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A Numerical Method for Investigating Fractional Volterra-Fredholm Integro-Differential Model

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Abstract. In this article, we investigate the fractional Volterra-Fredholm integro-differential equations. These equations appear in several applications such as control theory, biology, and particle dynamics in physics. We derive a numerical method based on the operational matrix method to solve this class of integro-differential equations. We prove the existence and uniqueness of the exact solution. Additionally, we demonstrate the uniform convergence of the numerical solutions to the exact solution. We present several numerical examples to show the numerical efficiency of the proposed method. In the first example, we choose a linear problem and find that the approximate solution converges to the exact solution when the number of block pulse functions is very large. In the next two examples, we consider the nonlinear case and compute the L_2-local truncation error since exact solutions are not available. The error was of order 10^{-12}. Furthermore, we sketch the graph of the approximate solutions for different values of the fractional derivative to observe the influence of the fractional derivative on the profile of the solutions. Theoretical and numerical results show that the proposed method is accurate and can be applied to other nonlinear problems in science.

2020 Mathematics Subject Classifications: 74S40, 65D15

Key Words and Phrases: Block pulse function, Fractional derivative, Nonlinear dynamics, Volterra-Fredholm integro-differential

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1. Introduction

Fractional calculus, a branch of mathematical analysis, extends the concepts of differentiation and integration to non-integer orders. The numerous applications it has found in physics, chemistry, biology, and control theory have garnered considerable attention.

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For example, fractional calculus is applied in anomalous diffusion, viscoelasticity, and electromagnetism in physics, as well as control systems and signal processing in engineering. For more details, see [9, 10, 14, 16]. Incorporating fractional derivatives and integrals is one of the valuable tools FC provides for simulating complex systems with memory and long-term interactions. This theory has revolutionized the approach to many previously challenging problems and has found applications in modeling anomalous diffusion, viscoelasticity, fractional control systems, and more.

The Caputo derivative, introduced by Michele Caputo in 1967, stands out for its practicality. The Caputo derivative is more suitable for real-world simulations because it doesn't require the function to be different throughout the domain, unlike the Riemann-Liouville fractional derivative.

Phenomena like fractional control systems, viscoelasticity, and atypical diffusion are all modeled using the Caputo derivative in engineering and research. It is used for mathematical modeling of physical and biological systems such as heat transfer and groundwater flow. The following references are recommended for further reading [6, 11, 12].

A powerful technique for solving fractional differential equations is the operational matrix method. A linear combination of basis functions, usually starting with block pulse functions, is used to approximate the solution. These functions are constructed using operational matrixes, which represent differentiation and integration operations, and the coefficients of the linear combination are determined by resolving algebraic equations. The OMM has been successfully applied to various issues, including fractional delay equations [15], Ricatti equations [5], systems of differential equations [8], and nonlinear differential equations [1]. Researchers investigated alternative basis functions to enhance computational effectiveness and precision [2–4, 7].

In this article, we will study the following class of integro-differential equations:

$$D^{\mu}\Omega(t) = G_1(t,\Omega(t)) + G_2(t) + \int_0^t \Pi_1(t,s)G_3(\Omega(s))ds + \int_0^1 \Pi_2(t,s)G_4(\Omega(s))ds, \quad (1)$$

$$\Omega(0) = \omega_0, \quad (2)$$

where $t \in [0,1]$, $\Pi_1, \Pi_2 \in C([0,1]\times[0,1])$,$G_1 \in C([0,1]\times\Re)$, $G_3, G_4 \in C(\Re)$, $G_2 \in C([0,1])$, and $0 < \mu \leq 1$. The derivative her is in the Caputo sense.

