system under prepayments with shortages

to partially satisfy the backlogged demand. The remaining portion $(S - W_1)$ is saved in "RW" while "W_1" units are kept in "OW". The holding cost in the "RW" is apparently higher than that in the "OW" due to the "$RW's$" better facilities, and as a result, the "RW"'s products will reportedly be taken first. In the time interval $[0, f_1]$, the inventory decreases due to customer demand. But in the time interval $[f_1, t_a]$ the inventory level decreases due to both the constant depreciation rate "β" and the customer demand $D(p)$. It becomes zero in RW at the time $t = t_a$. The inventory level in OW, however, drops as a result of a constant rate of deterioration "α" within the range $[f_1, t_a]$. The levels of inventory are zero for RW and greater than zero for OW in the time interval $[t_a, t_b]$. Individual time intervals may be considered to be $[0, f_1]$, $[f_1, t_a]$, $[t_a, t_b]$ and $[t_b, T]$.

Here there will be two cases, the first case is the deterioration rate starts when the level of inventory of RW is in use in the interval $(0 \leq f_1 \leq t_a)$, and in the second case the deterioration rate starts when the level of inventory of OW is in use in the interval $(t_a \leq f_1 \leq t_b)$.

3.1. For CASE I: $(0 \leq f_1 \leq t_a)$

Following are the differential equations that explain how the levels of inventory for OW and RW differ:
For various time periods, differential equations for RW are:

$$\frac{dI_r(t)}{dt} = -(a - bI_r(t)), \qquad 0 < t < f_1 \tag{1}$$

$$\frac{dI_r(t)}{dt} = -(a - bI_r(t)) - \beta I_r(t), \qquad f_1 < t < t_a \tag{2}$$

subject to the conditions:

$$I_r(t) = \begin{cases} S - W_1, & at\ t = 0 \\ 0, & at\ t = t_a \end{cases} \tag{3}$$

On solving the above differential Equations:

$$I_r(t) = \frac{a}{b} - \left\{\frac{a}{b} - (S - W_1)\right\} e^{bt}, \qquad 0 < t < f_1 \tag{4}$$

$$I_r(t) = \frac{a}{b-\beta}\{(1 - e^{-(b-\beta)(t_a - t)})\}, \qquad f_1 < t < t_a. \tag{5}$$

Furthermore, the following differential Equations can be used to represent the inventory level $I_o(t)$ in OW at any instant "t"

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t), \qquad f_1 < t < t_a \tag{6}$$

$$\frac{dI_o(t)}{dt} = -(a - bI_o(t)) - \alpha I_o(t), \qquad t_a < t < t_b \tag{7}$$

$$\frac{dI_o(t)}{dt} = -\eta(a - bI_o(t)), \qquad t_b < t < T \tag{8}$$

subject to the conditions:

$$I_o(t) = \begin{cases} W_1, & at\ t = f_1 \\ 0, & at\ t = t_b \\ -R, & at\ t = T \end{cases} \tag{9}$$

On solving the above differential Equations:

$$I_o(t) = W_1 e^{-\alpha(t - f_1)}, \qquad f_1 < t < t_a \tag{10}$$

$$I_o(t) = \frac{a}{b-\alpha}\{(1 - e^{-(b-\alpha)(t_b - t)})\}, \qquad t_a < t < t_b \tag{11}$$

$$I_o(t) = \frac{a}{b} - \left\{\frac{a}{b} + R\right\} e^{-\eta b(T-t)}, \qquad t_b < t < T \tag{12}$$

By considering the continuity at $t = f_1$, $t = t_a$ and $t = t_b$, we can write:

$$S = W_1 - \frac{a}{2 - e^{bf_1}}\left[\frac{1}{b-\beta}\left\{1 - e^{-(b-\beta)(t_a - f_1)}\right\} + \frac{1}{b}e^{bf_1}\right] \tag{13}$$

$$t_b = t_a - \frac{1}{b-\alpha}\left[\ln\left(1 - \frac{W_1(b-\alpha)}{a}e^{-\alpha(t_a - f_1)}\right)\right] \tag{14}$$

$$R = \frac{a}{b}\left[e^{\eta b(T-t_b)} - 1\right] \tag{15}$$

Here, we discuss how the model's inventory-related costs originated based on the assumptions:

(*a*) Ordering Cost: A

(*b*) Purchase Cost: $C_p(S+R) = [C_{pL}(S+R), C_{pU}(S+R)] =$

$$\left[C_{pL}\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(a-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)}\right)\right.$$

$$+ C_{pL}\left(\frac{a\left(\beta e^{-bf_1} - be^{-bt_a-\beta f_1+\beta t_a}\right)}{b(b-\beta)} + W_1\right),$$

$$C_{pU}\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(a-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)}\right)$$

$$\left. + C_{pU}\frac{a\left(\beta e^{-bf_1} - be^{-bt_a-\beta f_1+\beta t_a}\right)}{b(b-\beta)} + W_1\right]$$

