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On the Characterizations of Approach Groups

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Abstract. In this paper, we present several characterization theorems on approach groups, and ultra approach groups. In so doing, we first give necessary and sufficient conditions for an approach structure to be compatible with group structure. We show that every ultra approach group is ultra-uniformizable. Secondly, starting with an approach space, and its natural neighborhood system on a group, we characterize the resulting neighborhood approach group. Finally, we show that the category of ultra approach-Cauchy group is a topological category, and more importantly, we show that the category of ultra approach-Cauchy groups and the category of strongly normal ultra approach-limit groups are isomorphic.

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1. Introduction

It is observed in [15], and elsewhere that **TOP**, the category of topological spaces, is simultaneously bireflectively and bicoreflectively embedded in **AP**, the category of approach spaces. This shows, however, that it makes not much difference notions like limits, colimits, initial structure that we may consider either in **TOP** or in **AP**. But it does make difference whether we make initial structures of ∞pq-metric approach spaces in $\mathbf{pqMET}^{\infty}$, the set of all ∞pq-metrics or in **AP**. This is so, because of the facts that in the first place, the domain of the ordinary metric space object $(X, d : X \times X \to [0, \infty])$ belonging to the category $\mathbf{pqMET}^{\infty}$ and the distance space object $\left(X, \Delta_d : X \times 2^X \to [0, \infty]\right)$ known as metric distance space, being member of the category **AP**, are essentially different; and, in the second place, they are also different from categorical viewpoint.
Given the importance of the preceding paragraph, a vast scale of research articles appeared over the years on studying various aspects of approach spaces, and their equivalence structures. A tiny part of the work cited in this paper cf. [4, 6–8, 13, 14, 17, 18, 20], whereas

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the vast majority of research carried out in these ever-growing area are not mentioned here just because the present paper is not directly linked to those works. It is pointed out in [9, 14, 19, 21], the importance of non-archimedean approach structures or ultra approach structures. We find it interesting to study the compatibility of the non-Archimedean approach structures with group structures, particularly, ultra approach-Cauchy structures, ultra approach limit structures, and so on; although we do not intend to study non-Archimedean metric group or ultra metric group structures here in this paper.
The idea of approach group along with its uniformization first appeared in [18], and later, we modified this concept further in [4] in a wider context. Furthermore, we identified this approach group with some other structures cf. [2, 3]. In this paper, we characterize approach groups vis-à-vis ultra approach groups, and study some of their related results. Thus, we concentrate on three main issues in relation with approach groups, such as, (a) ultra approach groups and some of their characterizations including ultra uniformization of ultra approach groups which however have not been considered in [18] although the idea of ultra approach spaces, and ultra uniform spaces are crept inside in [14] in addition to some other papers; (b) considering natural connection of neighborhood system with approach spaces, we characterize approach group by compatible neighborhood system on group structure; (c) considering approach-Cauchy structures, we show that for a group, there is a one-to-one correspondence between ultra approach-Cauchy group structures and strongly normal ultra approach limit group structures, the idea of strong normality first appeared in [5]; in fact, we prove here that the category of ultra approach-Cauchy groups and the category of strongly normal ultra approach limit groups are isomorphic.
We arrange these findings as follows. In Section 2, we consider some basic facts that are used in the sequel. We present the notion of ultra approach groups in Section 3, provide characterization theorems, and ultra uniformization of ultra approach groups. Using the notion of neighborhood system as defined in approach space, we give characterization theorem on approach group in Section4. Relation between approach groups and approach limit groups are discussed in Section 5. Finally, we describe the connection between ultra approach-Cauchy groups and strongly normal approach-Cauchy groups in Section 6.

2. Preliminaries

We denote the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on a set X by $\mathbb{F}(X)$. The *point filter* of a point $x \in X$ is defined by $\dot{x} = \{A \subseteq X \ : \ x \in A\}$, or by $[x]$. The set $\mathbb{F}(X)$ is ordered by set inclusion, i.e., we write $\mathbb{F} \leq \mathbb{G}$ if $\mathbb{F} \subseteq \mathbb{G}$.

If $(\mathbb{F}_j)_{j \in J}$ is a family of filters on a set X, then for a filter $\mathbb{U}$ on J, the compressed operator $\kappa\left(\mathbb{U}, (\mathbb{F}_j)_{j\in J}\right)$ is defined by [15]

$$\kappa\left(\mathbb{U}, (\mathbb{F}_j)_{j\in J}\right) = \bigvee_{V\in\mathbb{U}} \bigwedge_{j\in V} \mathbb{F}_j.$$

If $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, then the product filter $\mathbb{F} \times \mathbb{G} =< \{F \times G | F \in \mathbb{F}, G \in \mathbb{G}\} >$, i.e., we have $\{F \times G | F \in \mathbb{F}, G \in \mathbb{G}\}$ as a basis for $\mathbb{F} \times \mathbb{G}$.

For $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ we define

$$\mathbb{F} \odot \mathbb{G} = m(\mathbb{F} \times \mathbb{G}) \text{ and } \mathbb{F}^{-1} = i(\mathbb{F}).$$

Noting that $m(F \times G) = \{xy | x \in F, y \in G\} = F \odot G$, we have $\{F \odot G | F \in \mathbb{F}, G \in \mathbb{G}\}$ as a basis for $\mathbb{F} \odot \mathbb{G}$. Similarly, we find $\{F^{-1} | F \in \mathbb{F}\}$ as a basis for $\mathbb{F}^{-1}$, where $F^{-1} = \{x^{-1} | x \in F\}$.

Throughout the text for a group $(X, \cdot)$, we consider e as the identity element.

Lemma 1. *Let X and Y be groups, $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathbb{F}(X)$ and $f : X \longrightarrow Y$ a group homomorphism, then we have*
(i) $\mathbb{F} \odot \mathbb{F}^{-1} \leq \dot{e}$ *and* $\mathbb{F}^{-1} \odot \mathbb{F} \leq \dot{e}$;
(ii) $\dot{x} \odot (\dot{x})^{-1} = (\dot{x})^{-1} \odot \dot{x} = \dot{e}$;
(iii) $\dot{\widehat{xy}} = \dot{x} \odot \dot{y}$;
(iv) $\dot{\widehat{x^{-1}}} = (\dot{x})^{-1}$;
(v) $(\mathbb{F} \odot \mathbb{G}) \odot \mathbb{H} = \mathbb{F} \odot (\mathbb{G} \odot \mathbb{H})$;
(vi) $(\mathbb{F}^{-1})^{-1} = \mathbb{F}$;
(vii) $(\mathbb{F} \odot \mathbb{G})^{-1} = \mathbb{G}^{-1} \odot \mathbb{F}^{-1}$;
(viii) $\dot{e} \odot \mathbb{F} = \mathbb{F} \odot \dot{e} = \mathbb{F}$;
(ix) $(\mathbb{F} \wedge \mathbb{G})^{-1} = \mathbb{F}^{-1} \wedge \mathbb{G}^{-1}$;
(x) $(\mathbb{F} \wedge \mathbb{G}) \odot \mathbb{H} = (\mathbb{F} \odot \mathbb{H}) \wedge (\mathbb{G} \odot \mathbb{H})$;
(xi) $\mathbb{F} \leq \dot{x} \odot \mathbb{G} \Leftrightarrow (\dot{x})^{-1} \odot \mathbb{F} \leq \mathbb{G}$ *(resp.* $\mathbb{F} \leq \mathbb{G} \odot \dot{x} \Leftrightarrow \mathbb{F} \odot (\dot{x})^{-1} \leq \mathbb{G}$*)* ;
(xii) $f(\mathbb{F} \odot \mathbb{G}) = f(\mathbb{F}) \odot f(\mathbb{G})$;
(xiii) $f(\mathbb{F}^{-1}) = (f(\mathbb{F}))^{-1}$.

A subset $\Omega \subset [0, \infty]^X$ is called an ideal in $[0, \infty]^X$ if for any $\xi_1, \xi_2 \in \Omega$, $\xi_1 \vee \xi_2 \in \Omega$ and that for any $\xi \in \Omega$ with $\nu \leq \xi$ implies $\nu \in \Omega$, where the lattice $[0, \infty]^X$ is equipped with the point-wise order.

Definition 1. *[15] A collection of ideals $\Omega = (\Omega(x))_{x \in X}$ in $[0, \infty]^X$ indexed by the points of X is called an* approach system *on X if and only if the following conditions are fulfilled:*
(AS1) $\forall x \in X$, $\forall \nu \in \Omega(x)$: $\nu(x) = 0$.
(AS2) $\forall x \in X, \forall \nu \in [0, \infty]^X, \forall \epsilon > 0, \forall N < \infty$, *there exists* $\nu_\epsilon^N \in \Omega(x)$ *such that* $\nu \wedge N \leq \nu_\epsilon^N + \epsilon$ *implies* $\nu \in \Omega(x)$.
(AS3) $\forall x \in X, \forall \nu \in \Omega(x), \forall \epsilon > 0, N < \infty$ *there exists* $(\nu_z) \in \prod_{z \in X} \Omega(z)$ *such that for any* $y, z \in X$: $\nu(y) \wedge N \leq \nu_x(z) + \nu_z(y) + \epsilon$.

For any $x \in X$, $\nu \in \Omega(x)$ is called a local distance in x, and the value $\nu(t)$ of a local distance $\nu \in \Omega(x)$ at a point $t \in X$ is interpreted as *the distance from x to t according to ν*. Each local distance makes its own measurement of the distance other points in the space are away from the given point.
A subset $\mathbb{B} \subset [0, \infty]^X$ is called a ideal basis in $[0, \infty]^X$ if for any $\beta_1, \beta_2 \in \mathbb{B}$ there is a $\beta \in \mathbb{B}$ such that $\beta_1 \vee \beta_2 \leq \beta$.

