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On Inner Products Derived From the Standard n-Inner Product on an Inner Product Space

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Abstract. In this paper, we study relations between inner products derived from the standard n-inner product defined on an inner product space. In particular, we are interested in knowing when orthogonality with respect to the original inner product is preserved by the derived inner product.

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1. Introduction

Let X be a real vector space of dimension $d \geq n$ and let $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ be a function such that for every $x_0, x_1, \dots, x_n, x_{n+1} \in X$ and $\alpha \in \mathbb{R}$ we have

(I1) $\langle x_1, x_1 | x_2, \dots, x_n \rangle \geq 0$ and $\langle x_1, x_1 | x_2, \dots, x_n \rangle = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \dots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \dots, x_{i_n} \rangle$ for any permutation; $\{i_1, i_2, \dots, i_n\}$ of $(1, \dots, n)$;

(I3) $\langle x_0, x_1 | x_2, \dots, x_n \rangle = \langle x_1, x_0 | x_2, \dots, x_n \rangle$;

(I4) $\langle \alpha x_0, x_1 | x_2, \dots, x_n \rangle = \alpha \langle x_0, x_1 | x_2, \dots, x_n \rangle$;

(I5) $\langle x_0 + x_{n+1}, x_1 | x_2, \dots, x_n \rangle = \langle x_0, x_1 | x_2, \dots, x_n \rangle + \langle x_{n+1}, x_1 | x_2, \dots, x_n \rangle$.

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The function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ is called an n-inner product introduced by Misiak in 1989 [14]. Here, the pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an n-inner product space. If $(X, \langle \cdot, \cdot \rangle)$ is a real inner product space of dimension $d \geq n$, we can define the standard n-inner product by

$$\langle x_0, x_1 | x_2, \ldots, x_n \rangle = \begin{vmatrix} \langle x_0, x_1 \rangle & \langle x_0, x_2 \rangle & \ldots & \langle x_0, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \ldots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \ldots & \langle x_n, x_n \rangle \end{vmatrix}.$$

So, a real inner product space of dimension $d \geq n$ with the n-inner product defined above is an example of an n-inner product space, which we can call a standard n-inner product space.

Next, from n-inner product space we can derive n-norm, defined by

$$\|x_1, x_2, \ldots, x_n\| = \langle x_1, x_1 | x_2, \ldots, x_n \rangle^{\frac{1}{2}}.$$

Furthermore, an n-norm on X is a function $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ such that for every $x_0, x_1, \ldots, x_n, x_{n+1} \in X$ and $\alpha \in \mathbb{R}$, the function satisfying the following properties:

(N1) $\|x_1, x_2, \ldots, x_n\| \geq 0$ and $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(N2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$;

(N4) $\|x_0 + x_1, x_2, \ldots, x_n\| \leq \|x_0, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x_n\|$.

Geometrically, $\|x_1, x_2, \ldots, x_n\|$ represents the generalized volume of an n-dimensional parallelepiped spanned by $x_1, x_2, \ldots, x_n$. Then,

$$\frac{\langle x_0, x_1 | x_2, \ldots, x_n \rangle}{\|x_0, x_2, \ldots, x_n\| \, \|x_1, x_2, \ldots, x_n\|}$$

is the cosine of the angle between two parallelepipeds spanned by $x_0, x_2, \ldots, x_n$ and $x_1, x_2, \ldots, x_n$. See [9, 10, 15] for more properties of n-inner products. The related results may also be found in [3–6, 13, 16, 17].