(*c*) Holding Cost:

$$\left[\int_0^{f_1} h_L(t) I_r(t)\,dt + \int_{f_1}^{t_a} h_L(t)\, I_r(t)\,dt + \int_0^{f_1} h_L(t) I_o(t)\,dt + \int_{f_1}^{t_a} h_L(t) I_o(t)\,dt\right.$$

$$+ \int_{t_a}^{t_b} h_L(t) I_o(t)\,dt, \quad \int_0^{f_1} h_U(t) I_r(t)\,dt + \int_{f_1}^{t_a} h_U(t)\, I_r(t)\,dt + \int_0^{f_1} h_U(t) I_o(t)\,dt$$

$$\left. + \int_{f_1}^{t_a} h_U(t) I_o(t)\,dt + \int_{t_a}^{t_b} h_U(t)\, I_o(t) dt\right]$$

$$= \left[\frac{-\frac{e^{bf_1}(a+b(W_1-S))(b(f_1p+h_L)-p)}{b^2} + \frac{(bh_L-p)(a+b(W_1-S))}{b^2} + \frac{1}{2}af_1{}^2p + af_1h_L}{b}\right.$$

$$- \frac{a\left(\frac{e^{(b-\beta)(f_1-t_a)}(p-(b-\beta)(f_1p+h_L))}{(b-\beta)^2} + \frac{(b-\beta)(h_L+pt_a)-p}{(b-\beta)^2} + \frac{f_1{}^2p}{2} + f_1h_L - h_Lt_a - \frac{pt_a{}^2}{2}\right)}{b-\beta}$$

$$- \frac{a\left(\frac{e^{(b-\alpha)(t_a-t_b)}(q-(b-\alpha)(h_L+qt_a))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_L+qt_b)-q}{(b-\alpha)^2}+h_Lt_a-h_Lt_b+\frac{qt_a{}^2}{2}-\frac{qt_b{}^2}{2}\right)}{b-\alpha}$$

$$+\frac{W_1\left(-e^{\alpha(f_1-t_a)}(\alpha h_L+\alpha qt_a+q)+\alpha f_1q+\alpha h_L+q\right)}{\alpha^2}+\frac{1}{2}f_1W_1(f_1q+2h_L),$$

$$\frac{-\frac{e^{bf_1}(a+b(W_1-S))(b(f_1p+h_U)-p)}{b^2}+\frac{(bh_U-p)(a+b(W_1-S))}{b^2}+\frac{1}{2}af_1{}^2p+af_1h_U}{b}$$

$$-\frac{a\left(\frac{e^{(b-\beta)(f_1-t_a)}(p-(b-\beta)(f_1p+h_U))}{(b-\beta)^2}+\frac{(b-\beta)(h_U+pt_a)-p}{(b-\beta)^2}+\frac{f_1{}^2p}{2}+f_1h_U-h_Ut_a-\frac{pt_a{}^2}{2}\right)}{b-\beta}$$

$$-\frac{a\left(\frac{e^{(b-\alpha)(t_a-t_b)}(q-(b-\alpha)(h_U+qt_a))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_U+qt_b)-q}{(b-\alpha)^2}+h_Ut_a-h_Ut_b+\frac{qt_a{}^2}{2}-\frac{qt_b{}^2}{2}\right)}{b-\alpha}$$

$$+\frac{W_1\left(-e^{\alpha(f_1-t_a)}(\alpha h_U+\alpha qt_a+q)+\alpha f_1q+\alpha h_U+q\right)}{\alpha^2}+\frac{1}{2}f_1W_1\left(f_1q+2h_U\right)\Bigg]$$

(d) Deterioration Cost:

$$\Bigg[D_{1L}\int_{f_1}^{t_a}\theta I_r(t)\,dt+D_{2L}\int_{f_1}^{t_a}\gamma I_o(t)\,dt+D_{2L}\int_{t_a}^{t_b}\gamma I_o(t)\,dt,$$

$$D_{1U}\int_{f_1}^{t_a}\theta I_r(t)\,dt+D_{2U}\int_{f_1}^{t_a}\gamma I_o(t)\,dt+D_{2U}\int_{t_a}^{t_b}\gamma I_o(t)\,dt\Bigg]$$

$$=\Bigg[\frac{aD_{1L}\theta\left(e^{(b-\beta)(f_1-t_a)}+b(t_a-f_1)+\beta f_1-\beta t_a-1\right)}{(b-\beta)^2}$$

$$+\frac{\gamma D_{2L}W_1\left(e^{\alpha(f_1-t_a)}-e^{\alpha(f_1-t_b)}\right)}{\alpha}-\frac{\gamma D_{2L}W_1\left(e^{\alpha(f_1-t_a)}-1\right)}{\alpha},$$

$$\frac{aD_{1U}\theta\left(e^{(b-\beta)(f_1-t_a)}+b(t_a-f_1)+\beta f_1-\beta t_a-1\right)}{(b-\beta)^2}$$

$$+\frac{\gamma D_{2U}W_1\left(e^{\alpha(f_1-t_a)}-e^{\alpha(f_1-t_b)}\right)}{\alpha}-\frac{\gamma D_{2U}W_1\left(e^{\alpha(f_1-t_a)}-1\right)}{\alpha}\Bigg]$$

(e) Shortage Cost:

$$\Bigg[-c_L\int_{t_b}^{T}I_o(t)\,dt,\quad -c_U\int_{t_b}^{T}I_o(t)\,dt\Bigg]$$

$$= \left[-\frac{c_L\left((a+bR)e^{b\eta(t_b-T)}+a(b\eta(T-t_b)-1)-bR\right)}{b^2\eta}, \right.$$
$$\left. -\frac{c_U\left((a+bR)e^{b\eta(t_b-T)}+a(b\eta(T-t_b)-1)-bR\right)}{b^2\eta} \right]$$

(f) Capital Cost:

$$\left[\left(I_c \left[\frac{kC_{pL}(S+R)}{n} \, \frac{M}{n} \, (1+2+3+.........+n) \right] \right), \right.$$
$$\left. \left(I_c \left[\frac{kC_{pU}(S+R)}{n} \, \frac{M}{n} \, (1+2+3+.........+n) \right] \right) \right]$$

$$= \left[\frac{C_{pL}I_ckM(n+1)\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)} \right)}{2n} \right.$$
$$+ \frac{C_{pL}I_ckM(n+1)W_1}{2n}$$
$$+ \frac{C_{pL}I_ckM(n+1)(a)\,\beta e^{-bf_1-be^{-bt_a-\beta f_1+\beta t_a}}}{2nb(b-\beta)},$$
$$\frac{C_{pU}I_ckM(n+1)\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)} \right)}{2n}$$
$$+ \frac{C_{pU}I_ckM(n+1)W_1}{2n}$$
$$\left. + \frac{C_{pU}I_ckM(n+1)(a)\,\beta e^{-bf_1-be^{-bt_a-\beta f_1+\beta t_a}}}{2nb(b-\beta)} \right]$$