Definition 2. *[15] A collection of ideal bases $\mathbb{B} = (\mathbb{B}(x))_{x \in X}$ in $[0, \infty]^X$ is called an* approach basis *if and only if the following statements are true.*
(AB1) $\forall x \in X$, $\forall \beta \in \mathbb{B}(x)$: $\beta(x) = 0$.

(AB2) $\forall x \in X$, $\forall \beta \in \mathbb{B}(x)$, $\forall \epsilon > 0, \forall N < \infty$, *there exists* $(\beta_z)_{z\in X} \in \prod_{z\in X} \mathbb{B}(z)$ *such that* $\forall y, z \in X\colon \beta(y) \wedge N \leq \beta_x(z) + \beta_z(y) + \epsilon$.

Definition 3. *[15] If* Ω *is an approach system, then* $\mathbb{B} = (\mathbb{B}(x))_{x\in X}$ *is called a basis for* Ω *if and only if the following are satisfied.*
(Ab1) $\forall x \in X$, $\mathbb{B}(x)$ *is a basis for an ideal.*
(Ab2) $\forall x \in X$: $\Omega(x) = \widehat{\mathbb{B}(x)}$, *where* $\widehat{\mathbb{B}(x)} = \{\nu \in [0,\infty]^X | \forall \epsilon > 0, \forall N \leq \infty \exists \xi \in \mathbb{B}(x) : \nu \wedge N \leq \xi + \epsilon\}$.

Definition 4. *[15] A function* $\lambda : \mathbb{F}(X) \longrightarrow [0,\infty]^X$ *is called a limit operator if and only if the following conditions are fulfilled:*
(AL1) $\forall x \in X : \lambda(\dot{x})(x) = 0$;
(AL2) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X), x \in X : \mathbb{F} \leq \mathbb{G}$ *implies* $\lambda(\mathbb{G})(x) \leq \lambda(\mathbb{F})(x)$;
(AL3) $\forall (\mathbb{F}_j)_{j\in J} \in \mathbb{F}(X)^J, x \in X : \lambda\left(\bigwedge_j \mathbb{F}_j\right)(x) = \bigvee_j \lambda(\mathbb{F}_j)(x)$;
(AL4) $\forall \mathbb{G} \in \mathbb{F}(X), (\mathbb{F}_y)_{y\in X} \in \mathbb{F}(X)^X, x \in X : \lambda\left(\kappa\left(\mathbb{G}, (\mathbb{F}_y)_{y\in X}\right)\right)(x) \leq \lambda(\mathbb{G})(x) + \bigvee_{y\in X} \lambda(\mathbb{F}_y)(y)$.
Then the pair (X, λ) *is called an* approach space.

A map $f : (X, \lambda) \longrightarrow (Y, \lambda')$ between two approach spaces is called a *contraction* if $\lambda'(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$, $\forall x \in X$.

The category of approach spaces and contraction mappings is denoted by **AP**.

Theorem 1. *[15] Let* $\left(f_j : X \longrightarrow (X_j, \Omega^j)_{j\in J}\right)$ *be a structured source in* ***AP***. *Then an approach basis on* X *for the unique initial lift of this source in* ***AP*** *is given by*

$$\mathbb{B}(x) = \{\bigvee_{j\in K} \nu_j \circ f_j | \, K \in 2^{(J)}, \forall j \in K : \nu_j \in \Omega^j(f_j(x))\}, \forall x \in X.$$

Given an approach space (X, λ), one defines for $\alpha \in [0,\infty]$ and $x \in X$, the α-neighborhood filter at $x \in X$ given in [13] by

$$\mathbb{U}^x_\alpha = \bigwedge\{\mathbb{F} \in \mathbb{F}(X) | \lambda(\mathbb{F})(x) \leq \alpha\}.$$

Remark 1. *In an approach space* (X, λ), $\forall \mathbb{F} \in \mathbb{F}(X)$, $\forall x \in X$ *and* $\forall \alpha \in [0,\infty]$:

$$\lambda(\mathbb{F})(x) \leq \alpha \Leftrightarrow \mathbb{F} \geq \mathbb{U}^x_\alpha.$$

[13]

Theorem 2. *[13] Let* $(X, \lambda) \in |$ ***AP***$|$. *The system* $\mathcal{U} = (\mathbb{U}^x_\alpha)_{x\in X, \alpha\in[0,\infty]}$ *has the following properties:*
(U0) $\mathbb{U}^x_\alpha \in \mathbb{F}(X)$ *for all* $x \in X$ *and* $\alpha \in [0,\infty]$;
(U1) $\mathbb{U}^x_\alpha \leq \dot{x}$, *for all* $x \in X$ *and* $\alpha \in [0,\infty]$;
(U2) $\mathbb{U}^x_{\alpha+\beta} \leq \kappa\left(\mathbb{U}^x_\beta, ((\mathbb{U}^y_\alpha)_{y\in X})\right)$, *for all* $x \in X$ *and* $\alpha, \beta \in [0,\infty]$;
(U3) $0 \leq \alpha \leq \beta$ *implies* $\mathbb{U}^x_\beta \leq \mathbb{U}^x_\alpha$;
(U4) For all $\emptyset \neq A \subset [0,\infty]$: $\bigvee_{\alpha\in A} \mathbb{U}^x_\alpha = \mathbb{U}^x_{\wedge A}$.
We call the system $\mathcal{U}$ *as the corresponding neighborhood system of the approach space* (X, λ).

If $(X,\cdot)$ is a group and $\nu \in [0,\infty]^X$, then for any $x \in X$, we write $x \odot \nu : X \longrightarrow [0,\infty]$, $y \mapsto x \odot \nu(y) = \nu(xy)$. Similarly, we also write $\nu \odot x : X \longrightarrow [0,\infty]$, $y \mapsto \nu \odot x(y) = \nu(yx)$. If $\nu \in [0,\infty]^X$, then $\nu^{-1} : X \to [0,\infty]$ is defined by $\nu^{-1}(x) = \nu(x^{-1})$. If $\mathbb{B} \subset [0,\infty]^X$, put $x \odot \mathbb{B} = \{x \odot \nu | \, \nu \in \mathbb{B}\}$. Note that $\langle x \odot \mathbb{B} \rangle = x \odot \langle \mathbb{B} \rangle$.
Now for the convenience of the reader, we recall some essential categorical terms that are needed in the sequel, for the details, we refer to [1].
A *functor* $\mathcal{F} \colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is a morphism between categories, consists of mappings between objects of $\mathfrak{C}$ and objects of $\mathfrak{D}$(sometimes we write as $|\mathfrak{C}|$ to denote the objects of $\mathfrak{C}$) and the mapping between morphisms of $\mathfrak{C}$ and morphisms of $\mathfrak{D}$ such that (i) if $f \colon S \longrightarrow T$, then $\mathcal{F}(f) \colon \mathfrak{C}(S) \longrightarrow \mathfrak{D}(T)$; (ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$, whenever $f \circ g$ is defined; (iii) $\mathcal{F}(id_S) = id_{\mathcal{F}(S)}$. The functor $\mathcal{F}$ is called an *embedding* if it is injective on objects. If $\mathfrak{E}$ is a category, then by a *concrete category over* $\mathfrak{E}$, we understand a pair $(\mathfrak{G}, \mathcal{F})$, where $\mathfrak{C}$ is a category and $\mathcal{F} \colon \mathfrak{G} \longrightarrow \mathfrak{E}$ is a faithful functor.
A *construct* is a concrete category over **SET**, the category of sets, and we consider the objects of a construct as structured set (S,ξ), and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, that is, for any source $(f_j \colon S \longrightarrow (S_j, \varsigma_j))_{j\in J}$, there is a unique structure ς on S such that a mapping $g \colon (T,\beta) \longrightarrow (S,\varsigma)$ is a morphism if and only if for each $j \in J$ the composition $f_j \circ g \colon (T,\beta) \longrightarrow (S_j,\varsigma_j)$ is a morphism, where (T,β) is a structured set.
A functor $\mathcal{F} \colon \mathfrak{C} \longrightarrow \mathfrak{D}$ between categories $\mathfrak{C}$ and $\mathfrak{D}$ is called an *isomorphism* if there is a functor $\mathcal{H} \colon \mathcal{D} \longrightarrow \mathcal{C}$ such that $\mathcal{H} \circ \mathcal{F} = id_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{H} = id_{\mathcal{D}}$. Two categories $\mathfrak{C}$ and $\mathfrak{D}$ are said to be *isomorphic* if there is an isomorphism.

3. Approach groups, characterizations and ultra uniformization

Definition 5. *[15] Let X be a set. A family of ideals $(\Omega(x))_{x\in X}$ in $[0,\infty]^X$ is called an* ultra approach system *on X if and only if for all $x \in X$, the following properties are satisfied:*
(AS1) $\forall \nu \in \Omega(x)$*:* $\nu(x) = 0$.
(AS2) $\forall \nu \in [0,\infty]^X$*:* $(\forall \epsilon > 0, \forall N < \infty, \exists \nu_\epsilon^N \in \Omega(x)$ *s.t.* $\nu \wedge N \leq \nu_\epsilon^N + \epsilon)$ *implies* $\nu \in \Omega(x)$.
(AS3) $\forall \nu \in \Omega(x), \forall \epsilon > 0, \forall N < \infty, \exists$ *a family* $(\Omega(z))_{z\in X}$ *such that* $\nu_z \in \Omega_z, \forall z \in X$ *and that* $\forall y \in X, \nu(y) \wedge N \leq \nu_x(z) \vee \nu_z(y) + \epsilon$.

