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A Specific Class of Harmonic Meromorphic Functions Associated with the Mittag- Leffler Transformation

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Abstract. This article uses the Mittag-Leffler transformation to explore a specific category of harmonic meromorphic functions. The Mittag-Leffler transformation is a crucial tool for analysing meromorphic functions and provides essential properties and insights into their behavior. The main focus of this study is on harmonic meromorphic functions that can be represented by the Mittag-Leffler transformation. Furthermore, this research introduces an innovative derivative operator that incorporates this transformation into the domain of harmonic meromorphic functions. The Mittag-Leffler transformation is widely recognised as a powerful technique for analysing various mathematical functions, especially those with fractional order derivatives. It improves our understanding of harmonic meromorphic functions and their inherent characteristics. The research findings highlight the effectiveness of this new derivative operator in unraveling the complexities of these functions. They provide valuable insights into their behavior and fundamental traits. Additionally, the study offers coefficient inequalities, the distortion theorem, distortion bounds, extreme points, convex combinations, and convolution analyses specifically tailored to functions within this particular class.

2020 Mathematics Subject Classifications: 30C41, 30C50

Key Words and Phrases: Analytic functions, Meromorphic functions, Starlike functions, Multiplier transformation , Mittag-Leffler function, Hadamard products

1. Introduction

We begin by considering the open unit disk in the complex plane, denoted by $\diamondsuit = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, and the class of Meromorphic functions Σ defined as follows:

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$$f(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} a_\mu \varsigma^\mu, a_\mu \in \mathbb{C}. \tag{1}$$

where f is analytic in the punctured open unit disk $\diamondsuit^* = \diamondsuit \setminus \{0\} = \{\varsigma \in C : 0 < |\varsigma| < 1\}$. The class Σ was investigated and studied by Clunie [13].

For $f \in \Sigma$ of the form (1) and $g \in \Sigma$ given by

$$g(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} b_\mu \varsigma^\mu, b_\mu \in \mathbb{C}$$

.

the Hadamard product or convolution of f and g is defined as follows (see [1, 3]):

$$(f * g)(\varsigma) = (g * f)(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} a_\mu b_\mu \varsigma^\mu.$$

The well-known multiplier transformation operator $I_1(r, \lambda) : \Sigma \to \Sigma$ has been studied by Cho and Srivastava [12], Cho and Kim [11], and recently by Atshan and Joudah [6]. It is defined as follows:

$$I_1(r, \lambda) f(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} \left(\frac{\mu + \delta}{1 + \delta} \right)^r a_\mu \varsigma^\mu \quad (\delta \geq 0, \varsigma \in \diamondsuit^*). \tag{2}$$

For $\alpha, \eta \in \mathbb{C}$, Wiman [23] introduced the generalised Mittag–Leffler function $E_{\alpha,\eta}(\varsigma)$ which is given by:

$$E_\alpha(\varsigma) = \sum_{\mu=0}^{\infty} \frac{\varsigma^\mu}{\Gamma(\alpha\mu + 1)} \tag{3}$$

and

$$E_{\alpha,\eta}(\varsigma) = \sum_{\mu=0}^{\infty} \frac{\varsigma^\mu}{\Gamma(\alpha\mu + \eta)}, \qquad \Re e\{\alpha, \eta\} > 0. \tag{4}$$

The function given by (4) is not within the class Σ. Based on this, the function is then normalised as follows [14]:

$$\Omega_{\alpha,\eta}(\varsigma) = \varsigma^{-1} \Gamma(\eta) E_{\alpha,\eta}(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)}{\Gamma(\alpha(\mu + 1) + \eta)} \varsigma^\mu. \tag{5}$$

In recent years, there has been a growing interest in Mittag–Leffler-type functions, driven by their increasing range of applications in probability, applied problem-solving, statistical analysis, and distribution theory, among other domains. More information about the utilization of Mittag–Leffler functions can be found in references [3, 4, 7, 8, 16, 20, 22].

Much of our research involving Mittag–Leffler functions focuses on aspects of convexity, close-to-convexity, and starlikeness. Recent studies on the $E_{\alpha,\eta}(\varsigma)$ Mittag–Leffler function can be found in [9]. Additionally, [21] has presented findings related to partial sums for $E_{\alpha,\eta}(\varsigma)$.

Motivated by Challab and Darus [10], we define the linear derivative operator $S_{\eta}^{\alpha}[r,\delta,\lambda]:\Sigma\to\Sigma$ by

$$\begin{aligned} S_{\eta}^{\alpha}[r,\delta,\lambda]f(\varsigma) =&(1-\lambda)(I_1(r,\delta)f(\varsigma)*\Omega_{\alpha,\eta}(\varsigma)+\lambda\varsigma((I_1(r,\delta)f(\varsigma)*\Omega_{\alpha,\eta}(\varsigma))' \\ =&\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)}\left(\frac{\mu+\delta}{1+\delta}\right)^r a_{\mu}\varsigma^{\mu}, \end{aligned} \tag{6}$$

where $\delta\geq 0$, $r\in N$, $0\leq\lambda\leq 1$, $\alpha,\eta\in\mathbb{C}$ and $I_1(r,\lambda)f(\varsigma)$ of the form (2).