Consequently, the total cyclic cost per unit of time is

$$TC1 = \frac{1}{T}\left[\begin{array}{l} \langle \text{Ordering Cost} \rangle + \langle \text{Purchase Cost} \rangle + \langle \text{Holding Cost} \rangle \\ + \langle \text{Deterioration Cost} \rangle + \langle \text{Shortage Cost} \rangle + \langle \text{Capital Cost} \rangle \end{array} \right]$$

$$TC1 = [TC_L, TC_U]$$

where
TC_L

$$
\begin{aligned}
&= \frac{A_L}{T} \\
&+ \frac{1}{T}\left[\frac{1}{2}f_1W_1(f_1q+2h_L)\right] \\
&+ \frac{1}{T}\left[\frac{-\frac{e^{bf_1}(a+b(W_1-S))(b(f_1p+h_L)-p)}{b^2}+\frac{(bh_L-p)(a+b(W_1-S))}{b^2}+\frac{1}{2}af_1{}^2p+af_1h_L}{b}\right] \\
&- \frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\alpha)(t_a-t_b)}(q-(b-\alpha)(h_L+qt_a))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_L+qt_b)-q}{(b-\alpha)^2}\right)}{b-\alpha}\right] \\
&- \frac{1}{T}\left[\frac{h_Lt_a-h_Lt_b+\frac{qt_a{}^2}{2}-\frac{qt_b{}^2}{2}}{b-\alpha}\right] \\
&+ \frac{1}{T}\left[\frac{W_1\left(-e^{\alpha(f_1-t_a)}(\alpha h_L+\alpha qt_a+q)+\alpha f_1q+\alpha h_L+q\right)}{\alpha^2}\right] \\
&- \frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\beta)(f_1-t_a)}(p-(b-\beta)(f_1p+h_L))}{(b-\beta)^2}+\frac{(b-\beta)(h_L+pt_a)-p}{(b-\beta)^2}\right)}{b-\beta}\right] \\
&- \frac{a}{T}\left[\frac{\frac{f1^2p}{2}+f_1h_L-h_Lt_a-\frac{pt_a{}^2}{2}}{b-\beta}\right]+\frac{C_{pL}W_1}{T} \\
&+ \frac{1}{T}\left[C_{pL}\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)}\right)\right] \\
&+ \frac{aC_{pL}\left(\beta e^{-bf_1}-be^{-bt_a-\beta f_1+\beta t_a}\right)}{Tb(b-\beta)} \\
&+ \frac{1}{T}\left[\frac{C_{pL}I_ckM(n+1)W_1}{2n}\right] \\
&+ \frac{1}{T}\left[\frac{C_{pL}I_ckM(n+1)(a)\,\beta e^{-bf_1-be^{-bt_a-\beta f_1+\beta t_a}}}{2nb(b-\beta)}\right]
\end{aligned}
$$

$$+\frac{1}{T}\left[\frac{C_{pL}I_ckM(n+1)\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)e^{-bt_a+\beta t_a}\right)}{b(b-\beta)}\right)}{2n}\right]$$

$$-\frac{1}{T}\left[\frac{\gamma D_{2L}W_1\left(e^{\alpha(f_1-t_a)}-e^{\alpha(f_1-t_b)}\right)}{\alpha}+\frac{\gamma D_{2L}W_1\left(e^{\alpha(f_1-t_a)}-1\right)}{\alpha}\right]$$

$$+\left[\frac{ac_L\left(\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{Tb^2\eta}\right.$$

$$\left.-\frac{ac_L\left(\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}-t_a+t_b\right)\right)\right)}{Tb^2\eta}\right]$$

$$+\frac{1}{T}\left[\frac{ac_Lb\eta(t_b-T)}{b^2\eta}\right]+\frac{1}{T}\left[\frac{aD_{1L}\theta\left(e^{(b-\beta)(f_1-t_a)}+b(t_a-f_1)+\beta f_1-\beta t_a-1\right)}{(b-\beta)^2}\right]$$

and
TC_U

$$=\frac{A_U}{T}$$

$$+\frac{1}{T}\left[\frac{1}{2}f_1W_1(f_1q+2h_U)+\frac{-\frac{e^{bf_1}(a+b(W_1-S))(b(f_1p+h_U)-p)}{b^2}}{b}\right]$$

$$+\frac{1}{T}\left[\frac{\frac{(bh_U-p)(a+b(W_1-S))}{b^2}+\frac{1}{2}af_1{}^2p+af_1h_U}{b}\right]$$

$$
- \frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\alpha)(t_a-t_b)}(q-(b-\alpha)(h_U+qt_a))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_U+qt_b)-q}{(b-\alpha)^2}+h_Ut_a-h_Ut_b+\frac{qt_a{}^2}{2}-\frac{qt_b{}^2}{2}\right)}{b-\alpha}\right]
$$

$$
+ \frac{1}{T}\left[\frac{W_1\left(-e^{\alpha(f_1-t_a)}(\alpha h_U+\alpha q t_a+q)+\alpha f_1 q+\alpha h_U+q\right)}{\alpha^2}\right]
$$

$$
- \frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\beta)(f_1-t_a)}(p-(b-\beta)(f_1p+h_U))}{(b-\beta)^2}+\frac{(b-\beta)(h_U+pt_a)-p}{(b-\beta)^2}+\frac{f1^2p}{2}+f_1h_U-h_Ut_a-\frac{pt_a{}^2}{2}\right)}{b-\beta}\right]
$$