Definition 6. *[15] If (X,Ω) and (X',Ω') are approach spaces (resp.ultra approach spaces), then a map $f : (X,\Omega) \to (X',\Omega')$ is called* contracting *at $x \in X$ if and only if for all $\nu' \in \Omega'(f(x))$, for all $\epsilon > 0$ and for all $N < \infty$ there exists a $\nu \in \Omega(x)$ such that $(\nu' \circ f) \wedge N \leq \nu + \epsilon$.*
The map f is called a contraction *if and only if it is contracting in each $x \in X$.*

Definition 7. *[14] A family of ideals Ξ in $[0,\infty]^{X\times X}$ is called an* ultra approach uniformity *on X if and only if the following properties are satisfied:*
(uAU1) $\forall \xi \in \Xi, \forall x \in X$*:* $\xi(x,x) = 0$.
(uAU2) $\forall \xi \in [0,\infty]^{X\times X}$*:* $(\forall \epsilon > 0, \forall N < \infty, \exists \xi_\epsilon^N \in \Xi$ *s.t.* $\xi \wedge N \leq \xi_\epsilon^N + \epsilon)$ *implies* $\xi \in \Xi$.
(uAU3) $\forall \xi \in \Xi$*:* $\xi^s \in \Xi$*, where* $\xi^s(x,y) = \xi(y,x)$*,* $\forall (x,y) \in X \times X$.

(uAU4) $\forall \xi \in \Xi, \forall \epsilon > 0, \forall N \leq \infty, \exists \xi^N \in \Xi$ *s.t.* $\forall x, y, z \in X$: $\xi(x,z) \wedge N \leq \xi^N(x,y) \vee \xi^N(y,z) + \epsilon$.

If (X, Ξ) is an ultra approach uniform space, then the underlying approach system of Ξ is given by $\Omega(x) = \{\xi(x,.) | \xi \in \Xi\}$, or by Ω_x.

Definition 8. *Let* $(X, \cdot)$ *be a group, and* $\Omega = (\Omega(x))_{x \in X}$ *be a family of* ultra approach system *on* X*. Then the triple* $\left(X, \cdot, \Omega = (\Omega(x))_{x \in X}\right)$ *is called an* ultra approach group *if and only if the following conditions are fulfilled:*
(UAG1) the mapping $m : X \times X \to X, (x,y) \mapsto xy$ *is a contraction;*
(UAG2) the inversion $\jmath : X \to X, x \mapsto x^{-1}$ *is a contraction.*

Lemma 2. *If* $\left(X, \cdot, \Omega = (\Omega(x))_{x \in X}\right)$ *is an approach group, then for any* $a \in X$*,* $\mathcal{L}_a$ *the left (resp.* $\mathcal{R}_a$ *the right) translation is a homeomorphism. That is, bijective and bi-contraction.*

Proof. This goes almost in the same way as in the proof of Proposition 2.3 [18] with $\mathcal{L}_a : X \to X$, $x \mapsto ax$.

Proposition 1. *Let* $\left(X, \cdot, \Omega = (\Omega(x))_{x \in X}\right)$ *be an approach group. Then for any* $x \in X$*,* $\Omega(x) = \{\nu \circ \mathcal{L}_{x^{-1}} | \nu \in \Omega(e)\}$ *(respectively,* $\Omega(x) = \{\nu \circ \mathcal{R}_{x^{-1}} | \nu \in \Omega(e)\}$*).*

Proof. Let $x \in X$. If $\nu \in \Omega(e)$, then $\nu \in \Omega\left(\mathcal{L}_{x^{-1}}(x)\right)$. Since $\mathcal{L}_{x^{-1}}$ is a contraction, by definition, we have $\nu \circ \mathcal{L}_{x^{-1}} \in \Omega(x)$. Conversely, if $\nu \circ \mathcal{L}_{x^{-1}} \in \Omega(x)$, then $\nu \circ \mathcal{L}_{x^{-1}} \in \Omega\left(\mathcal{L}_x(e)\right)$, which by definition of contraction yields that $(\nu \circ \mathcal{L}_{x^{-1}}) \circ \mathcal{L}_x \in \Omega(e)$. But $(\nu \circ \mathcal{L}_{x^{-1}}) \circ \mathcal{L}_x = \nu$, and hence $\nu \in \Omega(e)$. This shows that $\nu \in \Omega(e) \Leftrightarrow \nu \circ \mathcal{L}_{x^{-1}} \in \Omega(x)$. The other part follows exactly the same way.

Theorem 3. *Let* $(X, \cdot)$ *be a group and* $\Omega = (\Omega(x))_{x \in X}$ *be an approach system on* X*. Then the triple* $\left(X, \cdot, \Omega = (\Omega(x))_{x \in X}\right)$ *is an approach group if and only if the following are fulfilled:*
(a) $\forall x \in X$: $\Omega(x) = \{x^{-1} \odot \nu | \nu \in \Omega(e)\}$*, where* $x^{-1} \odot \nu = \nu \circ \mathcal{L}_{x^{-1}}$ *(alternatively,* $\Omega(x) = \{\nu \odot x^{-1} | \nu \in \Omega(e)\}$*, where* $\nu \odot x^{-1} = \nu \circ \mathcal{R}_{x^{-1}}$*);*
(b) $\forall \nu \in \Omega(e)$*,* $\forall \epsilon > 0$*,* $\forall N < \infty$*, there exists* $\mu \in \Omega(e)$*,* $\nu^{-1} \wedge N \leq \mu + \epsilon$*, i.e.* $\jmath : X \to X, x \mapsto x^{-1}$ *is contracting at* e*;*
(c) $\forall \nu \in \Omega(e), \forall \epsilon > 0, \forall N < \infty$*, there exists* $\mu \in \Omega(e)$ *such that*
$\forall x, y \in X$: $\nu(xy) \wedge N \leq \mu(x) \vee \mu(y) + \epsilon$*, i.e.* $m : (x,y) \mapsto xy$ *is contracting at* $(e,e) \in X \times X$*;*
(d) $\forall \nu \in \Omega(e)$*,* $\forall \epsilon > 0$*,* $\forall N < \infty$*,* $\forall x \in X$*, there exists* $\mu \in \Omega(e)$ *such that* $\left(x \odot \nu \odot x^{-1}\right) \wedge N \leq \mu + \epsilon$*, i.e.* $Int_x : z \mapsto xzx^{-1}$ *is contracting at* e*.*

Proof. If $\left(X, \cdot, \Omega = (\Omega(x))_{x \in X}\right)$ is an approach group, then (a) follows from the Proposition 1, and (b) follows from the Definition 3.1 [15], while (c) follows the Definition. As for (d), we employ the Theorem 3.4(b) [15] since $Int_x = \mathcal{L}_x \circ \mathcal{R}_{x^{-1}}$, and both the translations are contraction maps. To show the converse, assume that (a)-(d) are true. First, we show that the inversion map $\jmath : X \to X, x \mapsto x^{-1}$ is a contraction. Let $x \in X$, $\nu \in \Omega\left(\jmath(x)\right)$,

$\epsilon > 0$, and $N < \infty$, we want to find a $\theta \in \Omega(x)$ (say) such that $(\nu \circ \jmath) \wedge N \leq \theta + \epsilon$. Since $\nu \in \Omega(\jmath(x))$, there exists a $\mu \in \Omega(e)$ such that $\nu = x \odot \mu$. Consequently, due to axiom (b), there exists a $\mu_1 \in \Omega(e)$ such that $\mu^{-1} \wedge N \leq \mu_1 + \epsilon$. Then for any $z \in X$, $(\nu \circ \jmath)(z) \wedge N = [(x \odot \mu) \circ \jmath](z) \wedge N = \mu(xz^{-1}) \wedge N = \mu^{-1}(zx^{-1}) \wedge N \leq \mu_1(zx^{-1}) + \epsilon = (\mu_1 \odot x^{-1})(z) + \epsilon \Rightarrow (\nu \circ \jmath) \wedge N \leq \theta + \epsilon$, with $\theta := \mu_1 \odot x^{-1}$, $\theta \in \Omega(x)$. Thus we have proved that for any $\nu \in \Omega(\jmath(x))$, $\forall \epsilon > 0, \forall N < \infty$, there exists a $\theta \in \Omega(x)$ such that $(\nu \circ \jmath) \wedge N \leq \theta + \epsilon$, that is, $\jmath : X \to X, x \mapsto x^{-1}$ is contracting at x, and hence it is contracting in each x, and so, the inversion $\jmath : X \to X$ is a contraction. To prove that the map $m : (x, y) \mapsto xy$ is contacting in $(a, b) \in X \times X$, we employ axioms (a)-(d) in conjunction with Theorem 3.4 [15] to the contracting maps $m, \mathcal{L}_{a^{-1}}$, $\mathcal{L}_{b^{-1}}$,$\mathcal{L}_a$ and Int_b which are respectively contracting at (e, e), a, b, and e, to get the compositions: $m(a, b) = [\mathcal{L}_a \circ Int_b \circ m \circ (\mathcal{L}_{a^{-1}} \times \mathcal{L}_{b^{-1}})](a, b) = ab$. Thus, we have the $m : X \times X \to X, (x, y) \mapsto xy$ is a contraction map.