Historically, numerous authors have introduced and developed several concepts of orthogonality in 2-normed spaces and 2-inner product spaces [1, 2, 7, 11, 12, 15]. Just as the concepts of orthogonality in normed spaces draw inspiration from those in inner product spaces, the notions of orthogonality in 2-normed spaces are similarly linked to those in 2-inner product spaces. In [11], it is shown that the standard definition of orthogonality in a 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$ with $\dim(X) \geq 3$, is as follows:

Definition 1 (G-orthogonality in 2-inner product spaces)**.** *Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product spaces. x_1 is G-orthogonal to x_2 if and only if there exists a subspace V of X with codim$(V) = 1$ such that $\langle x_1, x_2 | x \rangle = 0$ for all $x \in V$ (Denoted by $x_1 \perp_G x_2$).*

We can say this definition is standard because when X is a standard 2-inner product space, we have $x_1 \perp x_2$ if and only if $x_1 \perp_G x_2$. The definition of G-orthogonality provided above represents an enhancement over the definition proposed by Cho and Kim [2] and Godini [7] as demonstrated in [11]. On the other hand, Cho and Kim's concept of orthogonality and Godini can be seen as a development of Khan and Siddiqui's concept of orthogonality [12]. Furthermore, one can define the notion of G-orthogonality in n-inner product spaces as follows:

Definition 2 (G-orthogonality in n-inner product spaces)**.** *Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product spaces with $\dim(X) \geq n + 1$. x_1 is G-orthogonal to x_2 if and only if there exists a subspace V of X with codim$(V) = 1$ such that $\langle x_1, x_2 | x_3, \ldots, x_{n+1} \rangle = 0$ for all $x_3, \ldots, x_{n+1} \in V$ (Denoted by $x_1 \perp_G x_2$).*

With this definition, in a standard n-inner product space, G-orthogonality is also equivalent to the usual orthogonality (with respect to the inner product). In other words, $x_1 \perp x_2$ if and only if $x_1 \perp_G x_2$.

We can see that the definition requires the assumption that the dimension of X is greater than n. In [11], it is shown that if we define G-orthogonality in the standard 2-inner product space X of dimension 2, any pair of linearly independent vectors becomes G-orthogonal. Similarly, if we define G-orthogonality for a standard n-inner product space X of dimension n, any pair of linearly independent vectors also becomes G-orthogonal. Indeed, this fact is undesirable. It is necessary to adopt a different approach to establish orthogonality in n-inner product spaces of dimension n in a general sense.

Meanwhile, note that if $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is an arbitrary n-inner product space and $A = \{a_1, a_2, \ldots, a_n\}$ is a set of n linearly independent vectors in X, then one may observe that

$$\langle x, y \rangle_A := \sum_{\{i_2, \ldots i_n\} \subset \{1,2,\ldots,n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle$$

defines an inner product on X. There are n terms in the above sum, as there are n subsets of $\{1, 2, \ldots, n\}$ consisting of $n-1$ elements. If $\dim X = d < \infty$, we can also define an inner product by the above formula using a set of d linearly independent vectors in X (see [8]).

Thus, starting from an inner product, we can define the standard n-inner product, and then from the n-inner product, we can derive a new inner product. It is then interesting to investigate how the new inner product derived from standard n-inner product relates to the original inner product on $(X, \langle \cdot, \cdot \rangle)$. In particular, we would like to know whether or not the new inner product preserves orthogonality. This generally depends on the set A that we choose in the definition of the new inner product. In the next sections, we present necessary and sufficient conditions for the set A to give the positive answer. With this approach, we have an alternative way to establish the orthogonality of two vectors in arbitrary n-inner product spaces because we can define the inner product on n-inner product spaces.

2. The n-dimensional case

Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be a standard n-inner product space. As indicated in [10] and [11], the n-dimensional case is special. So we shall first pay attention to the case where $\dim X = n$. Our results are the following theorems.