$$
+ \frac{C_{pU}W_1}{T}
$$

$$
+ \frac{1}{T}\left[C_{pU}\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)}\right)\right]
$$

$$
+ \frac{1}{T}\left[C_{pU}\left(\frac{a\left(\beta e^{-bf_1}-be^{-bt_a-\beta f_1+\beta t_a}\right)}{b(b-\beta)}\right)\right]
$$

$$
+ \frac{1}{T}\left[\frac{C_{pU}I_ckM(n+1)\left(\frac{a\left((b-\beta)\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{b(b-\beta)}\right)}{2n}\right]
$$

$$
+ \frac{1}{T}\left[\frac{C_{pU}I_ckM(n+1)\left(\frac{a\left(\beta e^{-bf_1}-be^{-bt_a-\beta f_1+\beta t_a}\right)}{b(b-\beta)}+W_1\right)}{2n}\right]
$$

$$
- \frac{1}{T}\left[\frac{\gamma D_{2U}W_1\left(e^{\alpha(f_1-t_a)}-e^{\alpha(f_1-t_b)}\right)}{\alpha}+\frac{\gamma D_{2U}W_1\left(e^{\alpha(f_1-t_a)}-1\right)}{\alpha}\right]
$$

$$+\left[\frac{ac_U\left(\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}+T-t_a\right)\right)\right)}{Tb^2\eta}\right.$$

$$\left.-\frac{ac_U\left(\exp\left(b\eta\left(\frac{\log\left(\frac{W_1(\alpha-b)e^{\alpha(f_1-t_a)}}{a}+1\right)}{b-\alpha}-t_a+t_b\right)\right)\right)}{Tb^2\eta}\right]$$

$$+\frac{1}{T}\left[\frac{ac_Ub\eta(t_b-T)}{b^2\eta}\right]+\frac{1}{T}\left[\frac{aD_{1U}\theta\left(e^{(b-\beta)(f_1-t_a)}+b(t_a-f_1)+\beta f_1-\beta t_a-1\right)}{(b-\beta)^2}\right]$$

3.2. For CASE II: $(t_a \leq f_1 \leq t_b)$

For various time periods, differential equations for RW are:

$$\frac{dI_r(t)}{dt}=-(a-bI_r(t)), \qquad 0<t<t_a \tag{16}$$

subject to the conditions:

$$I_r(t)=\begin{cases} S-W_1, & at\ t=0 \\ 0, & at\ t=t_a \end{cases} \tag{17}$$

On solving the above differential Equations:

$$I_r(t)=\frac{a}{b}\left[1-e^{-b(t_a-t)}\right] \qquad 0<t<t_a \tag{18}$$

Furthermore, the following differential Equations can be used to represent the inventory level $I_o(t)$ in OW at any instant "t"

$$\frac{dI_o(t)}{dt}=-(a-bI_o(t)), \qquad t_a<t<f_1 \tag{19}$$

$$\frac{dI_o(t)}{dt}=-(a-bI_o(t))-\alpha I_o(t), \qquad f_1<t<t_b \tag{20}$$

$$\frac{dI_o(t)}{dt}=-\eta(a-bI_o(t)), \qquad t_b<t<T \tag{21}$$

subject to the Conditions:

$$I_o(t)=\begin{cases} W_1, & at\ t=t_a \\ 0, & at\ t=t_b \end{cases} \tag{22}$$

On solving the above differential Equations:

$$I_o(t) = \frac{a}{b} - \left[\frac{a}{b} - W_1\right] e^{b(t-t_a)}, \qquad t_a < t < f_1 \tag{23}$$

$$I_o(t) = \frac{a}{b-\alpha}\{(1 - e^{-(b-\alpha)(t_b-t)})\}, \qquad f_1 < t < t_b \tag{24}$$

$$I_o(t) = \frac{a}{b} - \left\{\frac{a}{b} + R\right\} e^{-\eta b(T-t)}, \qquad t_b < t < T \tag{25}$$

By considering the continuity at $t = t_a$, $t = f_1$ and $t = t_b$, we can write:

$$S = W_1 + \frac{a}{b}\left[1 - e^{-bt_a}\right] \tag{26}$$

$$R = \frac{a}{b}\left[e^{\eta b(T-t_b)} - 1\right] \tag{27}$$

$$t_b = f_1 - \frac{1}{b-\alpha}\left[\ln\left(1 - \frac{b-\alpha}{b} + \left(\frac{b-\alpha}{b} + \frac{(b-\alpha)W_1}{a}\right) e^{b(f_1-t_a)}\right)\right] \tag{28}$$

Here, we discuss how the model's inventory-related costs originated based on the assumptions:

(a) Ordering Cost: A

(b) Purchase Cost: $C_p(S+R)= [C_{pL}(S+R), \quad C_{pU}(S+R)]$

$$= \Bigg[C_{pL}\left[\frac{a\left(\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b}\right]$$

$$+ C_{pL}\left[\frac{a\left(\exp\left(-b\eta\left(f_1-T\right)\right)-1\right)}{b}\right] + C_{pL}\left[\frac{a - ae^{-bt_a}}{b} + W_1\right],$$

$$C_{pU}\left[\frac{a\left(\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b}\right]$$

$$+ C_{pU}\left[\frac{a\left(\exp\left(-b\eta\left(f_1-T\right)\right)-1\right)}{b}\right] + C_{pU}\left[\frac{a - ae^{-bt_a}}{b} + W_1\right]\Bigg]$$