Example 1. *If $r=0$, then $S_{\eta}^{\alpha}[r,\delta,\lambda]$ is reduced to*

$$S_{\eta}^{\alpha}[0,\delta,\lambda]f(\varsigma)=\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)}a_{\mu}\varsigma^{\mu}$$

introduced by Ghanim and Al-Janaby [15].

Example 2. *If $\alpha=0$, then $S_{\eta}^{\alpha}[r,\delta,\lambda]$ is reduced to*

$$S_{\eta}^{0}[r,\delta,\lambda]f(\varsigma)=\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}[1+\lambda(\mu-1)]^k\left(\frac{\mu+\delta}{1+\delta}\right)^r a_{\mu}\varsigma^{\mu}$$

Example 3. *If $\alpha=0$ and $k=0$, then $S_{\eta}^{\alpha}[r,\delta,\lambda]$ is reduced to*

$$S_{\eta}^{0}[r,\delta,\lambda]f(\varsigma)=\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}\left(\frac{\mu+\delta}{1+\delta}\right)^r a_{\mu}\varsigma^{\mu}$$

introduced by Atshan and Joudah [6].

Example 4. *If $\alpha=0, r=0$, then $S_{\eta}^{\alpha}[r,\delta,\lambda]$ is reduced to*

$$S_{\eta}^{0}[r,\delta,\lambda]f(\varsigma)=\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}[1+\lambda(\mu-1)]^k a_{\mu}\varsigma^{\mu}$$

introduced by Challab and Darus [10].

For $\varsigma\in\diamondsuit^*=\diamondsuit\setminus\{0\}$, by $\mathcal{M}_{\mathcal{H}}$, we denote the class of harmonic meromorphic functions of the form

$$f(\varsigma)=\hbar(\varsigma)+\overline{g(\varsigma)}=\frac{1}{\varsigma}+\sum_{\mu=1}^{\infty}a_{\mu}\varsigma^{\mu}+\sum_{\mu=1}^{\infty}\overline{b_{\mu}\varsigma^{\mu}}, \tag{7}$$

which are harmonic in the punctured unit disk $\diamondsuit\setminus\{0\}$, where h and g are analytic in $\diamondsuit^*$and $\diamondsuit$, respectively, and $\hbar$ has a simple pole at the origin with residue 1 here. This class was firstly studied by Jahangiri and Silverman [19], followed by Jahangiri et al. [18], Ahuja and Jahangiri [2] and others. We further denote by the subclass $\mathcal{M}_{\overline{\mathcal{H}}}$ of $\mathcal{M}_{\mathcal{H}}$ consisting of functions f of the form

$$f(\varsigma) = \frac{1}{\varsigma} + \sum_{\mu=1}^{\infty} |a_\mu| \varsigma^\mu + \sum_{\mu=1}^{\infty} |b_\mu| \varsigma^\mu, \quad (z \in \diamondsuit^*). \tag{8}$$

The function $f = \hbar + \overline{g}$, defined by the equation (7), can be classified as a harmonic meromorphic function that is locally univalent and sense-preserving in the region $\diamondsuit^*$ if and only if $\left|\frac{g'(\varsigma)}{\hbar'(\varsigma)}\right| < 1$.

The investigation of harmonic meromorphic functions within this specific context has been examined by Jahangiri and Silverman in their work [19].

A function $f \in \mathcal{M}_{\mathcal{H}}$ is said to be in the class $\mathcal{MS}^*_{\mathcal{H}}$ of meromorphically harmonic starlike functions in $\diamondsuit^*$, if it satisfies the condition

$$\Re e \left\{ \frac{\varsigma \hbar'(\varsigma) - \overline{\varsigma\, g'(\varsigma)}}{\hbar(\varsigma) + g(\varsigma)} \right\} > 0, \qquad (z \in \diamondsuit^*).$$

In other hand, a function $f \in \mathcal{M}_{\mathcal{H}}$ is said to be in the class $\mathcal{MC}_{\mathcal{H}}$ of meromorphically harmonic convex functions in $\diamondsuit^*$, if it satisfies the condition

$$\Re e \left\{ \frac{\varsigma \hbar''(\varsigma) + \hbar'(\varsigma) - \overline{\varsigma\, g''(\varsigma) + g'(\varsigma)}}{\hbar'(\varsigma) + g'(\varsigma)} \right\} > 0, \qquad (z \in \diamondsuit^*).$$

The classes $\mathcal{MS}^*_{\mathcal{H}}$ and $\mathcal{MC}_{\mathcal{H}}$, which consist of harmonic meromorphic starlike functions and harmonic meromorphic convex functions, have been the subject of study by Jahangiri and Silverman [19], Jahangiri [17], Atshan and Joudah [6], Ghanem and Al-Janaby [15], Elhaddad and Darus [14] and Alsoboh et al. [5].