The fractional Volterra-Fredholm integro-differential model (FVFIDM) represents a significant extension of classical Volterra and Fredholm integral equations by incorporating fractional calculus concepts. This model plays a crucial role in various fields such as physics, engineering, biology, and finance, where systems exhibit memory effects, long-range interactions, and complex dynamics. Volterra and Fredholm integrodifferential equations are two types of integral equations with important applications in various fields such as physics, engineering, and applied mathematics. Although they are related, they differ in their formulation and properties. Volterra integrodifferential equations involve an integral that extends over a variable limit of integration, typically from a fixed starting point to the independent variable, while Fredholm integrodifferential equations involve an integral with fixed limits of integration that do not depend on the independent variable.

Applications of FVFIDMs are widespread, ranging from modeling viscoelastic materials and anomalous diffusion processes to analyzing population dynamics in ecology and pricing financial derivatives. Understanding the behavior of FVFIDMs is essential for predicting and controlling complex systems in various scientific and engineering domains. Fore more details, see [6, 11, 12].

The structure of our paper is listed below. Several definitions and lemmas will be introduced in the next section. Section three will develop a version of the OMM to solve the proposed problem. Some theoretical outcomes, such as existence and uniqueness, estimations of errors, and convergence of the solution, will be demonstrated in section 4. Three illustrative examples will be presented in section 5. Numerical validation of the proposed method's convergence to the unique solution of our problems provided by these examples. Finally, we will draw conclusions and provide closing remarks in the final section.

2. Foundational Concepts

This section presents various fundamental concepts and results employed within this paper.

Definition 1. *[6, 11, 12] For $\mu \in (0,1)$ and $t > 0$, the Caputo derivative of $\Omega(t)$ is defined as follows*

$$D^{\mu}\Omega(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-\tau)^{-\mu}\Omega'(\tau)d\tau, \tag{3}$$

and the fractional integral operator is given by

$$I^{\mu}\Omega(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\Omega(\tau)d\tau. \tag{4}$$

The fractional fundamental theorem of calculus is given as

Lemma 1. *[13, 20] For $\mu > 0 \in (0,1)$, and $\Omega(t) \in C[0,1]$, the following equations hold*

$$I^{\mu}D^{\mu}\Omega(t) = \Omega(t) - \Omega(0), \tag{5}$$

$$D^{\mu}I^{\mu}\Omega(t) = \Omega(t). \tag{6}$$

Another important concept in this paper is the block pulse function (BPF) which is defined as follows.

Definition 2. *[15, 18, 19] Let M be a postive integer. $t_r = jr, j \in 0, 1, 2, ..., M-1, r = \frac{1}{M}$. Then, the j-block pulse function is give by*

$$\gamma_j(t) = \begin{cases} 1, & t \in [t_j, t_{j+1}) \\ 0, & [0,1] - [t_j, t_{j+1}) \end{cases}, \quad 0 \le s < M. \tag{7}$$

In the next theorem, we present two properties of the BPFs which are the disjoint and orthogonal properties.

Theorem 1. *[5, 17, 21] Then, for any $i, j \in \{0, 1, \ldots, M-1\}$, we have*

$$\gamma_i(t)\gamma_j(t) = \begin{cases} \gamma_i(t), & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \tag{8}$$

and

$$\int_0^1 \gamma_i(t)\gamma_j(t)dt = \begin{cases} r, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}. \tag{9}$$

For numerical purposes, we present the following important lemma.

Lemma 2. *[5, 17, 21] If $\Omega \in L^2[0,1]$, then*

$$\Omega(t) = \lim_{M\to\infty} \sum_{j=0}^{M-1} \omega_j \gamma_j(t), \tag{10}$$

where

$$\omega_j = \frac{1}{r} \int_{jr}^{(j+1)r} \Omega(t)dt. \tag{11}$$

For the approximation purposes, we choose M large enough. For such m, we can rewrite Ω in the matrix form as

$$\Omega(t) \approx \overline{\Omega}^T \overline{\gamma}(t) \tag{12}$$

where

$$\overline{\Omega} = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_{M-1} \end{pmatrix}, \overline{\gamma}(t) \begin{pmatrix} \gamma_0(t) \\ \gamma_1(t) \\ \vdots \\ \gamma_{M-1}(t) \end{pmatrix}. \tag{13}$$