Theorem 4. *Let $(X, \cdot)$ be a group, and $\mathcal{B}$ be a family of ideals in $[0, \infty]^X$ such that the following are fulfilled:*
(1) $\mathcal{B}$ is an ideal basis, such that $\forall\ \nu \in \mathcal{B}$: $\nu(e) = 0$;
(2) $\forall \nu \in \mathcal{B}$, $\forall \epsilon > 0$, $\forall N < \infty$, there exists $\mu \in \mathcal{B}$ such that $\nu^{-1} \wedge N \leq \mu + \epsilon$;
(3) $\forall \nu \in \mathcal{B}, \forall \epsilon > 0, \forall N < \infty$, there exists $\mu \in \mathcal{B}$ such that
$\forall x, y \in X$: $\nu(xy) \wedge N \leq \mu(x) \vee \mu(y) + \epsilon$;
(4) $\forall \nu \in \mathcal{B}$, $\forall \epsilon > 0$, $\forall N < \infty$, $\forall x \in X$, there exists $\mu \in \mathcal{B}$ such that
$(x \odot \nu \odot x^{-1}) \wedge N \leq \mu + \epsilon$.
Then there exists a unique approach system such that $\mathcal{B}$ is a basis for the approach system at e and compatible with group structure of X. This approach system is given by: $\mathcal{A}(x) = \langle \{x^{-1} \odot \nu |\, \nu \in \mathcal{B}\} \rangle = \langle \{\nu \odot x^{-1} |\, \nu \in \mathcal{B}\} \rangle$.

Proof. In view of the preceding theorem, we only prove (AS3), for this we proceed as follows.
Let $\xi = x^{-1} \odot \nu \in \mathcal{A}(x)$ with $\nu \in \mathcal{B}$, let $\epsilon > 0$, and $N < \infty$. Choose $\eta \in \mathcal{B}$ such that $\nu(xy) \wedge N \leq \eta(x) \vee \eta(y) + \epsilon$. If $\xi_z = z^{-1} \odot \eta$, then $\xi(y) \wedge N = (x^{-1} \odot \nu)(y) \wedge N = \nu(x^{-1}y) \wedge N = \nu(x^{-1}zz^{-1}y) \wedge N \leq \eta(x^{-1}z) \vee \eta(z^{-1}y) + \epsilon = \xi_x(z) \vee \xi_z(y) + \epsilon$.

Proposition 2. *Every ultra approach group is ultra approach uniformizable.*

Proof. Let $(X, \cdot, \Omega)$ be an ultra approach group. Define
$\nu_l : X \times X \to [0, \infty], (x, y) \mapsto \nu_l(x, y) = \nu(x^{-1}y)$
and
$\Gamma =< \{\nu_l \in [0, \infty]^{X \times X} |\, \nu \in \Omega_e\} >$.
(uAU1) Let $x \in X$ and $\gamma \in \Gamma$. Then there is a $\nu \in \Omega_e$ such that $\gamma(x, x) = \nu_l(x, x) = \nu(e) = 0$.
(uAU3) Let $\epsilon > 0$ and $\gamma \in \Gamma$. Then there exists $\nu \in \Omega_e$ such that $\gamma(x, y) = \nu_l(x, y) = \nu(x^{-1}y)$. Since by contraction of r, one obtains $\nu_l \circ r \in \Omega_e$, yields that $\nu^{-1} \in \Omega_e$. Thus, we have $\gamma^s(x, y) = \gamma(y, x) = \nu_l(y, x) = \nu^{-1}(x^{-1}y) = (\nu^{-1})_l(x, y)$. So, $\gamma^s \in \Gamma$.
(uAU4) Let $\gamma \in \Gamma$ be such that $\gamma(x, y) = \nu_l(x, y) = \nu(x^{-1}y)$. Then for each $\epsilon > 0$ and $N < \infty$, there exists $\nu_\epsilon^N \in \Omega_e$ such that $\nu(xy) \wedge N \leq \nu_\epsilon^N(x) \vee \nu_\epsilon^N(y)$. If we put

$\gamma_\epsilon^N(x,y) = \nu_\epsilon^N(x^{-1}y)$, then
$\gamma(x,y) \wedge N = \nu(x^{-1}y) \wedge N$
$= \nu\left(x^{-1}zz^{-1}y\right) \wedge N$
$\leq \nu_\epsilon^N(x^{-1}z) \vee \nu_\epsilon^N(z^{-1}y)$
$\leq \gamma_\epsilon^N(x,z) \vee \gamma_\epsilon^N(z,y) + \epsilon.$
The underlying ultra-approach structure of Γ is given by
$\Omega'_x = \{\gamma(x,.)|\gamma \in \gamma\}$
$= \{\gamma(.,x)|\gamma \in \Gamma\}$
$= \{\nu \circ \mathcal{L}_{x^{-1}}|\nu \in \Omega_e\}$.
In fact, $\gamma(x,.)(y) = \gamma(x,y) = \nu(x^{-1}y) = \nu \circ \mathcal{L}_{x^{-1}}(y)$, for any $y \in X$. Thus by Proposition 1, we have $\Omega'_x = \Omega_x$.

4. Characterization of approach groups by neighborhood systems

Definition 9. *Let $(X,\cdot)$ be a group, (X,λ) be an approach space, and $\mathcal{U} = (\mathbb{U}_\alpha^x)_{x\in X,\alpha\in[0,\infty]}$ be a corresponding neighborhood system of the approach space (X,λ). Then the triple $\left(X,\cdot,\mathcal{U} = (\mathbb{U}_\alpha^x)_{x\in X,\alpha\in[0,\infty]}\right)$ is called a* neighborhood approach group *if and only if the following are fulfilled:*
(NAGM) $\mathbb{U}_{\alpha\vee\beta}^{xy} \leq \mathbb{U}_\alpha^x \odot \mathbb{U}_\beta^y$, $\forall x,y \in X$ *and* $\forall \alpha,\beta \in [0,\infty]$.
(NAGI) $\mathbb{U}_\alpha^{x^{-1}} \leq (\mathbb{U}_\alpha^x)^{-1}$.

Theorem 5. *Let $(X,\cdot)$ be a group and $\mathcal{U} = (\mathbb{U}_\alpha^x)_{\alpha\in[0,\infty],x\in X}$ be the neighborhood approach system corresponding to approach space (X,λ). Then $\left(X,\cdot,\mathcal{U} = (\mathbb{U}_\alpha^x)_{\alpha\in[0,\infty],x\in X}\right)$ is a neighborhood approach group if and only if the following axioms are fulfilled.*
(1) $\mathbb{U}_\alpha^e \in \mathbb{F}(X)$, $\forall \alpha \in [0,1]$;
(2) $\mathbb{U}_\alpha^e \leq \dot{e}$, $\forall \alpha \in [0,\infty]$;
(3) $\mathbb{U}_{\alpha+\beta}^e \leq \kappa\left(\mathbb{U}_\beta^e, (\mathbb{U}_\alpha^y)_{y\in X}\right)$, $\forall \alpha,\beta \in [0,\infty]$;
(4) if $0 \leq \alpha \leq \beta$, *then* $\mathbb{U}_\beta^e \leq \mathbb{U}_\alpha^e$;
(5) $\mathbb{U}_\alpha^e = \bigvee_{\alpha<\beta} \mathbb{U}_\beta^e$;
(6) $\mathbb{U}_{\alpha\vee\beta}^e \leq \mathbb{U}_\alpha^e \odot \mathbb{U}_\beta^e$, $\forall \alpha,\beta \in [0,\infty]$;
(7) $\mathbb{U}_\alpha^e \leq (\mathbb{U}_\alpha^e)^{-1}$;
(8) $\forall \alpha \in [0,1]$, $\forall x \in X$: $\mathbb{U}_\alpha^x = \dot{x} \odot \mathbb{U}_\alpha^e = \mathbb{U}_\alpha^e \odot \dot{x}$.

Proof. Let $\left(X,\cdot,\mathcal{U} = (\mathbb{U}_\alpha^x)_{\alpha\in[0,\infty],x\in X}\right)$ be a neighborhood approach group. Then conditions (1)-(7) follow immediately. We prove only (8). Since $\dot{x} \geq \mathbb{U}_\alpha^x$, we have $\dot{x} \odot \mathbb{U}_\alpha^e \geq \mathbb{U}_\alpha^x \odot \mathbb{U}_\alpha^e \geq \mathbb{U}_{\alpha\vee\alpha}^{xe} = \mathbb{U}_\alpha^x$. Next, we have: $\mathbb{U}_\alpha^x = \dot{e} \odot \mathbb{U}_\alpha^x = \left(\dot{x} \odot (\dot{x})^{-1}\right) \odot \mathbb{U}_\alpha^x = (\dot{x}) \odot \left((\dot{x})^{-1} \odot \mathbb{U}_\alpha^x\right) \geq \dot{x} \odot \left((\mathbb{U}_\alpha^x)^{-1} \odot \mathbb{U}_\alpha^x\right) \geq \dot{x} \odot \left(\mathbb{U}_\alpha^{x^{-1}} \odot \mathbb{U}_\alpha^x\right) \geq \dot{x} \odot \mathbb{U}_{\alpha\vee\alpha}^{x^{-1}x} = \dot{x} \odot \mathbb{U}_\alpha^e$. This ends the proof that $\mathbb{U}_\alpha^x = \dot{x} \odot \mathbb{U}_\alpha^e$. Similarly, one can obtain the right part. Hence the results follows.