Motivated by Challab and Darus [10], we define the linear operator for the harmonic meromorphic class of functions $f \in \mathcal{M}_{\mathcal{H}}$, by letting $k \geq 0$ and

$$I^k(S^\alpha_\eta[r,\delta,\lambda])f(\varsigma) = I^k(S^\alpha_\eta[r,\delta,\lambda])\hbar(\varsigma) + (-1)^k \overline{I^k(S^\alpha_\eta[r,\delta,\lambda])g(\varsigma)}, \tag{9}$$

where

$$I^k(S^\alpha_\eta[r,\delta,\lambda])\hbar(\varsigma) = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r a_\mu \varsigma^\mu, \tag{10}$$

and

$$I^k(S^\alpha_\eta[r,\delta,\lambda])g(\varsigma) = \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r b_\mu \overline{\varsigma}^\mu. \tag{11}$$

Utilising the operator $I^k(S^\alpha_\eta[r,\delta,\lambda])$, we introduce a class of generalised harmonic Meromorphically starlike functions within the region $\diamondsuit^*$ through the following definition.

Definition 1. *For $0 \leq \gamma < 1$, the class $\mathcal{MK}_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$ is used to denote harmonic meromorphic functions f defined as in* (1)*, and belonging to this class is upon the satisfaction of the condition:*

$$\mathfrak{Re}\left\{-\frac{\varsigma(I^k(S_\eta^\alpha[r,\delta,\lambda])\hbar(\varsigma))' - \overline{\varsigma(I^k(S_\eta^\alpha[r,\delta,\lambda])g(\varsigma))'}}{I^k(S_\eta^\alpha[r,\delta,\lambda])\hbar(\varsigma) + \overline{I^k(S_\eta^\alpha[r,\delta,\lambda])g(\varsigma)}}\right\} > \gamma, \qquad (\varsigma \in \diamondsuit^*). \tag{12}$$

Furthermore, by $\mathcal{TMK}_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma) \subset \mathcal{MK}_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$, we denote the subclass of harmonic Meromorphic functions $f_k = \hbar_k + \overline{g_k(\varsigma)}$ where $\hbar_k$ and g_k of the form

$$\hbar_k(\varsigma) = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty}|a_\mu|\varsigma^\mu \qquad \text{and} \qquad g_k(\varsigma) = (-1)^k\sum_{\mu=1}^{\infty}|b_\mu|\varsigma^\mu, \quad (z \in \diamondsuit^*). \tag{13}$$

2. Coefficient Inequalities

In our initial theorem, we establish the sufficient coefficient bounds applicable to functions f within the class $\mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$.

Theorem 1. *For $0 \leq \gamma < 1$, consider the function $f = \hbar + \overline{g}$ defined by* (7)*, subject to the condition*

$$\sum_{\mu=1}^{\infty}\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)}\left(\frac{\mu+\delta}{1+\delta}\right)^r\Big[(\mu+\gamma)|a_\mu| + (\mu-\gamma)|b_\mu|\Big] \leq 1-\gamma,$$

where $\mu \in 0,1,2,\ldots$. Then, f is harmonic univalent and sense-preserving in $\diamondsuit^*$, and $f \in \mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$.

Proof. Consider the function $f = \hbar + \overline{g}$ as defined in Equation (7), which satisfies the inequality given in Equation (1). Assuming that $0 < |\varsigma_1| \leq |\varsigma_2| < 1$, we can conclude that

$$\begin{aligned}
\big|f(\varsigma_1) - f(\varsigma_2)\big| &\geq \frac{|\varsigma_1-\varsigma_2|}{|\varsigma_1\varsigma_2|}\left(1 - |\varsigma_2|^2\sum_{\mu=1}^{\infty}(|a_\mu|+|b_\mu|)\frac{|{\varsigma_1}^\mu - {\varsigma_2}^\mu|}{|\varsigma_1-\varsigma_2|}\right)\\
&\geq \frac{|\varsigma_1-\varsigma_2|}{|\varsigma_1\varsigma_2|}\left(1 - |\varsigma_2|^2\sum_{\mu=1}^{\infty}(|a_\mu|+|b_\mu|)\left|\varsigma_1^{\mu-1} + \cdots + \varsigma_2^{\mu-1}\right|\right)\\
&\geq \frac{|\varsigma_1-\varsigma_2|}{|\varsigma_1\varsigma_2|}\left(1 - |\varsigma_2|^2\sum_{\mu=1}^{\infty}\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)}\left(\frac{\mu+\delta}{1+\delta}\right)^r\mu(|a_\mu|+|b_\mu|)\right)\\
&> |\varsigma_1-\varsigma_2|\left(1 - \sum_{\mu=1}^{\infty}\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)}\left(\frac{\mu+\delta}{1+\delta}\right)^r\left\{\left(\frac{\mu+\gamma}{1-\gamma}\right)|a_\mu| + \left(\frac{\mu+\gamma}{1-\gamma}\right)|b_\mu|\right\}\right).
\end{aligned}$$

By utilising condition (1), one can determine that the last expression is non-negative. Consequently, it can be deduced that f is univalent in $\diamondsuit^*$. To establish that f is sense-preserving in $\diamondsuit^*$, it suffices to demonstrate that $|\hbar'(\varsigma)| > |g'(\varsigma)|$ using the ordinary derivative. For $0 < |\varsigma| = \mathfrak{r} < 1$, this can be inferred from the utilization of (1).