(c) Holding Cost:

$$\Bigg[\int_0^{t_a} h_L(t)\, I_r(t)dt + \int_{t_a}^{f_1} h_L(t) I_o(t)\,dt + \int_{f_1}^{t_b} h_L(t) I_o(t)\,dt,$$

$$\int_0^{t_a} h_U(t)\, I_r(t)dt + \int_{t_a}^{f_1} h_U(t) I_o(t)\,dt + \int_{f_1}^{t_b} h_U(t) I_o(t)\,dt\Bigg]$$

$$= \Bigg[\frac{\frac{(bW_1-a)e^{b(f_1-t_a)}(b(f_1q+h_L)-q)}{b^2} - \frac{(bW_1-a)(b(h_L+qt_a)-q)}{b^2}}{b}$$

$$+ \frac{\frac{1}{2}af_1{}^2q + af_1h_L - ah_Lt_a - \frac{1}{2}aqt_a{}^2}{b}$$

$$- \frac{a\left(\frac{e^{-bt_a}(p-bh_L)}{b^2} + \frac{bh_L+bpt_a-p}{b^2} - \frac{1}{2}t_a(2h_L+pt_a)\right)}{b}$$

$$- \frac{a\left(\frac{e^{(b-\alpha)(f_1-t_b)}(q-(b-\alpha)(f_1q+h_L))}{(b-\alpha)^2} + \frac{(b-\alpha)(h_L+qt_b)-q}{(b-\alpha)^2}\right)}{b-\alpha}$$

$$+ \frac{a\left(\frac{f_1{}^2q}{2} + f_1h_L - h_Lt_b - \frac{qt_b{}^2}{2}\right)}{b-\alpha},$$

$$\frac{\frac{(bW_1-a)e^{b(f_1-t_a)}(b(f_1q+h_U)-q)}{b^2} - \frac{(bW_1-a)(b(h_U+qt_a)-q)}{b^2}}{b}$$

$$+ \frac{\frac{1}{2}af_1{}^2q + af_1h_U - ah_Ut_a - \frac{1}{2}aqt_a{}^2}{b}$$

$$- \frac{a\left(\frac{e^{-bt_a}(p-bh_U)}{b^2} + \frac{bh_U+bpt_a-p}{b^2} - \frac{1}{2}t_a(2h_U+pt_a)\right)}{b}$$

$$- \frac{a\left(\frac{e^{(b-\alpha)(f_1-t_b)}(q-(b-\alpha)(f_1q+h_U))}{(b-\alpha)^2} + \frac{(b-\alpha)(h_U+qt_b)-q}{(b-\alpha)^2}\right)}{b-\alpha}$$

$$+ \frac{a\left(\frac{f_1{}^2q}{2} + f_1h_U - h_Ut_b - \frac{qt_b{}^2}{2}\right)}{b-\alpha}\Bigg]$$

(d) Deterioration Cost:

$$\left[D_{2L}\int_{f_1}^{t_b} \gamma I_o(t)dt, \quad D_{2U}\int_{f_1}^{t_b} \gamma I_o(t)dt\right]$$

$$= \left[\frac{a\gamma D_{2L} \left(e^{(b-\alpha)(f_1 - t_b)} + b(t_b - f_1) + \alpha f_1 - \alpha t_b - 1 \right)}{(b-\alpha)^2}, \right.$$
$$\left. \frac{a\gamma D_{2U} \left(e^{(b-\alpha)(f_1 - t_b)} + b(t_b - f_1) + \alpha f_1 - \alpha t_b - 1 \right)}{(b-\alpha)^2} \right]$$

(e) Shortage Cost:

$$\left[-c_L \int_{t_b}^{T} I_o(t)dt, \ -c_U \int_{t_b}^{T} I_o(t)dt \right]$$
$$= \left[-\frac{c_L \left((a+bR)e^{b\eta(t_b - T)} + a(b\eta(T - t_b) - 1) - bR \right)}{b^2\eta}, \right.$$
$$\left. -\frac{c_U \left((a+bR)e^{b\eta(t_b - T)} + a(b\eta(T - t_b) - 1) - bR \right)}{b^2\eta} \right]$$

(f) Capital Cost:

$$\left[\left(I_c \left[\frac{kC_{pL}(S+R)}{n} \frac{M}{n} \ (1+2+3+ \ldots\ldots\ldots + n) \right] \right), \right.$$
$$\left. \left(I_c \left[\frac{kC_{pU}(S+R)}{n} \frac{M}{n} \ (1+2+3+ \ldots\ldots\ldots + n) \right] \right) \right]$$

$$= \left[\frac{C_{pL} I_c k M (n+1) \left(\frac{a \left(\exp \left(-b\eta \left(-\frac{\log \left(\frac{e^{-bta} \left(a\alpha \left(e^{bta} - e^{bf_1} \right) + abe^{bf_1} - bW_1 (b-\alpha) e^{bf_1} \right)}{ab} \right)}{b-\alpha} \right) \right) \right)}{b} \right)}{2n} \right.$$
$$+ \frac{C_{pL} I_c k M (n+1) \left(\frac{a(\exp(-b\eta(f_1 - T)) - 1)}{b} + S \right)}{2n},$$

$$\frac{C_{pU}I_ckM(n+1)\left(\frac{a\left(\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b}\right)}{2n}$$

$$\left.+\frac{C_{pU}I_ckM(n+1)\left(\frac{a(\exp(-b\eta(f_1-T))-1)}{b}+S\right)}{2n}\right]$$

Consequently, the total cyclic cost per unit of time is

$$TC2=\frac{1}{T}\left[\begin{array}{l}\langle\text{Ordering Cost}\rangle+\langle\text{Purchase Cost}\rangle+\langle\text{Holding Cost}\rangle\\+\langle\text{Deterioration Cost}\rangle+\langle\text{Shortage Cost}\rangle+\langle\text{Capital Cost}\rangle\end{array}\right]$$