Theorem 1. *Let $A = \{a_1, a_2, \ldots, a_n\} \subset X$ be an orthogonal set with $\|a_i\| = \alpha > 0$ for all $i = 1, 2, \ldots, n$. Then $\langle x, y\rangle_A = 0$ if and only if $\langle x, y\rangle = 0$ for all $x, y \in X$.*

Proof. For any subset $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$, we have

$$\langle x, y | a_{i_2}, a_{i_3}, \ldots, a_{i_n}\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, a_{i_2}\rangle & \ldots & \langle x, a_{i_n}\rangle \\ \langle a_{i_2}, y\rangle & \langle a_{i_2}, a_{i_2}\rangle & \ldots & \langle a_{i_2}, a_{i_n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_n}, y\rangle & \langle a_{i_n}, a_{i_2}\rangle & \ldots & \langle a_{i_n}, a_{i_n}\rangle \end{vmatrix}_{n\times n}$$

$$= \langle x, y\rangle \begin{vmatrix} \|a_{i_2}\|^2 & 0 & \ldots & 0 \\ 0 & \|a_{i_3}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \|a_{i_n}\|^2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$- \langle x, a_{i_2}\rangle \begin{vmatrix} \langle a_{i_2}, y\rangle & 0 & \ldots & 0 \\ \langle a_{i_3}, y\rangle & \|a_{i_3}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_n}, y\rangle & 0 & \ldots & \|a_{i_n}\|^2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$+ \cdots + (-1)^{n-1}\langle x, a_{i_n}\rangle \begin{vmatrix} \langle a_{i_2}, y\rangle & \|a_{i_2}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_{n-1}}, y\rangle & 0 & \ldots & \|a_{i_{n-1}}\|^2 \\ \langle a_{i_n}, y\rangle & 0 & \ldots & 0 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \langle x, y\rangle \prod_{j=2}^{n} \|a_{i_j}\|^2 - \langle x, a_{i_2}\rangle\langle a_{i_2}, y\rangle \prod_{j=3}^{n} \|a_{i_j}\|^2 - \cdots - \langle x, a_{i_n}\rangle\langle a_{i_n}, y\rangle \prod_{j=2}^{n-1} \|a_{i_j}\|^2$$

$$= \left[\langle x, y\rangle - \frac{\langle x, a_{i_2}\rangle\langle a_{i_2}, y\rangle}{\|a_{i_2}\|^2} - \frac{\langle x, a_{i_3}\rangle\langle a_{i_3}, y\rangle}{\|a_{i_3}\|^2} - \cdots - \frac{\langle x, a_{i_n}\rangle\langle a_{i_n}, y\rangle}{\|a_{i_n}\|^2}\right] \prod_{j=2}^{n} \|a_{i_j}\|^2 .$$

Let $\{1, 2, \dots n\} \backslash \{i_2, \dots i_n\} = \{i_1\}$. By Parseval's identity, we have $\langle x, y \rangle = \sum_{j=1}^{n} \frac{\langle x, a_{i_j} \rangle \langle a_{i_j}, y \rangle}{\|a_{i_j}\|^2}$ (since $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$ as sets). Hence it follows that

$$\langle x, y | a_{i_2}, a_{i_3}, \dots, a_{i_n} \rangle = \frac{\langle x, a_{i_1} \rangle \langle a_{i_1}, y \rangle}{\|a_{i_1}\|^2} \prod_{j=2}^{n} \|a_{i_j}\|^2 = \frac{\langle x, a_{i_1} \rangle \langle a_{i_1}, y \rangle}{\|a_{i_1}\|^4} \prod_{i=1}^{n} \|a_i\|^2 .$$

Summing the above expressions for $i_1 = 1, 2, \dots, n$, we get

$$\begin{aligned} \langle x, y \rangle_A &= \sum_{\{i_2, \dots i_n\} \subset \{1,2,\dots,n\}} \langle x, y | a_{i_2}, \dots, a_{i_n} \rangle \\ &= \left[\frac{\langle x, a_1 \rangle \langle a_1, y \rangle}{\|a_1\|^4} + \frac{\langle x, a_2 \rangle \langle a_2, y \rangle}{\|a_2\|^4} + \dots + \frac{\langle x, a_n \rangle \langle a_n, y \rangle}{\|a_n\|^4} \right] \prod_{i=1}^{n} \|a_i\|^2 . \end{aligned}$$