3. Method of Solution

In this section, we present the method of solution for problem (1)-(2). Let us assume that

$$I_1(t) = \int_0^t \Pi_1(t,s) G_3(\Omega(s))ds \tag{14}$$

and

$$I_2(t) = \int_0^1 \Pi_2(t,s) G_4(\Omega(s))ds. \tag{15}$$

Let us approximate $\Omega(t)$ by

$$\Omega(t) \approx \sum_{i=0}^{M-1} \omega_i \gamma_i(t). \tag{16}$$

Let

$$G_3(\Omega(t)) = \sum_{i=0}^{n_3} a_{3,i}\Omega^i(t), \quad G_4(\Omega(t)) = \sum_{i=0}^{n_4} a_{4,i}\Omega^i(t), \tag{17}$$

$$\Pi_1(t,s) = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} b_{1,i,j}\gamma_i(t)\gamma_j(s), \quad \Pi_2(t,s) = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} b_{2,i,j}\gamma_i(t)\gamma_j(s). \tag{18}$$

Using Equation (8) and by substituting Equation (16) into Equation (17), we get

$$\begin{aligned} G_3(\Omega(t)) &= \sum_{i=0}^{n_3} a_{3,i}\Omega^i(t) \\ &= \sum_{i=0}^{n_3} a_{3,i}\left(\sum_{j=0}^{M-1}\omega_j\gamma_j(t)\right)^i \\ &= \sum_{i=0}^{n_3} a_{3,i}\left(\sum_{j=0}^{M-1}\omega_j^i\gamma_j(t)\right) \\ &= \sum_{i=0}^{n_3}\sum_{j=0}^{M-1} a_{3,i}\omega_j^i\gamma_j(t) \end{aligned} \tag{19}$$

and

$$\begin{aligned} G_4(\Omega(t)) &= \sum_{i=0}^{n_4} a_{4,i}\Omega^i(t) \\ &= \sum_{i=0}^{n_4} a_{4,i}\left(\sum_{j=0}^{M-1}\omega_j\gamma_j(t)\right)^i \\ &= \sum_{i=0}^{n_4} a_{4,i}\left(\sum_{j=0}^{M-1}\omega_j^i\gamma_j(t)\right) \\ &= \sum_{i=0}^{n_4}\sum_{j=0}^{M-1} a_{4,i}\omega_j^i\gamma_j(t). \end{aligned} \tag{20}$$

Substitute Equation (19) and (18) into Equation (14) and using Equation (8) to get

$$\begin{aligned} I_1(t) &= \int_0^t \Pi_1(t,s)G_3(\Omega(s))ds \\ &= \int_0^t \left(\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} b_{1,i,j}\gamma_i(t)\gamma_j(s)ds\right)\left(\sum_{i=0}^{n_3}\sum_{j=0}^{M-1} a_{3,i}\omega_j^i\gamma_j(s)\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3}\sum_{k=0}^{M-1} b_{1,i,j}a_{3,l}\omega_k^l\gamma_j(t)\int_0^t \gamma_j(s)\gamma_k(s)ds \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l\gamma_j(t)\int_0^t \gamma_j(s)ds. \qquad (21)
\end{aligned}
$$

Using the definition of the BPFs, we can see that

$$
\begin{aligned}
\int_0^t \gamma_j(s)ds &= \begin{cases} \int_0^t 0ds, & t < jr \\ \int_{jr}^t 1ds, & r < t < (j+1)r \\ \int_{jr}^{(j+1)r} 1ds, & t \geq (j+1)r \end{cases} \\
&= \begin{cases} 0, & t < jr \\ (t-jr), & jr < t < (j+1)r \\ 1, & t \geq (j+1)r \end{cases}. \qquad (22)
\end{aligned}
$$