$TC2=[TC_L,TC_U]$

where

TC_L

$$=\frac{A_L}{T}$$

$$+\frac{1}{T}\left[\frac{C_{pL}I_ckM(n+1)\left(\frac{a\left(\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b}\right)}{2n}\right]$$

$$+\frac{1}{T}\left[\frac{C_{pL}I_ckM(n+1)\left(\frac{a(\exp(-b\eta(f_1-T))-1)}{b}+S\right)}{2n}\right]$$

$$-\frac{1}{T}\left[\frac{a\left(\frac{e^{-bt_a}(p-bh_L)}{b^2}+\frac{bh_L+bpt_a-p}{b^2}-\frac{1}{2}t_a(2h_L+pt_a)\right)}{b}\right]$$

$$+\frac{1}{T}\left[\frac{\frac{(bW_1-a)e^{b(f_1-t_a)}(b(f_1q+h_L)-q)}{b^2}-\frac{(bW_1-a)(b(h_L+qt_a)-q)}{b^2}}{b}\right]$$

$$+\frac{1}{T}\left[\frac{\frac{1}{2}a{f_1}^2q+af_1h_L-ah_Lt_a-\frac{1}{2}aq{t_a}^2}{b}\right]$$

$$-\frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\alpha)(f_1-t_b)}(q-(b-\alpha)(f_1q+h_L))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_L+qt_b)-q}{(b-\alpha)^2}+\frac{{f_1}^2q}{2}+f_1h_L-h_Lt_b-\frac{q{t_b}^2}{2}\right)}{b-\alpha}\right]$$

$$+\frac{1}{T}\left[\frac{a\gamma D_{2L}\left(e^{(b-\alpha)(f_1-t_b)}+b(t_b-f_1)+\alpha f_1-\alpha t_b-1\right)}{(b-\alpha)^2}\right]$$

$$-\frac{1}{T}\left[\frac{ac_L\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bta}\left(a\alpha\left(e^{bta}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}+f_1-T\right)\right)}{b^2\eta}\right]$$

$$\frac{1}{T}\left[\frac{\left(b\eta(T-t_b)\exp\left(b\eta\left(-\frac{\log\left(\frac{e^{-bta}\left(a\alpha\left(e^{bta}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b^2\eta}\right]$$

$$+\frac{1}{T}\left[\frac{\left(b\eta(T-t_b)\exp\left(b\eta\left(f_1-T\right)\right)+e^{b\eta(t_b-T)}-1\right)}{b^2\eta}\right]$$

and
TC_U

$$=\frac{A_U}{T}$$

$$+\frac{1}{T}\left[\frac{C_{pU}I_ckM(n+1)\left(\frac{a\left(\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}\right)\right)\right)}{b}\right)}{2n}\right]$$

$$+\frac{1}{T}\left[\frac{C_{pU}I_ckM(n+1)\left(\frac{a(\exp(-b\eta(f_1-T))-1)}{b}+S\right)}{2n}\right]$$

$$-\frac{1}{T}\left[\frac{a\left(\frac{e^{-bt_a}(p-bh_U)}{b^2}+\frac{bh_U+bpt_a-p}{b^2}-\frac{1}{2}t_a(2h_U+pt_a)\right)}{b}\right]$$

$$+\frac{1}{T}\left[\frac{\frac{(bW_1-a)e^{b(f_1-t_a)}(b(f_1q+h_U)-q)}{b^2}-\frac{(bW_1-a)(b(h_U+qt_a)-q)}{b^2}}{b}\right]$$

$$+\frac{1}{T}\left[\frac{\frac{1}{2}af_1{}^2q+af_1h_U-ah_Ut_a-\frac{1}{2}aqt_a{}^2}{b}\right]$$

$$-\frac{1}{T}\left[\frac{a\left(\frac{e^{(b-\alpha)(f_1-t_b)}(q-(b-\alpha)(f_1q+h_U))}{(b-\alpha)^2}+\frac{(b-\alpha)(h_U+qt_b)-q}{(b-\alpha)^2}+\frac{f_1{}^2q}{2}+f_1h_U-h_Ut_b-\frac{qt_b{}^2}{2}\right)}{b-\alpha}\right]$$

$$+\frac{1}{T}\left[\frac{a\gamma D_{2U}\left(e^{(b-\alpha)(f_1-t_b)}+b(t_b-f_1)+\alpha f_1-\alpha t_b-1\right)}{(b-\alpha)^2}\right]$$

$$-\frac{1}{T}\left[\frac{ac_U\exp\left(-b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}+f_1-T\right)\right)}{b^2\eta}\right]$$

$$\frac{1}{T}\left[\frac{\left(b\eta(T-t_b)\exp\left(b\eta\left(-\frac{\log\left(\frac{e^{-bt_a}\left(a\alpha\left(e^{bt_a}-e^{bf_1}\right)+abe^{bf_1}-bW_1(b-\alpha)e^{bf_1}\right)}{ab}\right)}{b-\alpha}+f_1-T\right)\right)\right)}{b^2\eta}\right]$$

$$+\frac{1}{T}\left[\frac{\left(e^{b\eta(t_b-T)}-1\right)}{b^2\eta}\right]$$

4. Numerical Examples

4.1. For CASE I: When $(0 \leq f_1 \leq t_a)$

Considering the values as $A_L = 495$, $S = 115$, $f_1 = 0.1912$, $h_L = 10$, $q = 12$, $W_1 = 100$, $p = 12$, $b = 0.5$, $t_a = 0.2228$, $t_b = 0.8818$, $\alpha = 0.1$, $C_{pL} = 8$, $\beta = 0.08$, $I_c = 0.25$, $k = 0.4$, $M = 0.25$, $a = 200$, $\theta = 0.08$, $D_{1L} = 190$, $D_{2L} = 190$, $T = 1.1046$, $\gamma = 0.06$, $\eta = 0.8$, $c_L = 95$, $n = 15$.