However, we are assuming that $\|a_i\| = \alpha$ for all $i = 1, 2, \dots, n$ and so we obtain

$$\begin{aligned} \langle x, y \rangle_A &= \left[\frac{\langle x, a_1 \rangle \langle a_1, y \rangle}{\alpha^4} + \frac{\langle x, a_2 \rangle \langle a_2, y \rangle}{\alpha^4} + \dots + \frac{\langle x, a_n \rangle \langle a_n, y \rangle}{\alpha^4} \right] \prod_{i=1}^{n} \alpha^2 \\ &= \frac{\langle x, y \rangle}{\alpha^2} \alpha^{2n} = \alpha^{2(n-1)} \langle x, y \rangle. \end{aligned}$$

Since $\alpha \neq 0$, we conclude that $\langle x, y \rangle_A = 0$ if and only if $\langle x, y \rangle = 0$, which proves the theorem.

Theorem 2. *Let $A = \{b_1, b_2, \dots, b_n\}$ be a set of n linearly independent vectors in X. Then $\langle x, y \rangle_A = \langle x, y \rangle$ if and only if A is an orthonormal basis for X.*

Proof. The sufficient part follows immediately from the previous theorem. For the necessary part, suppose that $\langle x, y \rangle_A = \langle x, y \rangle$ for all $x, y \in X$. To prove that A is an orthonormal basis for X, let us first compute (b_i, b_j) for $i \neq j$. We have

$$\langle b_i, b_j \rangle = \langle b_i, b_j \rangle_A = \sum_{\{i_2, \dots i_n\} \subset \{1,2,\dots,n\}} \langle b_i, b_j | b_{i_2}, \dots, b_{i_n} \rangle.$$

For any $\{i_2, \dots, i_n\} \subset \{1, 2, \dots, n\}$, observe that $b_i \in \{b_{i_2}, \dots, b_{i_n}\}$ or $b_j \in \{b_{i_2}, \dots, b_{i_n}\}$, because $\{b_{i_2}, \dots, b_{i_n}\}$ consists of $n-1$ elements of A. Consequently , $\langle b_i, b_j | b_{i_2}, \dots, b_{i_n} \rangle = 0$, because two rows or two columns in the determinant will be identical. Since this is true for any $\{i_2, \dots, i_n\} \subset \{1, 2, \dots, n\}$, we conclude that

$$\langle b_i, b_j \rangle = \sum_{\{i_2, \dots i_n\} \subset \{1,2,\dots,n\}} \langle b_i, b_j | b_{i_2}, \dots, b_{i_n} \rangle = 0.$$

Let us now compute $\langle b_i, b_i \rangle$ for $i = 1, 2, \dots, n$. Notice that if $b_i \in \{b_{i_2}, \dots, b_{i_n}\}$, then we have $\langle b_i, b_i | b_{i_2}, \dots, b_{i_n} \rangle = 0$. Meanwhile, if $b_i \notin \{b_{i_2}, \dots, b_{i_n}\}$ then —by the properties of the standard $n-$product— we have

$$\langle b_i, b_i | b_{i_2}, \dots, b_{i_n} \rangle = \langle b_1, b_1 | b_2, \dots, b_n \rangle = \begin{vmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_n \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_n, b_1 \rangle & \langle b_n, b_2 \rangle & \dots & \langle b_n, b_n \rangle \end{vmatrix} = \prod_{j=1}^{n} \|b_j\|^2 .$$

Hence, we obtain

$$\langle b_i, b_i \rangle_A = \sum_{\{i_2, \dots i_n\} \subset \{1,2,\dots,n\}} \langle b_i, b_i | b_{i_2}, \dots, b_{i_n} \rangle = \prod_{j=1}^{n} \|b_j\|^2 .$$

By our hypothesis, $\|b_i\|^2 = \|b_i\|_A^2 = \prod_{j=1}^{n} \|b_j\|^2$. This holds only if $\|b_i\| = 1$ for all $i = 1, 2, \dots, n$.

To sum up, we have proved that $\langle x, y \rangle_A = \langle x, y \rangle$ if and only if A is an orthonormal basis for X.