Using Lemma (2), we have

$$
\int_0^t \gamma_j(s)ds = \sum_{k=0}^{M-1} \theta_k \gamma_k(t) \qquad (23)
$$

where

$$
\theta_k = \begin{cases} 0, & k < j \\ \frac{r}{2}, & k = j \\ r, & j < k < M \end{cases}. \qquad (24)
$$

Thus, Equation (21) becomes

$$
\begin{aligned}
I_1(t) &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l\gamma_j(t)\left(\sum_{k=0}^{M-1}\theta_k\gamma_k(t)\right) \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l\theta_j\gamma_j(t) \\
&= P_1(\overline{\Omega})^T\overline{\gamma}(t) \qquad (25)
\end{aligned}
$$

where

$$
(P_1(\overline{\Omega}))_j = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l\theta_j. \qquad (26)
$$

Similarly, Substitute Equation (19) and (20) into Equation (15) and using Equation (8) to get

$$
I_2(t) = \int_0^1 \Pi_2(t,s)G_4(\Omega(s))ds
$$

$$\begin{aligned}
&= \int_0^1 \left(\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} b_{2,i,j}\gamma_i(t)\gamma_j(s)ds\right)\left(\sum_{i=0}^{n_4}\sum_{j=0}^{M-1} a_{4,i}\omega_j^i\gamma_j(s)\right) \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_4}\sum_{k=0}^{M-1} b_{2,i,j}a_{4,l}\omega_k^l\gamma_j(t)\int_0^1 \gamma_j(s)\gamma_k(s)ds \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_4} b_{2,i,j}a_{4,l}\omega_j^l\gamma_j(t)\int_0^1 \gamma_j(s)ds. \qquad (27)
\end{aligned}$$

Using the definition of the BPFs, we can see that

$$\int_0^1 \gamma_j(s)ds = r. \qquad (28)$$

Using Lemma (2), we have

$$\int_0^1 \gamma_j(s)ds = \sum_{k=0}^{M-1} r\gamma_k(t). \qquad (29)$$

Thus, Equation (27) becomes

$$\begin{aligned}
I_2(t) &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l\gamma_j(t)\left(\sum_{k=0}^{M-1} r\gamma_k(t)\right) \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l r\gamma_j(t) \\
&= P_2(\overline{\Omega})^T\overline{\gamma}(t) \qquad (30)
\end{aligned}$$

where

$$(P_2(\overline{\Omega}))_j = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{l=0}^{n_3} b_{1,i,j}a_{3,l}\omega_j^l r. \qquad (31)$$

Let

$$\begin{aligned}
G_1(t,\Omega(t)) &= \sum_{i=0}^{n_1} b_{1,i}(t)\Omega^i(t) \\
&= \sum_{i=0}^{n_1}\left(\sum_{j=0}^{M-1} a_{1,i,j}\gamma_j(t)\right)\left(\sum_{j=0}^{M-1}\omega_j\gamma_j(t)\right)^i \\
&= \sum_{i=0}^{n_1}\left(\sum_{j=0}^{M-1} a_{1,i,j}\gamma_j(t)\right)\left(\sum_{j=0}^{M-1}\omega_j^i\gamma_j(t)\right) \\
&= \sum_{i=0}^{n_1}\sum_{j=0}^{M-1} a_{1,i,j}\omega_j^i\gamma_j(t)
\end{aligned}$$

$$= P_3(\overline{\Omega})^T\overline{\gamma}(t) \quad (32)$$

where

$$(P_3(\overline{\Omega}))_j = \sum_{i=0}^{n_1} a_{1,i,j}\omega_j^i. \quad (33)$$

Finally, we write $G_2(t)$ as

$$\begin{aligned} G_2(t) &= \sum_{i=0}^{M-1} a_{2,i}\gamma_i(t), \\ &= P_4^T\overline{\gamma}(t) \end{aligned} \quad (34)$$

where

$$(P_4)_i = a_{2,i}. \quad (35)$$

Now, we will derive the the Riemann-Liouville fractional integral operator which plays an important role in our derivation.