$$\begin{aligned}
\left|\hbar'(\varsigma)\right| &= \left|\frac{-1}{\varsigma^2} + \sum_{\mu=1}^{\infty} \mu\, a_\mu \varsigma^{\mu-1}\right| \geq \left|\frac{-1}{\varsigma^2} + \sum_{\mu=1}^{\infty} \frac{\mu\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r a_\mu \varsigma^{\mu-1}\right| \\
&\geq \left|\frac{-1}{\varsigma^2}\right| - \sum_{\mu=1}^{\infty} \frac{\mu\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r |a_\mu||\varsigma|^{\mu-1} \\
&\geq \frac{1}{\mathfrak{r}}^2 - \sum_{\mu=1}^{\infty} \frac{\mu\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r |a_\mu|\mathfrak{r}^{\mu-1} \\
&\geq 1 - \sum_{\mu=1}^{\infty} \frac{\mu\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r |a_\mu| \\
&\geq 1 - \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \left(\frac{\mu+\gamma}{1-\gamma}\right) |a_\mu| \\
&\geq \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \left(\frac{\mu-\gamma}{1-\gamma}\right) |b_\mu| \\
&> \left|\frac{-1}{\varsigma^2} + \sum_{\mu=1}^{\infty} \mu\, b_\mu \varsigma^{\mu-1}\right| \geq \left|g'(\varsigma)\right|.
\end{aligned}$$

In order to prove that $f \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$, it suffices to show that

$$\mathfrak{Re}\left\{ -\frac{\varsigma\big(I^k(S_\eta^\alpha[r,\delta,\lambda]\hbar(\varsigma))\big)' - \overline{\varsigma\big(I^k(S_\eta^\alpha[r,\delta,\lambda]g(\varsigma)\big)'}}{I^k\big(S_\eta^\alpha[r,\delta,\lambda]\hbar(\varsigma)\big) + \overline{I^k\big(S_\eta^\alpha[r,\delta,\lambda]g(\varsigma)\big)}} - \gamma \right\} > 0, \qquad (\varsigma \in \diamondsuit^*).$$

Since, $\Re e(\rho(\varsigma)) > 0$ if and only if $\left|\frac{\rho(\varsigma)-1}{\rho(\varsigma)+1}\right| < 1$ for an analytic function $\rho(\varsigma) = 1 + c_1\varsigma + c_2\varsigma^2 + \dots$.

We let

$$M(\varsigma) = \left\{ \begin{array}{c} -\varsigma(I^k(S_\eta^\alpha[r,\delta,\lambda]\hbar(\varsigma)))' + \overline{\varsigma(I^k(S_\eta^\alpha[r,\delta,\lambda]g(\varsigma))'} - \gamma I^k(S_\eta^\alpha[r,\delta,\lambda]\hbar(\varsigma) \\ -\gamma \overline{I^k(S_\eta^\alpha[r,\delta,\lambda]g(\varsigma)} \end{array} \right\} \tag{14}$$

and

$$N(\varsigma) = I^k(S_\eta^\alpha[r,\delta,\lambda]\hbar(\varsigma) + \overline{I^k(S_\eta^\alpha[r,\delta,\lambda]g(\varsigma)}. \tag{15}$$

Then, we have to show that

$$\Xi(\varsigma) = |M(\varsigma) + N(\varsigma)| - |M(\varsigma) - N(\varsigma)| > 0. \tag{16}$$

Now, by substituting equations (9) and (10) into the left-hand side of inequality (16), we obtain

$$\Xi(\varsigma) \geq \left(\begin{array}{c} \frac{2-2\gamma}{\mathfrak{r}} - 2\sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu+\gamma)|a_\mu|\mathfrak{r}^\mu \\ -2\sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu-\gamma)|b_\mu|\mathfrak{r}^\mu \end{array} \right)$$

$$\geq 2(1-\gamma) \left(\begin{array}{c} 1 - \sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \left(\frac{\mu+\gamma}{1-\gamma}\right)|a_\mu|\mathfrak{r}^\mu \\ -\sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \left(\frac{\mu-\gamma}{1-\gamma}\right)|b_\mu|\mathfrak{r}^\mu \end{array} \right).$$

The positivity of this expression is assured by the fulfillment of condition (1), thereby concluding the proof.

In the subsequent theorem, it is demonstrated that the condition (1) is necessary for the inclusion of f in the class $\mathcal{T}MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$.

Theorem 2. *Let* $(0 \leq \gamma < 1)$ *and* $f_k = \hbar_k + \overline{g_k} \in \mathcal{T}M_H$ *is given by (13). Then* $f_k \in \mathcal{T}MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ *if and only if the inequality*

$$\sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \Big[(\mu+\gamma)|a_\mu| + (\mu-\gamma)|b_\mu| \Big] \leq 1-\gamma,$$

is satisfied.