3. The higher dimensional case

Let us now consider the case where $n+1 \le d = \dim\ X < \infty$. Let $A := \{a_1, a_2, \dots, a_d\}$ be a set of linearly independent vectors in X. (What happens if we use only n vectors will be discussed later, together with the case where $d = \infty$.) We define the following inner product on X:

$$\langle x, y \rangle_A := \sum_{\{i_2, \dots i_n\} \subset \{1,2,\dots,n\}} \langle x, y | a_{i_2}, \dots, a_{i_n} \rangle.$$

Note that there are $\binom{d}{n-1}$ terms in the above sum. Analogous to Theorem 2.1, we have the following theorem.

Theorem 3. *Let $A = \{a_1, a_2, \dots, a_d\} \subset X$ be an orthogonal set with $\|a_i\| = \alpha > 0$ for all $i = 1, 2, \dots, d$. Then $\langle x, y \rangle_A = 0$ if and only if $\langle x, y \rangle = 0$ for all $x, y \in X$.*

Proof. Let $I_d := \{1, 2, \dots, d\}$. For any $I_0 := \{i_2, \dots, i_n\} \subset I_d$, let $I_1 := I_d \setminus I_0$. Then, we have

$$\begin{aligned} \langle x, y | a_{i_2}, \dots a_{i_n} \rangle &= \sum_{i \in I_1} \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\|a_i\|^2} \prod_{i \in I_0} \|a_i\|^2 \\ &= \left[\langle x, y \rangle - \sum_{i \in I_0} \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\|a_i\|^2} \right] \prod_{i \in I_0} \|a_i\|^2 . \end{aligned}$$

Summing over all subsets $I_0 \subset I_d$ and using the assumption that $\|a_i\| = \alpha$ for all $i = 1, 2, \ldots n$, we obtain

$$\begin{aligned}\langle x, y\rangle_A &= \sum_{I_0 \subset I_d} \left[\langle x, y\rangle - \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right]\alpha^{2(n-1)} \\ &= \sum_{I_0 \subset I_d} \alpha^{2(n-1)}\langle x, y\rangle - \sum_{I_0 \subset I_d}\sum_{i \in I_0} \alpha^{2(n-1)}\frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\alpha^2}.\end{aligned}$$

The first sum on the right hand side is equal to $\binom{d}{n-1}\alpha^{2(n-1)}\langle x, y\rangle$. For the second sum, we observe that each expression $\langle x, a_i\rangle\langle a_i, y\rangle$ occurs precisely $\binom{d-1}{n-2}$ times for all $i \in I_d$.

Hence, by Parseval's identity, the second sum is equal $\binom{d-1}{n-2}\alpha^{2(n-1)}\langle x, y\rangle$. Therefore we get

$$\langle x, y\rangle_A = \left[\binom{d}{n-1} - \binom{d-1}{n-2}\right]\alpha^{2(n-1)}\langle x, y\rangle = \binom{d-1}{n-1}\alpha^{2(n-1)}\langle x, y\rangle,$$

which gives us the desired conclusion.

Corollary 1. *If $A = \{a_1, a_2, \ldots, a_d\}$ is an orthonormal basis for X, then $\langle x, y\rangle_A = \binom{d-1}{n-1}\langle x, y\rangle$ for all $x, y \in X$.*

Remark 1. The converse of the above corollary does not hold. To give an example, let $d = \dim X = 3$ and $n = 2$. Let $A = \{a_1, a_2, a_3\}$ be linearly independent set in X. Suppose that $\langle x, y\rangle_A = 2\langle x, y\rangle$ for all $x, y \in X$. We would like to check whether we have $\|a_i\| = 1$ for $i = 1, 2, 3$ and $\langle a_i, a_j\rangle = 0$ for $i \neq j$. Notice that

$$\langle x, y\rangle_A = \sum_{i=1}^{3}\langle x, y|a_i\rangle = \sum_{i=1}^{3}\begin{vmatrix}\langle x, y\rangle & \langle x, a_i\rangle \\ \langle a_i, y\rangle & \langle a_i, a_i\rangle\end{vmatrix}$$