Conversely, assume that all the conditions (1)-(8) are true.
We need to show that $\left(X,\cdot,\mathcal{U} = (\mathbb{U}_\alpha^x)_{\alpha\in[0,\infty],x\in X}\right)$ is a neighborhood approach group. As

it is already a neighborhood approach space, we first prove the condition (NAGM). Let $\alpha, \beta \in [0, \infty]$ and $x, y \in X$. Then by using Lemma 1 repeatedly we get:
$\mathbb{U}^{xy}_{\alpha\vee\beta} = \dot{\widehat{xy}} \odot \mathbb{U}^{e}_{\alpha\vee\beta}$
$= (\dot{x} \odot \dot{y}) \odot \mathbb{U}^{e}_{\alpha\vee\beta}$
$= \dot{x} \odot \left(\dot{y} \odot \mathbb{U}^{e}_{\alpha\vee\beta}\right)$
$\leq \dot{x} \odot \left(\dot{y} \odot \left(\mathbb{U}^{e}_{\alpha} \odot \mathbb{U}^{e}_{\beta}\right)\right)$ (by (6))
$= \dot{x} \odot \left((\dot{y} \odot \mathbb{U}^{e}_{\alpha}) \odot \mathbb{U}^{e}_{\beta}\right)$
$= \dot{x} \odot \left((\mathbb{U}^{e}_{\alpha} \odot \dot{y}) \odot \mathbb{U}^{e}_{\beta}\right)$ (applying (8))
$= (\dot{x} \odot \mathbb{U}^{e}_{\alpha}) \odot \left(\dot{y} \odot \mathbb{U}^{e}_{\beta}\right)$
$= \mathbb{U}^{x}_{\alpha} \odot \mathbb{U}^{y}_{\beta}$ (again by applying (8)).
To prove (NAGI), note that by applying (7), (8) and Lemma 1, we have:
$(\mathbb{U}^{x}_{\alpha})^{-1} = (\dot{x} \odot \mathbb{U}^{e}_{\alpha})^{-1} = (\mathbb{U}^{e}_{\alpha})^{-1} \odot (\dot{x})^{-1} \geq \mathbb{U}^{e}_{\alpha} \odot (\dot{x})^{-1} = \mathbb{U}^{x^{-1}}_{\alpha}$.

5. Ultra approach limit group and its relationship with neighborhood approach group

Definition 10. *[4] Let $(X, \cdot)$ be a group and (X, λ) be an ultra approach limit space. We call the triple $(X, \cdot, \lambda)$ an* ultra-approach limit group *if the following axioms are satisfied:*
(uALM) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X), x, y \in X : \lambda(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$.
(uALI) $\forall \mathbb{F} \in \mathbb{F}(X), x \in X : \lambda(\mathbb{F}^{-1})(x^{-1}) \leq \lambda(\mathbb{F})(x)$.

One can notice from [4] that the conditions (uALM) and (uALI) can be replaced by a single condition, i.e., for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X), x, y \in X : \lambda(\mathbb{F} \odot \mathbb{G})(xy^{-1}) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$.

Theorem 6. *If $(X, \cdot, \lambda)$ is an ultra approach limit group, then $\left(X, \cdot, \mathcal{U}_\lambda = (\mathbb{U}^{x}_{\alpha})_{\alpha\in[0,\infty], x\in X}\right)$ is a neighborhood approach group, where $\mathbb{U}^{x}_{\alpha} = \bigwedge\{\mathbb{F} \in \mathbb{F}(X) \,|\, \lambda(\mathbb{F})(x) \leq \alpha\}$, for any $\alpha \in [0, \infty]$ and $x \in X$.*
Conversely, if $\left(X, \cdot, \mathcal{U} = (\mathbb{U}^{x}_{\alpha})_{\alpha\in[0,\infty], x\in X}\right)$ is a neighborhood approach group, then $(X, \cdot, \lambda_{\mathcal{U}})$ is an ultra-approach limit group, where $\lambda_{\mathcal{U}}(\mathbb{F})(x) = \bigwedge\{\alpha \in [0, \infty] \,|\, \mathbb{U}^{x}_{\alpha} \leq \mathbb{F}\}$, for any $\mathbb{F} \in \mathbb{F}(X)$, and $x \in X$.

Proof. Let $(X, \cdot, \lambda)$ be an ultra approach limit group. For $\alpha, \beta \in [0, \infty]$, and $x, y \in X$, we put $\mathbb{F} = \mathbb{U}^{x}_{\alpha}$ and $\mathbb{G} = \mathbb{U}^{y}_{\beta}$. Then
$\lambda\left(\mathbb{U}^{x}_{\alpha} \odot \mathbb{U}^{y}_{\beta}\right)(xy)$
$= \lambda(\mathbb{F} \odot \mathbb{G})(xy)$
$\leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$ (by (uALM))
$\leq \alpha \vee \beta$.
This implies that $\lambda\left(\mathbb{U}^{x}_{\alpha} \odot \mathbb{U}^{y}_{\beta}\right)(xy) \leq \alpha \vee \beta$ which in view of Remark 1 yields that $\mathbb{U}^{x}_{\alpha} \odot$

$\mathbb{U}^y_\beta \geq \mathbb{U}^{xy}_{\alpha\vee\beta}$, i.e., the condition (NAGM) is proved. The condition (NAGI) is an immediate consequence of Lemma 3.7 [13].
Conversely, in view of Lemma 3.4[13], we only show condition (uALM). Assume that (NAGM) is true. Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, and $x, y \in X$. Then
$\lambda_{\mathcal{U}}(\mathbb{F})(x) \vee \lambda_{\mathcal{U}}(\mathbb{G})(y)$
$= (\bigwedge\{\alpha \in [0,\infty] \,|\, \mathbb{U}^x_\alpha \leq \mathbb{F}\}) \vee \left(\bigwedge\{\beta \in [0,\infty] | \mathbb{U}^y_\beta \leq \mathbb{G}\}\right)$
$= \bigwedge\{\alpha \vee \beta \in [0,\infty] \,|\, \mathbb{U}^x_\alpha \leq \mathbb{F}, \mathbb{U}^y_\beta \leq \mathbb{G}\}$ (by using Lemma 2.8[4])
$\geq \bigwedge\{\alpha \vee \beta \in [0,\infty] \,|\, \mathbb{U}^{xy}_{\alpha\vee\beta} \leq \mathbb{F} \odot \mathbb{G}\}$ (as because $\mathbb{U}^{xy}_{\alpha\vee\beta} \leq \mathbb{U}^x_\alpha \odot \mathbb{U}^y_\beta \leq \mathbb{F} \odot \mathbb{G}$, and with the assumption that $\mathbb{F} \odot \mathbb{G}$ exists, so is $\mathbb{U}^x_\alpha \odot \mathbb{U}^y_\beta$ by using Lemma 4.2[4])
$= \lambda_{\mathcal{U}}(\mathbb{F} \odot \mathbb{G})(xy)$, showing the condition (uALM) is proved. The last condition follows immediately by using (NAGI) coupled with Lemma 3.7[13].

Corollary 1. *Let* $\left(X, \cdot, \mathcal{U} = (\mathbb{U}^x_\alpha)_{\alpha\in[0,\infty], x\in X}\right)$ *be a neighborhood approach group. Then for any* $\alpha \in [0,\infty]$ *and* $x \in X$*:* $\mathbb{U}^x_\alpha = \dot{x} \odot \mathbb{U}^e_\alpha = \mathbb{U}^e_\alpha \odot \dot{x}$.

Proof. Let $x \in X$ and $\alpha \in [0,\infty]$. Then in view of the preceding theorem, Lemma 1 and Lemma 3.8[4], we have
$\mathbb{U}^x_\alpha = \bigwedge\{\mathbb{F} \in \mathbb{F}(X) \,|\, \lambda(\mathbb{F})(x) \leq \alpha\}$
$= \bigwedge\{\mathbb{F} \in \mathbb{F}(X) \,|\, \lambda\left((\dot{x})^{-1} \odot \mathbb{F}\right)(e) \leq \alpha\}$
$= \bigwedge\{\mathbb{F} \in \mathbb{F}(X) \,|\, (\dot{x})^{-1} \odot \mathbb{F} \geq \mathbb{U}^e_\alpha\}$ (by Remark 1)
$= \bigwedge\{\mathbb{F} \in \mathbb{F}(X) \,|\, \mathbb{F} \geq \dot{x} \odot \mathbb{U}^e_\alpha\}$
$= \dot{x} \odot \mathbb{U}^e_\alpha$.
Similarly, one can show that $\mathbb{U}^x_\alpha = \mathbb{U}^e_\alpha \odot \dot{x}$.

If we denote **uApLimGrp** as the category of all ultra-approach limit groups and **NApGrp**, the category of neighborhood approach groups associated with approach spaces, then if follows from [13] in conjunction with the Lemma 3.7[13] and the Theorem 6 above, these two categories are isomorphic, we leave details for the interested reader. However, the functors in question, say for instance, $\mathfrak{F}$ and $\mathfrak{G}$ are connected as described below:

$$\mathfrak{F} : \begin{cases} \textbf{uApLimGrp} & \longrightarrow & \textbf{NApGrp} \\ (X, \cdot, \lambda) & \longmapsto & (X, \cdot, \mathcal{U}_\lambda) \\ f & \longmapsto & f \end{cases}$$

and

$$\mathfrak{G} : \begin{cases} \textbf{NApGrp} & \longrightarrow & \textbf{uApLimGrp} \\ (X, \cdot, \mathcal{U}) & \longmapsto & (X, \cdot, \lambda_{\mathcal{U}}) \\ f & \longmapsto & f \end{cases}$$

6. Ultra approach-Cauchy groups, Ultra approach limit groups and Strongly normal ultra approach limit

In the light of the Section 5[4], we add some results in this section; specifically, our main aim here is to show that the categories **uApChyGrp** (the category of ultra approach-

Cauchy groups) and **SNuApLimGrp** (the category of strongly normal ultra approach limit groups) are isomorphic.