Proof. Considering Theorem 1, it is adequate to demonstrate the validity of the "if" part. Assume that $f_k \in \mathcal{T}MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$. Then

$$\mathfrak{Re}\left\{ -\frac{\varsigma(I^k(S^\alpha_\eta[r,\delta,\lambda]\hbar_k(\varsigma)))' - (-1)^k \overline{\varsigma(I^k(S^\alpha_\eta[r,\delta,\lambda]g_k(\varsigma)))'} + \gamma I^k(S^\alpha_\eta[r,\delta,\lambda]\hbar_k(\varsigma) + (-1)^k\gamma\overline{I^k(S^\alpha_\eta[r,\delta,\lambda]g_k(\varsigma)}}{I^k(S^\alpha_\eta[r,\delta,\lambda]\hbar_k(\varsigma) + (-1)^k\overline{I^k(S^\alpha_\eta[r,\delta,\lambda]g_k(\varsigma)}} \right\}$$

$$= \mathfrak{Re}\left\{ \frac{\frac{1-\gamma}{\varsigma} - \left(\sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu+\gamma)\, a_\mu \varsigma^\mu + \sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu-\gamma)\, b_\mu \varsigma^\mu \right)}{\frac{1}{\varsigma} + \sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r |a_\mu|\varsigma^\mu - \sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r |b_\mu|\varsigma^s} \right\} > 0. \tag{17}$$

The equation (17) must be satisfied for all $\varsigma \in \diamondsuit^*$. When we select the value of ς on the positive real axis, with $0 < \varsigma = \mathfrak{r} < 1$, we obtain

$$\frac{\left(\begin{array}{c} 1-\gamma - \sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu+\gamma)|a_\mu|\mathfrak{r}^{\mu+1} \\ -\sum\limits_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r (\mu+\gamma)|b_\mu|(\mu+\gamma)\mathfrak{r}^{\mu+1} \end{array} \right)}{\left\{ \begin{array}{c} 1 + \sum\limits_{\mu=1}^{\infty} [1+(\mu-1)\lambda]^k \left(\frac{\mu+\delta}{1+\delta}\right)^r \Omega_{\mu+1}(\alpha,\eta)|a_\mu|\mathfrak{r}^{\mu+1} - \\ \sum\limits_{\mu=1}^{\infty} [1+(\mu-1)\lambda]^k \left(\frac{\mu+\delta}{1+\delta}\right)^r \Omega_{\mu+1}(\alpha,\eta)|b_\mu|\mathfrak{r}^{\mu+1} \end{array} \right\}} > \gamma. \tag{18}$$

If condition (2) is not fulfilled, then as $\mathfrak{r} \to 1^-$, the numerator of (18) becomes negative. As a result, there exists a value $\varsigma_0 = \mathfrak{r}_0$ within the interval $(0,1)$ such that the left-hand side of inequality (18) is negative. This contradicts the condition stated in (2), thus completing the proof.

For $k = 0$ in Theorem 2, we have the following corollary.

Corollary 1. *For $f = h + \overline{g}$ of the form (7). Then, $f \in \mathcal{MK}_{\mathcal{H}}(0, r, \alpha, \eta, \delta, \lambda, \gamma)$ if and only if the inequality*

$$\sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \Big[(\mu+\gamma)\,|a_\mu| + (\mu-\gamma)\,|b_\mu| \Big] \le 1-\gamma,$$

is satisfied.

Corollary 2. *[17] For $f = h + \overline{g}$ of the form (7). Then, $f \in \mathcal{MK}_{\mathcal{H}}(0, r, \alpha, \eta, \delta, \lambda, \gamma)$ if and only if the inequality*

$$\sum_{\mu=1}^{\infty} \Big[(\mu+\gamma)\,|a_\mu| + (\mu-\gamma)\,|b_\mu| \Big] \le 1-\gamma,$$

is satisfied.

The ensuing theorem establishes a growth property for the class $\mathcal{T}MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$.

Theorem 3. *Let $f_k(\varsigma) = \hbar_k(\varsigma) + \overline{g_k(\varsigma)} \in \mathcal{T}MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ of the form (13), then we have for $|z| = \mathfrak{r} < 1$:*

$$\frac{1}{\mathfrak{r}} - \frac{\Gamma(3\alpha+\eta)(1-\gamma)\mathfrak{r}^2}{\Gamma(\eta)(1+\lambda)^k\,(2-\gamma)\left(\frac{2+\delta}{1+\delta}\right)^r} \le |f_k(\varsigma)| \le \frac{1}{\mathfrak{r}} + \frac{\Gamma(3\alpha+\eta)(1-\gamma)\mathfrak{r}^2}{\Gamma(\eta)(1+\lambda)^k\,(2-\gamma)\left(\frac{2+\delta}{1+\delta}\right)^r}.$$