for all $x, y \in X$. From the hypothesis, we have $\langle a_i, a_j\rangle_A = 2\langle a_i, a_j\rangle$ for $i, j = 1, 2, 3$, which may be rewritten as

$$\begin{aligned}\|a_1\|^2\|a_2\|^2 - \langle a_1, a_2\rangle^2 + \|a_1\|^2\|a_3\|^2 - \langle a_1, a_3\rangle^2 &= 2\|a_1\|^2, \\ \|a_1\|^2\|a_2\|^2 - \langle a_1, a_2\rangle^2 + \|a_2\|^2\|a_3\|^2 - \langle a_2, a_3\rangle^2 &= 2\|a_2\|^2, \\ \|a_1\|^2\|a_3\|^2 - \langle a_1, a_3\rangle^2 + \|a_2\|^2\|a_3\|^2 - \langle a_2, a_3\rangle^2 &= 2\|a_3\|^2, \\ \langle a_1, a_2\rangle\|a_3\|^2 - \langle a_1, a_3\rangle\langle a_2, a_3\rangle &= 2\langle a_1, a_2\rangle, \\ \langle a_1, a_3\rangle\|a_2\|^2 - \langle a_1, a_2\rangle\langle a_2, a_3\rangle &= 2\langle a_1, a_3\rangle, \\ \langle a_2, a_3\rangle\|a_1\|^2 - \langle a_1, a_2\rangle\langle a_1, a_3\rangle &= 2\langle a_2, a_3\rangle.\end{aligned}$$

Let $A := \|a_1\|, B := \|a_2\|, C := \|a_3\|, D := \langle a_1, a_2\rangle, E := \langle a_1, a_3\rangle, F := \langle a_2, a_3\rangle$. Then

$$\begin{aligned} A^2B^2 - D^2 + A^2C^2 - E^2 &= 2A^2, \\ A^2B^2 - D^2 + B^2C^2 - F^2 &= 2B^2, \\ A^2C^2 - E^2 + B^2C^2 - F^2 &= 2C^2, \\ DC^2 - EF &= 2D, \\ EB^2 - DF &= 2E, \\ FA^2 - DE &= 2F. \end{aligned}$$

Observe that $A = B = C = 1, D = E = F = 0$ satisfy the above equations simultaneously. We shall see that there are other possible solutions with $D, E, F \neq 0$.

Multiplying both sides in the last three equations by D, E, and F (respectively) and rearranging the terms, we obtain

$$D^2 + E^2 = A^2B^2 + A^2C^2 - 2A^2, \tag{1}$$
$$D^2 + F^2 = A^2B^2 + B^2C^2 - 2B^2, \tag{2}$$
$$E^2 + F^2 = A^2C^2 + B^2C^2 - 2C^2, \tag{3}$$
$$C^2D^2 - 2D^2 = DEF, \tag{4}$$
$$B^2E^2 - 2E^2 = DEF, \tag{5}$$
$$A^2F^2 - 2F^2 = DEF. \tag{6}$$

From $(1), (2)$, and (3), we get

$$D^2 = A^2B^2 - A^2 - B^2 + C^2, \tag{7}$$
$$E^2 = A^2C^2 - A^2 + B^2 - C^2, \tag{8}$$
$$F^2 = B^2C^2 + A^2 - B^2 - C^2. \tag{9}$$

From $(4), (5)$, and (6), we get

$$(C^2 - 2)D^2 = (B^2 - 2)E^2 = (A^2 - 2)F^2 = DEF. \tag{10}$$

Substituting $(7), (8)$, and (9) into (10), we obtain

$$\begin{aligned} (B^2 - C^2)(A^2 + B^2 + C^2 - 4) &= 0, \\ (A^2 - C^2)(A^2 + B^2 + C^2 - 4) &= 0, \\ (A^2 - B^2)(A^2 + B^2 + C^2 - 4) &= 0. \end{aligned}$$

Now one may check that $A = B = C = \frac{4}{3}, D + E + F = -\frac{2}{3}$ satisfy the above equations simultaneously. This tells us that A is not necessarily an orthonormal basis.