Definition 11. *[17] Let X be a set. A mapping $\Upsilon\colon \mathbb{F}(X) \longrightarrow [0,\infty]$ is called an* approach-Cauchy structure *if and only if the following conditions are fulfilled:*
(AChy1) $\Upsilon(\dot{x}) = 0$;
(AChy2) $\mathbb{F} \leq \mathbb{G}$ *implies* $\Upsilon(\mathbb{G}) \leq \Upsilon(\mathbb{F})$ *for all* $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$;
(AChy3) $\Upsilon(\mathbb{F} \wedge \mathbb{G}) \leq \Upsilon(\mathbb{F}) + \Upsilon(\mathbb{G})$.
Then the pair (X, Υ) is called an approach-Cauchy space.
A mapping $f\colon (X,\Upsilon) \longrightarrow (X',\Upsilon')$ between approach-Cauchy spaces is called approach-Cauchy contraction *if and only if for all $\mathbb{F} \in \mathbb{F}(X)$, $\Upsilon'(f(\mathbb{F})) \leq \Upsilon(\mathbb{F})$.*
*The category of all approach-Cauchy spaces and approach-Cauchy contractions is denoted by **ApChy**.*

Definition 12. *Let $(X,\cdot)$ be a group and (X,Υ) be an approach-Cauchy space. Then the triple $(X,\cdot,\Upsilon)$ is called an* approach-Cauchy group *if and only if for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$: $\Upsilon(\mathbb{F} \odot \mathbb{G}^{-1}) \leq \Upsilon(\mathbb{F}) + \Upsilon(\mathbb{G})$.*
*The category of all approach-Cauchy groups and approach-Cauchy contractions which are homomorphisms denoted by **ApChyGrp**.*

Definition 13. *[4, 17] A map $\Upsilon : \mathbb{F}(X) \to [0,\infty]$ is called an ultra approach-Cauchy structure on X if and only if the following axioms are fulfilled:*
(uAChy1) $\forall x \in X$: $\Upsilon(\dot{x}) = 0$;
(uAChy2) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ *with* $\mathbb{F} \leq \mathbb{G}$, $\Upsilon(\mathbb{G}) \leq \Upsilon(\mathbb{F})$,
(uAChy3) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, *if* $\mathbb{F} \vee \mathbb{G}$ *exists, then* $\Upsilon(\mathbb{F} \cap \mathbb{G}) \leq \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$.
A mapping $f\colon (X,\Upsilon) \longrightarrow (Y,\Upsilon')$ between ultra approach-Cauchy spaces is called Ultra approach-Cauchy contraction or Cauchy contraction *if and only if for all $\mathbb{F} \in \mathbb{F}(X)$, $\Upsilon'(f(\mathbb{F})) \leq \Upsilon(\mathbb{F})$.*
*The category of all ultra approach-Cauchy spaces and contractions is denoted by **uApChy**.*

Definition 14. *Let $\Upsilon : \mathbb{F}(X) \to [0,\infty]$ be an ultra approach-Cauchy structure on a group $(X,\cdot)$, then the triple $(X,\cdot,\Upsilon)$ is called an* ultra approach-Cauchy group *if and only if the mapping $h : (X \times X, \Upsilon \times \Upsilon) \to (X,\Upsilon), (x,y) \mapsto x^{-1}y$ is a contraction, (or, equivalently, $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\Upsilon\left(\mathbb{F} \odot \mathbb{G}^{-1}\right) \leq \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$.)*
Note that $\forall \mathbb{F} \in \mathbb{F}(X)$, $\Upsilon(\mathbb{F}^{-1}) \leq \Upsilon(\mathbb{F})$, and $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\Upsilon(\mathbb{F} \odot \mathbb{G}) \leq \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$, are also hold good.
*The category of ultra approach-Cauchy groups and Cauchy contractive homomorphisms is denoted by **uApChyGrp**.*

Proposition 3. ***uApChyGrp** is a topological category.*

Proof. Let $(X,\cdot)$ be a group, $f_j\colon X \longrightarrow X_j$ a group homomorphism, and $(X_j, \cdot, (\Upsilon_j)_{j\in J})$ be a family of ultra approach-Cauchy groups. Consider a source $\mathcal{S} = (f_j\colon X \longrightarrow (X_j, \cdot, \Upsilon_j))_{j\in J}$. Then in view of the Section 5[17] the ultra approach-Cauchy structure on X is given for any $\mathbb{F} \in \mathbb{F}(X)$, by $\Upsilon(\mathbb{F}) = \bigvee_{j\in J} \Upsilon_j(f(_j(\mathbb{F})))$. It is proved in [17] that the source $\mathcal{S}$ has

initial structure and that the pair (X, Υ) is an ultra approach-Cauchy space. It remains to be checked the following single condition:
Thus, for any $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, we have
$\Upsilon\left(\mathbb{F} \odot \mathbb{G}^{-1}\right) = \bigvee_{j\in J} \Upsilon_{j\in J}\left(f_j(\mathbb{F} \odot \mathbb{G}^{-1})\right) = \bigvee_{j\in J} \Upsilon_j\left(f_j(\mathbb{F}) \odot f_j(\mathbb{G})^{-1}\right)$
$\leq \bigvee \Upsilon_{j\in J}(f_j(\mathbb{F})) \vee \bigvee_{j\in J} \Upsilon_j(f_j(\mathbb{G})) = \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$, for all $j \in J$.
Finally, we show that for any ultra approach-Cauchy group $(Y, \cdot, \Upsilon')$, a group homomorphism $g\colon (Y, \cdot, \Upsilon') \longrightarrow (X, \cdot, \Upsilon)$ is a contraction if and only if for all $j \in J$, the map $f_j \circ g\colon (Y, \cdot, \Upsilon') \longrightarrow (X_j, \cdot, \Upsilon_j)$ is a contraction. But the composition $f_j \circ g$ is clearly contraction, while the contraction of g follows at once from [17].

Recall that an ultra approach limit group, [3], is a triple $(X, \cdot, \lambda)$ consisting of a group $(X, \cdot)$ and an ultra approach limit structure $\lambda : \mathbb{F}(X) \to [0, \infty]^X$ meaning for all $x \in X$, $\lambda(\dot{x}) = 0$, for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$ implies $\lambda(\mathbb{G}) \leq \lambda(\mathbb{F})$ and $\lambda(\mathbb{F} \wedge \mathbb{G}) = \lambda(\mathbb{F}) \vee \lambda(\mathbb{G})$, such that the group operation $h : X \times X \to X, (x, y) \mapsto xy^{-1}$ is a contraction, i.e., $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\lambda\left(\mathbb{F} \odot \mathbb{G}^{-1}\right)(xy^{-1}) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$. With each ultra approach limit group $(X, \cdot, \lambda)$, there is associated a natural ultra approach-Cauchy structure defined as follows:
$\Upsilon_\lambda : \mathbb{F}(X) \to [0, \infty], \mathbb{F} \mapsto \Upsilon_\lambda(\mathbb{F}) = \lambda\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)(e) \vee \lambda\left(\mathbb{F} \odot \mathbb{F}^{-1}\right)(e)$.
On the other hand, every ultra approach Cauchy group $(X, \cdot, \Upsilon)$ gives rise to an ultra approach limit structure given by:
$\lambda_\Upsilon : \mathbb{F}(X) \to [0, \infty]^X$, $\mathbb{F} \mapsto \lambda_\Upsilon(\mathbb{F})(x) = \Upsilon\left(\mathbb{F} \cap \dot{x}\right)$.

Lemma 3. *[4] Let $(X, \cdot, \lambda)$ be an ultra approach limit group, $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$. Then $\lambda(\mathbb{F})(x) = \lambda([x]^{-1} \odot \mathbb{F})(e) = \lambda(\mathbb{F} \odot [x]^{-1})(e)$.*

Proposition 4. *If $(X, \cdot, \lambda)$ is an ultra approach limit group, then*
$\Upsilon_\lambda : \mathbb{F}(X) \to [0, \infty]$ defined by
$\Upsilon_\lambda(\mathbb{F}) = \lambda\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)(e) \vee \lambda\left(\mathbb{F} \odot \mathbb{F}^{-1}\right)(e)$, $\forall \mathbb{F} \in \mathbb{F}(X)$
gives rise to an ultra approach-Cauchy structure.

Proof. (uAChy1) For any $x \in X$, $\Upsilon_\lambda(\dot{x}) = \lambda\left(\dot{x}^{-1} \odot \dot{x}\right)(e) \vee \lambda\left(\dot{x} \odot \dot{x}^{-1}\right)(e) = \lambda(\dot{e})(e) \vee \lambda(\dot{e})(e) = 0$.
(uAChy2) If $\mathbb{F} \leq \mathbb{G}$, then since $\mathbb{F}^{-1} \odot \mathbb{F} \leq \mathbb{G}^{-1} \odot \mathbb{G}$, we have $\Upsilon_\lambda(\mathbb{G}) \leq \Upsilon_\lambda(\mathbb{F})$.
(uAChy3) Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ such that $\mathbb{F} \vee \mathbb{G}$ exists, then $\mathbb{F}^{-1} \odot \mathbb{G} \leq \dot{e}$ and also, $\mathbb{G}^{-1} \odot \mathbb{F} \leq \dot{e}$, hence upon using these we have
$\Upsilon_\lambda(\mathbb{F}) \vee \Upsilon_\lambda(\mathbb{G})$
$= \lambda\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)(e) \vee \lambda\left(\mathbb{F} \odot \mathbb{F}^{-1}\right)(e) \vee \left(\mathbb{G}^{-1} \odot \mathbb{G}\right)(e) \vee \lambda\left(\mathbb{G} \odot \mathbb{G}^{-1}\right)(e)$
$= [\lambda\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)(e) \vee \left(\mathbb{G}^{-1} \odot \mathbb{G}\right)(e)] \bigvee [\lambda\left(\mathbb{F} \odot \mathbb{F}^{-1}\right)(e) \vee \lambda\left(\mathbb{G} \odot \mathbb{G}^{-1}\right)(e)]$
$\geq \lambda\left(\mathbb{F}^{-1} \odot \mathbb{G}\right)(e) \vee \lambda\left(\mathbb{G}^{-1} \odot \mathbb{F}\right)(e)$
Thus, we have
$\Upsilon_\lambda\left(\mathbb{F} \cap \mathbb{G}\right)$
$= \lambda\left(\left(\mathbb{F} \cap \mathbb{G}\right)^{-1} \odot \left(\mathbb{F} \cap \mathbb{G}\right)\right)(e) \vee \lambda\left(\left(\mathbb{F} \cap \mathbb{G}\right) \odot \left(\mathbb{F} \cap \mathbb{G}\right)^{-1}\right)(e)$
$= \lambda\left(\left(\mathbb{F}^{-1} \odot \mathbb{F}\right) \cap \left(\mathbb{F}^{-1} \odot \mathbb{G}\right) \cap \left(\mathbb{G}^{-1} \odot \mathbb{F}\right) \cap \left(\mathbb{G}^{-1} \odot \mathbb{G}\right)\right)(e) \bigvee$
$\lambda\left(\left(\mathbb{F} \odot \mathbb{F}^{-1}\right) \cap \left(\mathbb{F} \odot \mathbb{G}^{-1}\right) \cap \left(\mathbb{G} \odot \mathbb{F}^{-1}\right) \cap \left(\mathbb{G} \odot \mathbb{G}^{-1}\right)\right)(e)$