Proof. Taking the absolute value for $f_k(\varsigma)$ given by (13), we have

$$\begin{aligned}
|f_k(\varsigma)| &= \left| \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty} a_\mu \varsigma^\mu + (-1)^k \sum_{\mu=1}^{\infty} \overline{b_\mu \varsigma^\mu} \right| \\
&\le \frac{1}{\mathfrak{r}} + \sum_{\mu=1}^{\infty} (|a_\mu| + |b_k|)\mathfrak{r}^\mu \\
&\le \frac{1}{\mathfrak{r}} + \sum_{\mu=1}^{\infty} (|a_\mu| + |b_k|)\mathfrak{r} \\
&\le \ \frac{1}{\mathfrak{r}} + \frac{\Gamma(3\alpha+\eta)(1-\gamma)}{\Gamma(\eta)(1+\lambda)^k\left(\frac{2+\delta}{1+\delta}\right)^r(2-\gamma)} \left\{ \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k}{\Gamma(\alpha(\mu+1)+\eta)} \left(\frac{\mu+\delta}{1+\delta}\right)^r \left[\frac{\mu+\gamma}{1-\gamma}\ |a_\mu| + \frac{\mu-\gamma}{1-\gamma}\ |b_\mu| \right] \right\} \mathfrak{r} \\
&\le \frac{1}{\mathfrak{r}} + \frac{\Gamma(3\alpha+\eta)(1-\gamma)}{\Gamma(\eta)(1+\lambda)^k\,(2-\gamma)\left(\frac{2+\delta}{1+\delta}\right)^r}.
\end{aligned}$$

The second inequality, is the same of first inequality, so the proof is omitted. This proves the required result.

3. Extreme Points

Subsequently, we identify the extreme points of the closed convex hulls of the class $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$, designated as $clco\mathcal{TMK}_{\mathcal{H}}$.

Theorem 4. *Consider a function $f_k = \hbar_k + \overline{g_k}$ in the form (13). Then, $f \in clco\mathcal{TMK}_{\mathcal{H}}$ if and only if $f_{k,\mu}(\varsigma)$ can be represented as*

$$f_k(\varsigma) = \sum_{\mu=1}^{\infty} \varrho_\mu \hbar_{k,\mu}(\varsigma) + \Psi_\mu g_{k,\mu}(\varsigma),$$

where

$$\hbar_{k,0}(\varsigma) = \frac{(-1)^k}{\varsigma}, \qquad \hbar_{k,\mu}(\varsigma) = \frac{(-1)^k}{\varsigma} + \frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}\varsigma^\mu,$$

and

$$g_{k,0}(\varsigma) = \frac{(-1)^k}{\varsigma}, \qquad g_{k,\mu}(\varsigma) = \frac{(-1)^k}{\varsigma} + \frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}\overline{\varsigma}^\mu$$

where $\varrho_\mu \geq 0$, $\Psi_\mu \geq 0, \mu = 1, 2, \ldots$ and $\sum\limits_{\mu=0}^{\infty}(\varrho_\mu + \Psi_\mu) = 1$. The extreme points of the class $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ are $\{\hbar_{k,\mu}\}$ and $\{g_{k,\mu}\}$.

Proof. For $f(\varsigma) = \sum\limits_{\mu=0}^{\infty}(\varrho_\mu h_{k,\mu} + \Psi_\mu g_{k,\mu})$ where $\sum\limits_{\mu=0}^{\infty}(\varrho_\mu + \Psi_\mu) = 1$, we have

$$\begin{aligned} f_k(\varsigma) &= \varrho_0 \hbar_{0,\mu} + \Psi_0 g_{0,n} + \sum_{\mu=1}^{\infty}(\varrho_\mu \hbar_{k,\mu} + \Psi_\mu g_{k,\mu}) \\ &= \sum_{\mu=0}^{\infty} \tfrac{(-1)^k(\varrho_\mu+\Psi_\mu)}{\varsigma} + \sum_{\mu=1}^{\infty} \varrho_\mu \left(\tfrac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}\right)\varsigma^\mu \\ &\quad +(-1)^k \sum_{\mu=1}^{\infty} \Psi_\mu \left(\tfrac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}\right)\overline{\varsigma}^\mu \\ &= \tfrac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty} \left(\tfrac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}\right)\varrho_\mu\varsigma^\mu \\ &\quad +(-1)^k \sum_{\mu=1}^{\infty} \left(\tfrac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}\right)\Psi_\mu\overline{\varsigma}^\mu. \end{aligned}$$

This belong to $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ because

$$\begin{aligned} &\sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r}\left(\frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}\right)\varrho_\mu \\ &\sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r}\left(\frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}\right)\Psi_\mu \end{aligned}$$

$$= \sum_{\mu=1}^{\infty}(1-\gamma)\varrho_\mu + (1-\gamma)\Psi_\mu = (1-\gamma)\sum_{\mu=1}^{\infty}(\varrho_\mu + \Psi_\mu)$$
$$= (1-\gamma)(1-\varrho_0-\Psi_0) \le 1-\gamma.$$

Conversely, suppose that $f \in clco\mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$. For $\mu = 1,2,3,\ldots$, set

$$\varrho_\mu = \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}|a_\mu|, \qquad (0 \le \varrho_\mu \le 1)$$
$$\Psi_\mu = \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}|b_\mu|, \qquad (0 \le \Psi_\mu \le 1)$$

and

$$\varrho_0 + \Psi_0 = 1 - \sum_{\mu=1}^{\infty}\varrho_\mu - \sum_{\mu=1}^{\infty}\Psi_\mu..$$