After calculation the optimum value of $TC_L = 4054.93$

Considering the values as $A_U = 500$, $S = 115$, $f_1 = 0.1912$, $h_U = 12$, $q = 12$, $W_1 = 100$, $p = 12$, $b = 0.5$, $t_a = 0.2228$, $t_b = 0.8818$, $\alpha = 0.1$, $C_{pU} = 10$, $\beta = 0.08$, $I_c = 0.25$, $k = 0.4$, $M = 0.25$, $a = 200$, $\theta = 0.08$, $D_{1U} = 200$, $D_{2U} = 200$, $T = 1.1046$, $\gamma = 0.06$, $\eta = 0.8$, $c_U = 100$, $n = 15$.

After calculation the optimum value of $TC_U = 4606.39$

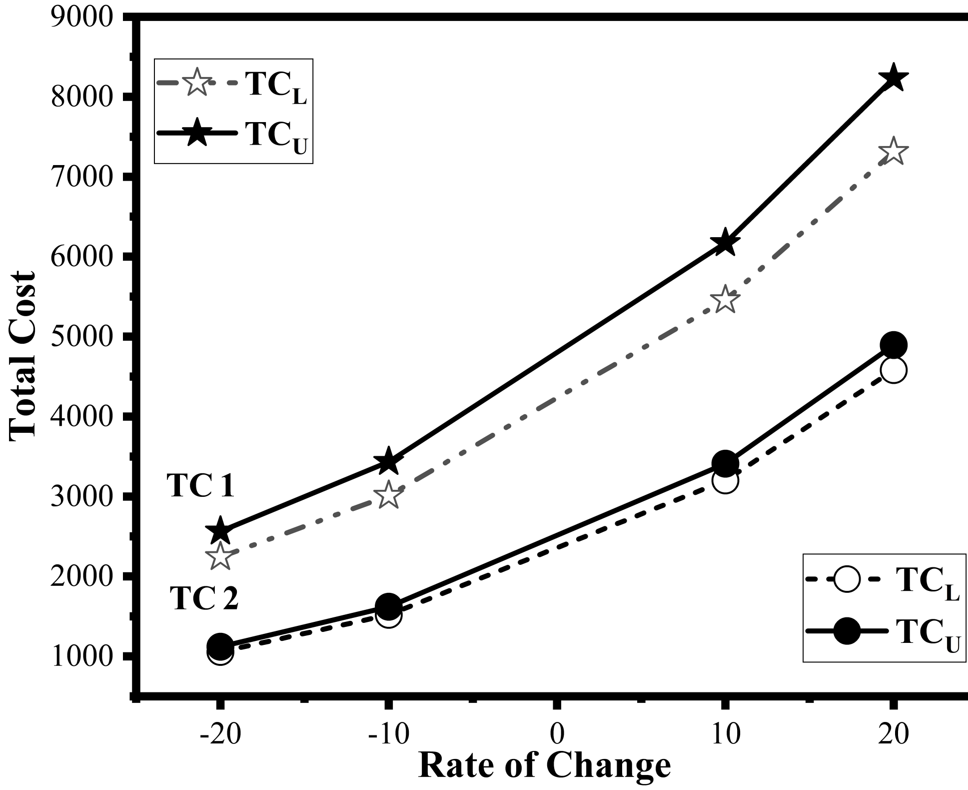
Hence, $TC1 = [TC_L, TC_U] = [4054.93, 4606.39]$

Analyzing holding cost for own warehouse with sensitivity

Table 1

Variation in parameter (%)	TC1= $[TC_L, TC_U]$	Cost Change in (%)	TC2= $[TC_L, TC_U]$	Cost Change in (%)
-20	[2243.52, 2563.80]	$[-1.8114, -2.0425]$	[1054.93, 1119.41]	$[-0.9096, -0.9794]$

-10	[3011.43, 3433.45]	[−1.0435, −1.1729]	[1520.46, 1620.71]	[−0.444, −0.4781]
10	[5456.06, 6171.91]	[1.4011, 1.5655]	[3200.69, 3410.11]	[1.2361, 1.3112]
20	[7310.93, 8232.89]	[3.256, 3.6265]	[4582.27, 4893.38]	[2.6177, 2.7945]



Observations from Table (1):

- Table 1 shows that the variation in the holding costs in its OW causes the cost per unit of time to change (increase/decrease).
- The graph makes it abundantly obvious that the cost per unit time in these two situations (both) rapidly rises when the holding cost of OW increases in value.
- However, whether the holding cost rises or falls, $TC1's$ value is always higher than $TC2's$ value.
- It is absolutely shown that maintaining a longer fixed Shelf-life is beneficial.

4.2. For CASE II: $(t_a \leq f_1 \leq t_b)$

Considering the values as $A_L = 495$, $S = 115$, $f_1 = 0.3567$, $h_L = 10$, $q = 12$, $W_1 = 100$, $p = 12$, $b = 0.5$, $t_a = 0.2228$, $t_b = 0.8818$, $\alpha = 0.1$, $C_{pL} = 8$, $\beta = 0.08$, $I_c = 0.25$, $k = 0.4$, $M = 0.25$, $a = 200$, $\theta = 0.08$, $D_{1L} = 190$, $D_{2L} = 190$, $T = 1.1046$, $\gamma = 0.06$, $\eta = 0.8$, $c_L = 95$, $n = 15$.

After calculation the optimum value of $TC_L = 1964.55$

Considering the values as $A_U = 500$, $S = 115$, $f_1 = 0.3567$, $h_U = 12$, $q = 12$, $W_1 = 100$, $p = 12$, $b = 0.5$, $t_a = 0.2228$, $t_b = 0.8818$, $\alpha = 0.1$, $C_{pU} = 10$, $\beta = 0.08$, $I_c = 0.25$, $k = 0.4$, $M = 0.25$, $a = 200$, $\theta = 0.08$, $D_{1U} = 200$, $D_{2U} = 200$, $T = 1.1046$, $\gamma = 0.06$, $\eta = 0.8$, $c_U = 95$, $n = 15$.