$= [\lambda\left(\mathbb{F}^{-1}\odot\mathbb{F}\right)(e)\vee\lambda\left(\mathbb{F}\odot\mathbb{F}^{-1}\right)(e)]\bigvee[\lambda\left(\mathbb{G}^{-1}\odot\mathbb{G}\right)(e)\vee\lambda\left(\mathbb{G}\odot\mathbb{G}^{-1}\right)(e)]$
$\bigvee[\lambda\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)(e)\vee\lambda\left(\mathbb{G}\odot\mathbb{F}^{-1}\right)(e)]\bigvee[\lambda\left(\mathbb{G}^{-1}\odot\mathbb{F}\right)(e)\vee\lambda\left(\mathbb{F}\odot\mathbb{G}^{-1}\right)(e)]$
$\leq \Upsilon_\lambda(\mathbb{F})\vee\Upsilon_\lambda(\mathbb{G})\vee\Upsilon_\lambda(\mathbb{G})\vee\Upsilon_\lambda(\mathbb{F}) = \Upsilon_\lambda(\mathbb{F})\vee\Upsilon_\lambda(\mathbb{G})$.

Definition 15. *An ultra approach limit group* $(X,\cdot,\lambda)$ *is called* strongly normal *if and only if for all* $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, $\lambda\left(\mathbb{F}\odot\mathbb{G}\odot\mathbb{F}^{-1}\right)(e)\leq\lambda\left(\mathbb{F}\odot\mathbb{F}^{-1}\right)(e)\vee\lambda\left(\mathbb{F}^{-1}\odot\mathbb{F}\right)(e)\vee\lambda(\mathbb{G})(e)$.

Proposition 5. *Let* $(X,\cdot,\lambda)$*, and* $(X',\cdot,\lambda')$ *be strongly normal ultra approach limit groups. If* $f\colon X\longrightarrow X'$ *is a group homomorphism, then the following assertions are equivalent:*
(i) $f\colon(X,\lambda)\longrightarrow(X',\lambda')$ *is a contraction;*
(ii) $f\colon(X,\Upsilon_\lambda)\longrightarrow(X',\Upsilon_{\lambda'})$ *is Cauchy contraction.*

Proof. Assume (i) holds. Then using Lemma 1, for any $\mathbb{F}\in\mathbb{F}(X)$,
$\Upsilon_{\lambda'}(f(\mathbb{F})) = \lambda'\left(f(\mathbb{F})^{-1}\odot f(\mathbb{F})\right)(f(e))\vee\lambda'\left(f(\mathbb{F})\odot f(\mathbb{F})^{-1}\right)(f(e)) = \lambda'\left(f(\mathbb{F}^{-1}\odot\mathbb{F})\right)(f(e))\vee$
$\lambda'\left(f(\mathbb{F}\odot\mathbb{F}^{-1})\right)(f(e))\leq\lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)\vee\lambda(\mathbb{F}\odot\mathbb{F}^{-1})(e) = \Upsilon_\lambda(\mathbb{F})$, i.e., $\Upsilon_{\lambda'}(f(\mathbb{F}))\leq\Upsilon_\lambda(\mathbb{F})$.
Now assume (ii), and let $\mathbb{F}\in\mathbb{F}(X)$, and $x\in X$. Then upon using Lemma 3, we get
$\lambda'(f(\mathbb{F}))(f(x)) = \lambda'\left([f(x)]^{-1}\odot f(\mathbb{F})\right)(f(e))\vee\lambda'\left(f(\mathbb{F})\odot[f(x)]^{-1}\right)(f(e))$
$\leq\lambda'\left(([f(x)]\wedge f(F))^{-1}\odot([f(x)]\wedge f(\mathbb{F})\right)(f(e))\vee\lambda'\left(f(\mathbb{F})\wedge[f(x)]\odot(f(\mathbb{F})\wedge[f(x)])^{-1}\right)(f(e))$
$=\Upsilon_{\lambda'}(f(\mathbb{F})\wedge[f(x)])\leq\Upsilon_\lambda(\mathbb{F}\wedge[x]) = \lambda((\mathbb{F}\wedge[x])^{-1}\odot(\mathbb{F}\wedge[x])(e)\vee\lambda((\mathbb{F}\wedge[x])\odot(\mathbb{F}\wedge[x])^{-1})(e)\leq$
$\lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e = x^{-1}x)\vee\lambda(\mathbb{F}\odot\mathbb{F})(e = xx^{-1})\leq\lambda(\mathbb{F})(x)\vee\lambda(\mathbb{F}^{-1})(x^{-1})\leq\lambda(\mathbb{F})(x)\vee\lambda(\mathbb{F})(x) =$
$\lambda(\mathbb{F})(x)$, which proves that $\lambda'(f(\mathbb{F}))(f(x))\leq\lambda(\mathbb{F})(x)$, i.e., $f\colon X\longrightarrow X'$ is a contraction.

Lemma 4. *If* $(X,\cdot,\Upsilon)$ *is a ultra approach-Cauchy group, then* $(X,\cdot,\lambda_\Upsilon)$ *is an ultra approach-limit group.*

Proof. In view of the Proposition 5.16 [17], we only need to show that the group operation $h: X\times X\to X, (x,y)\mapsto xy^{-1}$ is a contraction. If $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ and $x,y\in X$, then by the Lemma 1(iii), we have
$\left(\mathbb{F}\odot\mathbb{G}^{-1}\right)\cap\widehat{\dot{xy^{-1}}} = \left(\mathbb{F}\odot\mathbb{G}^{-1}\right)\cap\left(\dot{x}\odot\dot{y}^{-1}\right)\geq(\mathbb{F}\cap\dot{x})\odot\left(\mathbb{G}^{-1}\cap\dot{y}^{-1}\right)$.
Upon using (uAChy2), we get
$\Upsilon\left(\left(\mathbb{F}\odot\mathbb{G}^{-1}\right)\cap\left(\dot{x}\odot\dot{y}^{-1}\right)\right)$
$\leq\Upsilon\left((\mathbb{F}\cap\dot{x})\odot\left(\mathbb{G}^{-1}\cap\dot{y}^{-1}\right)\right)$
$=\Upsilon\left((\mathbb{F}\cap\dot{x})\odot(\mathbb{G}\cap\dot{y})^{-1}\right)$
$\leq\Upsilon(\mathbb{F}\cap\dot{x})\vee\Upsilon(\mathbb{G}\cap\dot{y})$
$=\lambda_\Upsilon(\mathbb{F})(x)\vee\lambda(\mathbb{G})(y)$.
Consequently, we have
$\lambda_\Upsilon\left(\mathbb{F}\odot\mathbb{G}^{-1}\right)(xy^{-1}) = \Upsilon\left(\left(\mathbb{F}\odot\mathbb{G}^{-1}\right)\cap\widehat{\dot{xy^{-1}}}\right)\leq\lambda_\Upsilon(\mathbb{F})(x)\vee\lambda_\Upsilon(\mathbb{G})(y)$.

Lemma 5. *If* $(X,\cdot,\lambda)$ *is an ultra approach limit group, then* $(X,\cdot,\Upsilon_\lambda)$ *is an ultra approach-Cauchy group if and only if it is strongly normal approach limit groups.*

Proof. Define $\Upsilon(\mathbb{F}) = \lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)\vee\lambda(\mathbb{F}\odot\mathbb{F}^{-1})(e)\vee\lambda(\mathbb{G})(e)$, for all $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ Assume that $(X,\cdot,\lambda)$ be a strongly normal approach limit group, we prove that $(X,\cdot,\Upsilon_\lambda)$