Theorem 2. *The operational matrix of I^μ is given by*

$$O_\mu = \frac{r^\mu}{\Gamma(\mu+2)}\begin{pmatrix} 1 & \sigma_1 & \sigma_2 & \dots & \sigma_{m-2} & \sigma_{M-1} \\ 0 & 1 & \sigma_1 & \dots & \sigma_{M-3} & \sigma_{M-2} \\ 0 & 0 & 1 & \ddots & \sigma_{M-4} & \sigma_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \sigma_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (36)$$

where $\sigma_\varsigma = (\varsigma+1)^{\mu+1} - 2\varsigma^{\mu+1} + (\varsigma-1)^{\mu+1}, \quad \varsigma = 1, 2, \dots, M-1.$

Proof. Let $l \in \{0, 1, \dots, M-1\}$. Then,

$$\begin{aligned} I^\mu\gamma_l(t) &= \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\gamma_l(s)ds \\ &= \begin{cases} 0, & t < lr \\ \frac{(t-lr)^\mu}{\Gamma(\mu+1)}, & lr \le t < (l+1)r \ . \\ \frac{(t-lr)^\mu - (t-lr-r)^\mu}{\Gamma(\mu+1)}, & (l+1)r \le t < 1 \end{cases} \end{aligned} \quad (37)$$

Let

$$I^\mu\gamma_l(t) = \sum_{i=0}^{M-1} c_{i,l}\gamma_i(t). \quad (38)$$

Then,

$$c_{i,l} = \frac{1}{r}\int_0^1 \left(I^\mu\gamma_l(t)\right)\gamma_i(t)dt$$

$$
\begin{aligned}
&= \frac{1}{r}\int_{ir}^{(i+1)r} \left(I^{\mu}\gamma_l(t)\right) dt \\
&= \begin{cases} \frac{r^{\mu}}{\Gamma(\mu+2)}, & 0 \le i = l \le M-1 \\ \frac{r^{\mu}\left((l-i+1)^{\mu+1}-2(l-i)^{\mu+1}+(l-i-1)^{\mu+1}\right)}{\Gamma(\mu+2)}, & 0 \le i < l \le M-1 \cdot \\ 0, & 0 \le l < i \le M-1 \end{cases}
\end{aligned} \tag{39}
$$

Let $\varsigma = j - i$ and $\sigma_{\varsigma} = (\varsigma+1)^{\mu+1} - 2\varsigma^{\mu+1} + (\varsigma-1)^{\mu+1}$. Then, the operational matrix of I^{μ} is

$$
O_{\mu} = \frac{r^{\mu}}{\Gamma(\mu+2)} \begin{pmatrix} 1 & \sigma_1 & \sigma_2 & \dots & \sigma_{m-2} & \sigma_{M-1} \\ 0 & 1 & \sigma_1 & \dots & \sigma_{M-3} & \sigma_{M-2} \\ 0 & 0 & 1 & \ddots & \sigma_{M-4} & \sigma_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \sigma_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{40}
$$

Now, we can rewrite Equation (1) as

$$
D^{\mu}\Omega(t) = \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T \overline{\gamma}(t). \tag{41}
$$

Using Lemma (1), we have

$$
\begin{aligned}
\Omega(t) &= \Omega(0) + \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T I^{\mu}\overline{\gamma}(t) \\
&= \omega_0 + \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T I^{\mu}\overline{\gamma}(t) \\
&= \left(\omega_0 P_5 + \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T O_{\mu}\right)\overline{\gamma}(t)
\end{aligned} \tag{42}
$$

where

$$
(P_5)_i = 1. \tag{43}
$$

Hence,

$$
\overline{\Omega}^T\overline{\gamma}(t) = \left(\omega_0 P_5 + \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T O_{\mu}\right)\overline{\gamma}(t). \tag{44}
$$