We now come to the case where $d = \dim\ X = \infty$. We assume that X is separable and $B := \{a_i : i = 1, 2, 3, \dots\}$ is an orthogonal basis for X. Thus for all $x, y \in X$, we have Parseval's identity that $\sum_{i=1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}$ will converge to $\langle x, y\rangle$. Next, let $A := \{a_1, a_2, \dots, a_n\}$, where the vectors a_i's are the first n vectors in B. We define

$$\langle x, y\rangle_A := \sum_{\{i_2,\dots,i_n\}\subset\{1,2,\dots,n\}} \langle x, y|a_{i_2}, \dots, a_{i_n}\rangle$$

for all $x, y \in X$. Then we have the following theorem.

Theorem 4. *For all $x, y \in X$, we have*

$$\langle x, y\rangle_A = \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^4} + n \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right] \prod_{j=1}^{n} \|a_j\|^2 .$$

In particular, if $\|a_i\| = \alpha$ for $i = 1, 2, \dots, n$ then

$$\langle x, y\rangle_A = \alpha^{2(n-1)} \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\alpha^2} + n \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right].$$

Proof. The proof is similar to the proof of Theorem 2.1 but this time we have

$$\langle x, y|a_{i_2}, \dots, a_{i_n}\rangle = \left[\frac{\langle x, a_{i_1}\rangle\langle a_{i_1}, y\rangle}{\|a_{i_1}\|^4} + \frac{1}{\|a_{i_1}\|^2} \sum_{i=n+1}^{\infty} \frac{\langle x, a_{i_1}\rangle\langle a_{i_1}, y\rangle}{\|a_i\|^2}\right] \prod_{j=1}^{n} \|a_j\|^2 ,$$

where $\{i_1\} = \{1, 2, \dots, n\} \setminus \{i_2, \dots, i_n\}$. Summing all these expression for all subsets $\{i_2, \dots, i_n\} \subset \{1, 2, \dots, n\}$, we obtain

$$\langle x, y\rangle_A = \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^4} + \sum_{i=1}^{n} \frac{1}{\|a_i\|^2} \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right] \prod_{j=1}^{n} \|a_j\|^2 .$$

In particular, if $\|a_i\| = \alpha$ for $i = 1, 2, \dots, n$, then we have

$$\langle x, y\rangle_A = \alpha^{2(n-1)} \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\alpha^2} + n \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right]$$

as claimed.

Remark 2. Note that if $\|a_i\| = 1$ for $i = 1, 2, 3, \dots$ (that is, B is an orthonormal basis for X), then the conclusion in the above theorem tells us that

$$\langle x, y\rangle_A = \sum_{i=1}^{n} \langle x, a_i\rangle\langle a_i, y\rangle + n \sum_{i=n+1}^{\infty} \langle x, a_i\rangle\langle a_i, y\rangle$$

for all $x, y \in X$.

Corollary 2. *Suppose that* $\|a_i\| = \alpha$ *for* $i = 1, 2, \ldots, n$. *For every* $x, y \in X$, *let* $x := x_A + x_A^\perp$ *and* $y := y_A + y_A^\perp$ *where* x_A *and* y_A *are the orthogonal projections of* x *and* y *on span* A *(respectively), and* $x_A^\perp$ *and* $y_A^\perp$ *are their complements (respectively). If* $\langle x_A, y_A\rangle_A = 0$ *and* $\langle x_A^\perp, y_A^\perp\rangle = 0$, *then* $\langle x, y\rangle = 0$. *Conversely, if* $\langle x_A, y_A\rangle = 0$ *and* $\langle x_A^\perp, y_A^\perp\rangle = 0$, *then* $\langle x, y\rangle_A = 0$.

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