Therefore, f_k can be written as

$$f_k(\varsigma) = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty}|a_\mu|\varsigma^\mu + (-1)^k\sum_{\mu=1}^{\infty}|b_\mu|\overline{\varsigma^\mu}$$
$$= \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty}\left(\frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}\right)\varrho_\mu\varsigma^\mu$$
$$+ (-1)^k\sum_{\mu=1}^{\infty}\left(\frac{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}\right)\Psi_\mu\overline{\varsigma}^\mu$$
$$= \frac{\varrho_0+\Psi_0}{\varsigma} + \sum_{\mu=1}^{\infty}\left(h_{k,n}(\varsigma) - \frac{(-1)^k}{\varsigma}\right)\varrho_\mu + \sum_{\mu=1}^{\infty}\left(g_{k,n}(\varsigma) - \frac{(-1)^k}{\varsigma}\right)\Psi_\mu$$
$$= \sum_{\mu=0}^{\infty}(\varrho_\mu\hbar_{k,\mu} + \Psi_\mu g_{k,\mu}).$$

4. Convex Combination and Convolution

Subsequently, we can establish that the class $\mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$ exhibits closure properties in relation to both convolution and convex combination.

Theorem 5. *For* $0 \le \beta \le \gamma < 1$. *Let* $f_k(\varsigma) \in \mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma)$ *and* $\xi_k(\varsigma) \in \mathcal{T}M_HMK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\beta,n)$, *then*
$(f_k * \xi_k)(\varsigma) \in \mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\gamma) \subseteq \mathcal{T}MK_{\mathcal{H}}(k,r,\alpha,\eta,\delta,\lambda,\beta)$.

Proof. The convolution, or the Hadamard product, of $f_k(\varsigma)$ and $\xi_k(\varsigma)$ is expressed as

$$(f_k * \beta_k)(\varsigma) = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty}|a_\mu||c_\mu|\varsigma^\mu + (-1)^k\sum_{\mu=1}^{\infty}|b_\mu||d_\mu|\varsigma^\mu.$$

We want to show that the coefficients of $f_k * \xi_k$ satisfy condition (2). For $\xi_k(\varsigma) \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \beta)$, we note that $|c_\mu| \leq 1$ and $|d_\mu| \leq 1$,

$$\sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\beta)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\beta)}|a_\mu||c_\mu| + \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\beta)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\beta)}|b_\mu||d_\mu|$$
$$\leq \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\beta)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\beta)}|a_\mu| + \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\beta)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\beta)}|b_\mu|$$
$$\leq \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}|a_\mu| + \sum_{\mu=1}^{\infty} \frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r(1-\gamma)}|b_\mu| \leq 1,$$

since $f_k(\varsigma) \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ and $0 \leq \beta \leq \gamma < 1$. Therefore, $(f * \xi)(\varsigma) \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma) \subseteq \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \beta)$.

Theorem 6. *Let $f_{m,k}$ defined as*

$$f_{m,k} = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty} |a_{\mu,m}|\varsigma^\mu + (-1)^k \sum_{\mu=1}^{\infty} |b_{\mu,m}|\bar{\varsigma}^\mu$$

be in class $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ for every $m = 1, 2, ..., l$, then the function

$$\Im_m(\varsigma) = \sum_{m=1}^{l} c_m f_{m,k}(\varsigma), \qquad (0 \leq c_m \leq 1), \tag{19}$$

are also in the class $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$, where $\sum\limits_{m=1}^{l} c_m = 1$.

Proof. According to the definition of $\Im_m(\varsigma)$ given by (19), we can write

$$\Im_m(\varsigma) = \frac{(-1)^k}{\varsigma} + \sum_{\mu=1}^{\infty} \left(\sum_{m=1}^{l} c_m |a_{\mu,m}| \right) \varsigma^\mu + (-1)^k \sum_{\mu=1}^{\infty} \left(\sum_{m=1}^{l} c_m |b_{\mu,m}| \right) \bar{\varsigma}^\mu.$$

Furthermore, for every $m = 1, 2, \cdots, l$, we have $f_{m,k} \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$. Then, by (2), we have

$$\mathcal{I} = \left(\begin{array}{l} \sum\limits_{\mu=1}^{\infty} \left(\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r} \right) \left\{ \sum\limits_{m=1}^{l} c_\mu |a_{\mu,m}| \right\} + \\ \sum\limits_{\mu=1}^{\infty} \left(\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r} \right) \left\{ \sum\limits_{m=1}^{l} c_m |b_{\mu,m}| \right\} \end{array} \right)$$
$$= \sum_{m=1}^{l} c_m \left\{ \sum_{\mu=1}^{\infty} \left(\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu+\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r} \right) |a_{\mu,m}| + \sum_{\mu=1}^{\infty} \left(\frac{\Gamma(\eta)[1+\lambda(\mu-1)]^k(\mu+\delta)^r(\mu-\gamma)}{\Gamma(\alpha(\mu+1)+\eta)(1+\delta)^r} \right) |b_{\mu,m}| \right\}$$
$$\leq \sum_{m=1}^{l} c_m(1-\gamma) \leq 1-\gamma.$$

Therefore, $\Im_m(\varsigma) \in \mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$.