Using the orthogonality property of the BPFs, Equation (44) becomes

$$
\omega_0 P_5 + \left(P_3(\overline{\Omega}) + P_4 + P_1(\overline{\Omega}) + P_2(\overline{\Omega})\right)^T O_{\mu} - \overline{\Omega} = 0. \tag{45}
$$

Then, we solve the algebraic System (45) to find the coefficients $\overline{\Omega}$ of the approximate solution.

We can summarize (OMM) as follows:

Algorithm 1

(i) Find the operational matrices for integral operators.

(ii) Approximate the solution in terms of the Block Pulse Functions (BPF).

(iii) Take the integral of both sides of the proposed problem.

(iv) Substitute the operational matrices to generate an algebraic system.

(v) Solve the algebraic system to obtain the coefficients of the solution.

4. Theoretical Results

In the first theorem, we want to prove that Problem (1)-(2) has a unique solution.

Theorem 3. *Let* $\Pi_1, \Pi_2 : [0,1] \times [0,1] \to \Re$ *and* $G_2 : [0,1] \to \Re$ *be continuous functions which are bounded by* Q_1 *and* Q_2*, respectively,* $G_3, G_4 : \Re \to \Re$ *be continuous Lipschitz functions with Lipschitz constants* L_3 *and* L_4*, respectively, and* $G_1 : [0,1] \times \Re \to \Re$ *be continuous Lipschitz function with respect to the second component with Lipschitz constant* L_1*. Then, the the following problem*

$$\begin{aligned} D^{\mu}\Omega(t) &= G_1(t,\Omega(t)) + G_2(t) + \int_0^t \Pi_1(t,s)G_3(\Omega(s))ds \\ &\quad + \int_0^1 \Pi_2(t,s)G_4(\Omega(s))ds, \qquad (46) \\ \Omega(0) &= \omega_0, \qquad (47) \end{aligned}$$

has a unique solution if

$$\frac{L_1(\mu+1) + (L_3Q_1 + L_4Q_2)}{\Gamma(\mu+2)} < 1. \qquad (48)$$

Proof. Take the fractional integral operator for both sides of Equation (46) to get

$$\begin{aligned} \Omega(t) - \omega_0 &= \frac{1}{\Gamma(\mu)} \int_0^t \left(G_1(z,\Omega(z)) + G_2(z) + \int_0^z \Pi_1(z,s)G_3(\Omega(s))ds \right)(t-z)^{\mu-1}dz \\ &+ \frac{1}{\Gamma(\mu)} \int_0^t \left(\int_0^1 \Pi_2(z,s)G_4(\Omega(s))ds \right)(t-z)^{\mu-1}dz. \qquad (49) \end{aligned}$$

Let

$$\begin{aligned} \wp(\Omega) &= \omega_0 + \frac{1}{\Gamma(\mu)} \int_0^t \left(G_1(z,\Omega(z)) + G_2(z) + \int_0^z \Pi_1(z,s)G_3(\Omega(s))ds \right)(t-z)^{\mu-1}dz \\ &+ \frac{1}{\Gamma(\mu)} \int_0^t \left(\int_0^1 \Pi_2(z,s)G_4(\Omega(s))ds \right)(t-z)^{\mu-1}dz. \qquad (50) \end{aligned}$$

Therefor,

$$|\wp(\Omega_1) - \wp(\Omega_2)| \leq \frac{1}{\Gamma(\mu)} \left| \int_0^t \left(G_1(z,\Omega_1(z)) - G_1(z,\Omega_2(z))\right)(t-z)^{\mu-1}dz \right|$$

$$+ \frac{1}{\Gamma(\mu)} \mid \int_0^t \int_0^z \Pi_1(z,s) \left(G_3(\Omega_1(s)) - G_3(\Omega_2(s))\right) ds (t-z)^{\mu-1} dz \mid$$
$$+ \frac{1}{\Gamma(\mu)} \mid \int_0^t \int_0^1 \Pi_2(z,s) \left(G_4(\Omega_1(s)) - G_4(\Omega_2(s))\right) ds (t-z)^{\mu-1} dz \mid .$$