is an ultra approach-Cauchy group.
In view of the Proposition 4, only we need to prove that the group operations are contractive. Since $(X, \cdot, \lambda)$ is strongly normal, we have for any $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$,
$\Upsilon(\mathbb{F} \odot \mathbb{G}) = \lambda((\mathbb{F} \odot \mathbb{G}) \odot (\mathbb{F} \odot \mathbb{G})^{-1})(e) \vee \lambda((\mathbb{F} \odot \mathbb{G})^{-1} \odot (\mathbb{F} \odot \mathbb{G}))(e)$
$= \lambda\left(\mathbb{F} \odot \mathbb{G} \odot \mathbb{G}^{-1} \odot \mathbb{F}^{-1}\right)(e) \vee \lambda(\mathbb{G}^{-1} \odot \mathbb{F}^{-1} \odot \mathbb{F} \odot \mathbb{G})(e) \leq \lambda(\mathbb{F}^{-1} \odot \mathbb{F})(e) \vee \lambda(\mathbb{F} \odot \mathbb{F}^{-1})(e) \vee$ $\lambda(\mathbb{G} \odot \mathbb{G}^{-1})(e) \vee \lambda((\mathbb{G} \odot \mathbb{G}))(e) \vee \lambda(\mathbb{G}^{-1} \odot \mathbb{G}^{-1}))(e) \vee \lambda(\mathbb{F}^{-1} \odot \mathbb{F})(e)$ (by applying λ to $[e] = [e] \odot [e] \leq \mathbb{G} \odot \mathbb{G})$
$\leq \lambda(\mathbb{F}^{-1} \odot \mathbb{F})(e) \vee \lambda(\mathbb{F} \odot \mathbb{F}^{-1})(e) \vee \lambda(\mathbb{G} \odot \mathbb{G}^{-1})(e) \vee \lambda(\mathbb{G}^{-1} \odot \mathbb{G})(e) = \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$, proving that $\Upsilon(\mathbb{F} \odot \mathbb{G}) \leq \Upsilon(\mathbb{F}) \vee \Upsilon(\mathbb{G})$. Finaly, for any $\mathbb{F} \in \mathbb{F}(X)$, we have
$\Upsilon(\mathbb{F}^{-1}) = \lambda((\mathbb{F}^{-1})^{-1} \odot \mathbb{F}^{-1})(e) \vee \lambda(\mathbb{F}^{-1} \odot (\mathbb{F}^{-1})^{-1})(e)$
$\lambda((\mathbb{F} \odot \mathbb{F}^{-1})(e) \vee \lambda(\mathbb{F}^{-1} \odot (\mathbb{F})(e) = \Upsilon(\mathbb{F})$. This ends the prove that $(X, \cdot, \Upsilon)$ is an ultra approach-Cauchy group.

Thus, there are two functors $\mathfrak{A}$ and $\mathfrak{B}$ which in conjunction with the Proposition 5 yields the following:

$$\mathfrak{A} : \begin{cases} \textbf{uApChyGrp} & \longrightarrow & \textbf{SNApLimGrp} \\ (X, \cdot, \Upsilon) & \longmapsto & (X, \cdot, \lambda_\Upsilon) \\ f & \longmapsto & f \end{cases}$$

and

$$\mathfrak{B} : \begin{cases} \textbf{SNApLimGrp} & \longrightarrow & \textbf{uApChyGrp} \\ (X, \cdot, \lambda) & \longmapsto & (X, \cdot, \Upsilon_\lambda) \\ f & \longmapsto & f \end{cases}$$

Theorem 7. ***uApChyGrp*** *is isomorphic to* ***SNApLimGrp****.*

Proof. Observed that

$$\textbf{SNApLimGrp} \xrightarrow{\mathfrak{B}} \textbf{uApChyGrp} \xrightarrow{\mathfrak{A}} \textbf{SNApLimGrp}:$$
$$(X, \cdot, \lambda) \longmapsto (X, \cdot, \Upsilon_\lambda) \longmapsto (X, \cdot, \lambda);$$

then one can check that $\mathfrak{A} \circ \mathfrak{B} = id_{\textbf{SNApLimGrp}}$, i.e., $\lambda_{\Upsilon_\lambda} = \lambda$. In fact, for any $\mathbb{F} \in \mathbb{F}(X)$, and $x \in X$, we have
$\lambda_{\Upsilon_\lambda}(\mathbb{F})(x) = \Upsilon_\lambda(\mathbb{F} \wedge [x]) = \lambda\left((\mathbb{F} \wedge [x])^{-1} \odot (\mathbb{F} \wedge [x])\right)(e) \vee \lambda\left((\mathbb{F} \wedge [x]) \odot (\mathbb{F} \wedge [x])^{-1}\right)(e)$
$= \lambda\left(([x]^{-1} \wedge \mathbb{F}^{-1}) \odot (\mathbb{F} \wedge [x])\right)(e) \vee \lambda\left((\mathbb{F} \wedge [x]) \odot (\mathbb{F}^{-1} \wedge [x]^{-1})\right)(e)$
$\geq \lambda\left(([x]^{-1} \wedge \mathbb{F}^{-1}) \odot (\mathbb{F} \wedge [x])\right)(e) \geq \lambda(\mathbb{F} \odot [x]^{-1}) = \lambda(\mathbb{F})(x)$, this is so, because of the fact that $[x]^{-1} \wedge \mathbb{F}^{-1}) \odot (\mathbb{F} \wedge [x]) \leq [x]^{-1} \odot \mathbb{F}$, the applying λ to get $\lambda_{\Upsilon_\lambda} \geq \lambda$. Conversely, for any $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$, we have
$\lambda_{\Upsilon_\lambda}(\mathbb{F})(x) = \Upsilon_\lambda(\mathbb{F} \wedge [x]) = \lambda\left((\mathbb{F} \wedge [x])^{-1} \odot (\mathbb{F} \wedge [x])\right) \vee \lambda\left((\mathbb{F} \wedge [x]) \odot (\mathbb{F} \wedge [x])^{-1}\right)$
$= \lambda\left((\mathbb{F} \wedge [x])^{-1} \odot (\mathbb{F} \wedge [x])\right) \vee \lambda\left((\mathbb{F} \wedge [x]) \odot (\mathbb{F}^{-1} \wedge [x]^{-1})\right)$.
Since in one hand, $(\mathbb{F} \wedge [x]) \odot (\mathbb{F}^{-1} \wedge [x]^{-1}) = (\mathbb{F} \odot \mathbb{F}^{-1}) \wedge (\mathbb{F} \odot [x]^{-1}) \wedge ([x] \odot \mathbb{F}^{-1}) \wedge [x] \odot [x]^{-1}$ and the other $(\mathbb{F} \wedge [x])^{-1} \odot (\mathbb{F} \wedge [x]) = (\mathbb{F}^{-1} \odot \mathbb{F}) \wedge (\mathbb{F}^{-1} \odot [x]) \wedge ([x]^{-1} \odot \mathbb{F}) \wedge ([x]^{-1} \odot [x])$, one obtains
$= \lambda\left((\mathbb{F} \wedge [x])^{-1} \odot (\mathbb{F} \wedge [x])\right) \vee \lambda\left((\mathbb{F} \wedge [x]) \odot (\mathbb{F}^{-1} \wedge [x]^{-1})\right)$
$\leq \lambda\left((\mathbb{F}^{-1} \odot \mathbb{F})(e = x^{-1}x) \vee \lambda(\mathbb{F}^{-1} \odot [x])(e) \wedge \lambda(\mathbb{F} \odot [x]^{-1})(x) \vee \lambda([e])(e)\right)$

$\vee \big(\lambda(\mathbb{F}\odot\mathbb{F}^{-1})(e=xx^{-1}) \vee \lambda(\mathbb{F}\odot[x]^{-1})(e) \vee \lambda(([x]\wedge\mathbb{F}^{-1})(e) \vee \lambda[e](e)\big) \leq \lambda(\mathbb{F})(x)$.
This is so, because of the fact that $\lambda(\mathbb{F}^{-1}\odot\mathbb{F})(x^{-1}x) \leq \lambda(\mathbb{F}^{-1})(x^{-1}) \vee \lambda(\mathbb{F}(x) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{F})(x) = \lambda(\mathbb{F})(x)$, and continuing in this way, we can do other parts including applying homogeneity. Thus, we can prove that $\lambda_{\Upsilon_\lambda} \leq \lambda$, and hence $\lambda_{\Upsilon_\lambda} = \lambda$. For the other direction, we look at the scheme below:

$$\mathbf{uApChyGrp} \xrightarrow{\mathfrak{B}} \mathbf{SNApLimGrp} \xrightarrow{\mathfrak{A}} \mathbf{uApChyGrp}:$$
$$(X,\cdot,\Upsilon) \longmapsto (X,\cdot,\lambda_\Upsilon) \longmapsto (X,\cdot,\Upsilon);$$

then one can check that $\mathfrak{B}\circ\mathfrak{A} = id_{\mathbf{uApChyGrp}}$, i.e., $\Upsilon_{\lambda_\Upsilon} = \Upsilon$. In fact, for any $\mathbb{F}\in\mathbb{F}(X)$, $\Upsilon_{\lambda_\Upsilon}(\mathbb{F}) = \lambda_\Upsilon(\mathbb{F}^{-1}\odot\mathbb{F})(e) \vee \lambda_\Upsilon(\mathbb{F}\odot\mathbb{F}^{-1})(e) = \Upsilon(\mathbb{F})$. Hence the result follows.

Corollary 2. *If the underlying group is Abelian, then **uApChyGrp** is isomorphic to **uApLimGrp**.*

7. Conclusion

In this paper, from categorical perspective, we considered two isomorphisms, one between the categories **uApLimGrp** (the category of all ultra-approach limit groups) and **NApGrp** (the category of neighborhood approach groups associated with approach spaces); another, between the categories **uApChyGrp** (the category of ultra approach-Cauchy groups) and **SNApLimGrp** (the category of strongly normal approach limit groups) besides discussing some characterizations of (ultra)-approach groups. Since ultra approach structures originated from the idea of non-archimedean structure or ultra metrics, we are interested to associate all of these structures in relation to ultra approach metric groups, and the ∞p-metrizability of ultra approach groups. These questions are yet to be established. We hope to settle these issues in one of our forthcoming papers.
It would be interesting to add some applications like those in [10–12] that are based on rough sets and their generalizations. Since as this stage we do not see any direct relationship of our work with rough sets, one of the reasons could be that *rough sets* are generalization of Zadeh's concept of fuzzy sets whereas our findings are based on non-fuzzy, rather it is based on non-Archimedean metric spaces; however, we will look into this extraordinary situations in our future research. However, for better understand the soft sets and their applications in medical science, we refer to some of the papers which are definitely interesting in their own right such as [10–12]. However, approach spaces have significant applications within mathematics, such as, functional analysis, and much beyond which can be find out in [16], where one can also find plenty of examples on approach spaces, and their connected branches. For further examples on approach groups, we refer to [4].

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