Corollary 3. *The class $\mathcal{TMK}_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$ is closed under convex combination.*

Conclusion

In the current study, we have introduced and examined the coefficient issues related to each of class $MK_{\mathcal{H}}(k, r, \alpha, \eta, \delta, \lambda, \gamma)$, which consists of harmonic meromorphic starlike functions. This class is utilised to describe a derivative operator that incorporates the Mittag-Leffler function as a multiplier transformation. This study derives coefficient inequalities, the distortion theorem, distortion bounds, extreme points, convex combination, and convolution for functions inside this particular class. The results obtained in this article can be generalised in the future using quantum calculus and other q-analogues of the fractional derivative operator.

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References

[1] Sarah Ahmed, Maslina Darus, and Georgia Irina Oros. Subordination results for the second-order differential polynomials of meromorphic functions. *Symmetry*, 14(12):2587, 2022.

[2] Om P Ahuja and Jay M Jahangiri. Certain meromorphic harmonic functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 25(1), 2002.

[3] Abdullah Alsoboh, Ala Amourah, Maslina Darus, and Carla Amoi Rudder. Studying the harmonic functions associated with quantum calculus. *Mathematics*, 11(10):2220, 2023.

[4] Abdullah Alsoboh and Maslina Darus. A q-starlike class of harmonic meromorphic functions defined by q-derivative operator. In *Mathematics and Computation*, pages 257–269, Singapore, 2023. Springer Nature Singapore.

[5] Abdullah Alsoboh, Maslina Darus, Ala Amourah, and Waggas Galib Atshan. A certain subclass of harmonic meromorphic functions with respect to k-symmetric points. *Int. J. Open Problems Complex Analysis*, 15(1), 2023.

[6] Waggas Galib Atshan and Ahmed Sallal Joudah. Subclass of meromorphic univalent functions defined by hadamard product with multiplier transformation. In *International Mathematical Forum*, volume 6, pages 2279–2292, 2011.

[7] Adel A Attiya. Some applications of mittag-leffler function in the unit disk. *Filomat*, 30(7):2075–2081, 2016.

[8] D Bansal and JK Prajapat. Certain geometric properties of the mittag-leffler functions. *Complex Variables and Elliptic Equations*, 61(3):338–350, 2016.

[9] Khalid Challab and Maslina Darus. On certain class of meromorphic harmonic concave functions defined by Salagean operator. *Journal of Quality Measurement and Analysis (JQMA)*, 11(1):49–60, 2015.

[10] Khalid Challab and Maslina Darus. On certain classes of meromorphic harmonic concave functions defined by al-oboudi operator. *Journal of Quality Measurement and Analysis (JQMA)*, 12(1-2):53–65, 2016.

[11] Nak Eun Cho and Tae Hwa Kim. Multiplier transformations and strongly close-to-convex functions. *Bulletin of the Korean Mathematical Society*, 40(3):399–410, 2003.

[12] NE Cho and HM Srivastava. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Mathematical and Computer Modelling*, 37(1-2):39–49, 2003.

[13] J Clunie. On meromorphic schlicht functions. *Journal of the London Mathematical Society*, 1(2):215–216, 1959.

[14] Suhila Elhaddad and Maslina Darus. On meromorphic functions defined by a new operator containing the mittag–leffler function. *Symmetry*, 11(2):210, 2019.

[15] F Ghanim and Hiba F Al-Janaby. Inclusion and convolution features of univalent meromorphic functions correlating with mittag-leffler function. *Filomat*, 34(7):2141–2150, 2020.

[16] Indranil Sen Gupta and Lokenath Debnath. Some properties of the mittag-leffler functions. *Integral Transforms and Special Functions*, 18(5):329–336, 2007.

[17] Jay Jahangiri et al. Harmonic meromorphic starlike functions. *Bulletin of the Korean Mathematical Society*, 37(2):291–301, 2000.

[18] Jay M Jahangiri, G Murugusundaramoorthy, and K Vijaya. Salagean-type harmonic univalent functions. *Southwest Journal of Pure and Applied Mathematics [electronic only]*, 2002:77–82, 2002.

[19] Jay M. Jahangiri and Herb Silverman. Meromorphic univalent harmonic functions with negative coefficients. *Bulletin of the Korean Mathematical Society*, 36(4):763–770, 1999.

[20] Gustave Mittag-Leffler. Sur la représentation analytique d'une branche uniforme d'une fonction monogène: cinquième note. *Acta mathematica*, 29(1):101–181, 1905.

[21] MS Rehman, Qazi Zahoor Ahmad, Hari M Srivastava, Bilal Khan, and Nazar Khan. Partial sums of generalized q-mittag-leffler functions. *Aims Math*, 5(1):408–420, 2019.

[22] Hari M Srivastava and Živorad Tomovski. Fractional calculus with an integral operator containing a generalized mittag–leffler function in the kernel. *Applied Mathematics and Computation*, 211(1):198–210, 2009.

[23] Adders Wiman. Über den fundamentalsatz in der teorie der funktionen $E_a(x)$. *Acta Mathematica*, 29:191–201, 1905.