After calculation the optimum value of $TC_U = 2098.82$

Hence, $TC2 = [TC_L, TC_U] = [1964.55, 2098.82]$

Analyzing holding cost for rented warehouse with sensitivity

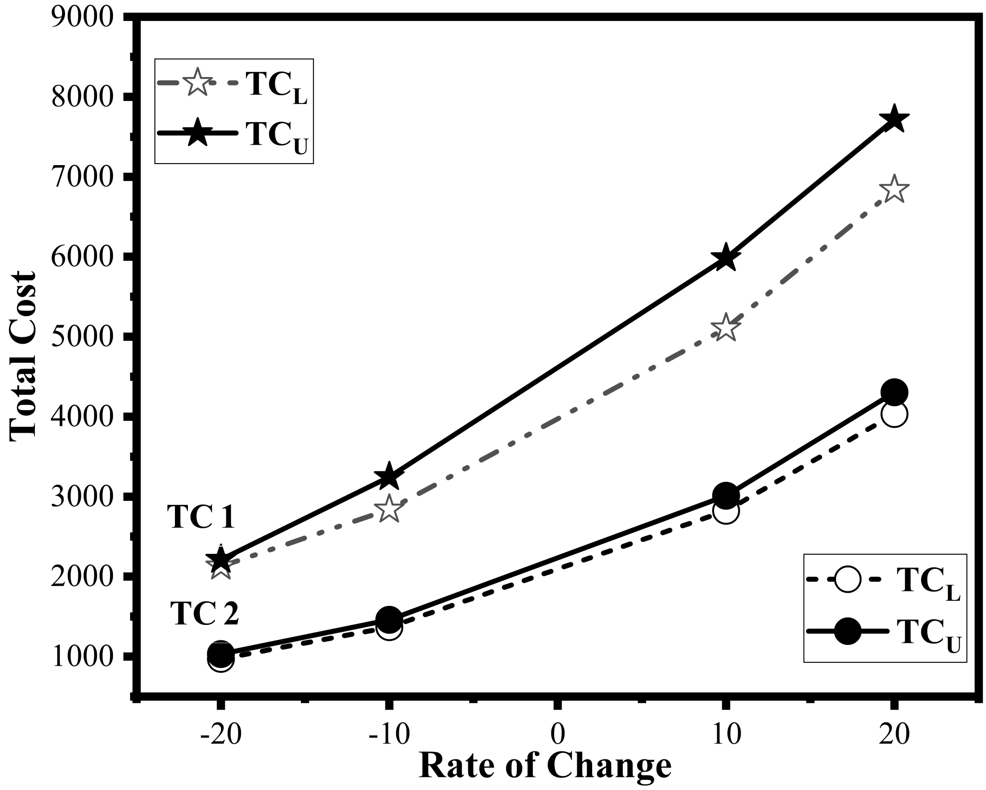
Table 2

Variation in parameter (%)	TC1= $[TC_L, TC_U]$	Cost Change in (%)	TC2= $[TC_L, TC_U]$	Cost Change in (%)
-20	[2128.31, 2213.13]	$[-1.9266, -2.3932]$	[970.90, 1031.37]	$[-0.9936, -1.0674]$
-10	[2839.56, 3246.74]	$[-1.2154, -1.3596]$	[1369, 1460.21]	$[-0.5955, -0.6386]$
10	[5106.55, 5983.91]	[1.0516, 1.3775]	[2823.81, 3010.41]	[0.8592, 0.9115]
20	[6836.86, 7717.87]	[2.7819, 3.1115]	[4032.35, 4305.67]	[2.0678, 2.2068]

Observations from Table (2):

- Table 2 shows that the variation in the holding costs in RW causes the cost per unit of time to change (increase/decrease).
- The graph makes it abundantly obvious that the cost per unit time in these two situations (both) rapidly rises when the holding cost of the RW increases in value.

- However, whether the holding cost rises or falls, $TC1's$ value is always higher than $TC2's$ value.
- It is absolutely shown that maintaining a longer fixed Shelf-life is beneficial.



5. Conclusions

In this study, for the first time, we have developed a two-warehouse inventory model with advance payment by considering interval-valued for ordering cost, purchase cost, shortage cost and deterioration cost. We also consider a fixed shelf life along with partially backlogging shortages. The rate of the demand depends on the amount of stock. The backlog rate changes over time. Based on the assumption, the cost function of this problem is a highly nonlinear constraint optimization problem.

When approaching non-linear inventory models, especially those involving optimization or simulation, several alternative solutions exist beyond using specific software like Mathematica. Mathematica's strength lies in its ability to handle symbolic computation, optimization, visualization, and integration across various mathematical domains. It is a powerful tool for analyzing and solving non-linear inventory models effectively. While generic algorithms are versatile, Mathematica's integrated environment and specialized functionalities provide a streamlined approach for researchers and practitioners in tackling complex inventory optimization problems. Thus, we utilise Mathematica to address this nonlinear optimisation problem in this study. A numerical example had been presented to demonstrate the computational result. Then, a sensitivity analysis had been performed to study the effect of changes of different parameters of the model on the optimal policy.

Future research may extend this work by examining factors such as time and advertisement-sensitive demand, trade credit, power-patterned demand, and displayed stock-dependent demand. In addition to these, the concept of this model can be expanded by incorporating several elements, such as advanced payment with discount policy, cash-follow analysis, flawed production process with rework policy, etc. Furthermore, this work's notion can be expanded to include fuzzy-valued or interval and fuzzy combined.

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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