Since G_1, G_3, G_4 are Lipschitz functions, then

$$\begin{aligned} |\wp(\Omega_1) - \wp(\Omega_2)| &\leq \frac{L_1 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t (t-z)^{\mu-1} dz \mid \\ &+ \frac{L_3 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t \int_0^z \Pi_1(z,s) ds (t-z)^{\mu-1} dz \mid \\ &+ \frac{L_4 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t \int_0^1 \Pi_2(z,s) ds (t-z)^{\mu-1} dz \mid . \end{aligned} \tag{51}$$

Since Π_1 and Π_2 are continuous on a compact set $[0,1] \times [0,1]$, then they are bounded by Q_1 and Q_2, respectively. Thus,

$$\begin{aligned} |\wp(\Omega_1) - \wp(\Omega_2)| &\leq \frac{L_1 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t (t-z)^{\mu-1} dz \mid \\ &+ \frac{L_3 Q_1 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t \int_0^z ds (t-z)^{\mu-1} dz \mid \\ &+ \frac{L_4 Q_2 \parallel \Omega_1 - \Omega_2 \parallel}{\Gamma(\mu)} \mid \int_0^t \int_0^1 ds (t-z)^{\mu-1} dz \mid \\ &\leq \left(\frac{L_1 t^{\mu}}{\Gamma(\mu+1)} + \frac{(L_3 Q_1 + L_4 Q_2) t^{\mu+1}}{\Gamma(\mu+2)} \right) \parallel \Omega_1 - \Omega_2 \parallel \\ &\leq \left(\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2)}{\Gamma(\mu+2)} \right) \parallel \Omega_1 - \Omega_2 \parallel . \end{aligned} \tag{52}$$

Since $\frac{L_1(\mu+1)+(L_3Q_1+L_4Q_2)}{\Gamma(\mu+2)} < 1$, then $\wp$ is contraction on Ω. By Banach fixed point theorem, Problem (46)-(47) has unique solution.

Next, our target is to prove that the sequence $\left\{\sum_{i=0}^{M-1} \omega_i \gamma_i(t)\right\}_{M=1}^{\infty}$ is uniformly convergent to the unique solution of Problem (1)-(2) on $[0,1]$.

Theorem 4. *Let $\Pi_1, \Pi_2 : [0,1] \times [0,1] \to \Re$ and $G_2 : [0,1] \to \Re$ be continuous functions which are bounded by Q_1 and Q_2, respectively, $G_3, G_4 : \Re \to \Re$ be continuous Lipschitz functions with Lipschitz constants L_3 and L_4, respectively, and $G_1 : [0,1] \times \Re \to \Re$ be continuous Lipschitz function with respect to the second component with Lipschitz constant L_1, then the sequence*

$$\left\{ \sum_{i=0}^{M-1} \omega_i \gamma_i(t) \right\}_{M=0}^{\infty} \tag{53}$$

converges to the unique solution of Problem (1)-(2) if

$$\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2)}{\Gamma(\mu+2)} < 1. \tag{54}$$

Proof. Let $\Omega(t)$ be the exact solution of Problem (1)-(2). Using Lemma (2), Ω can be written as

$$\Omega(t) = \sum_{i=0}^{\infty} \overline{\omega_i}\gamma_i(t). \tag{55}$$

For any $0 \leq i, j \leq M-1$, we have

$$\gamma_i(t_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{56}$$

Then,

$$\Omega(t_j) = \sum_{i=0}^{\infty} \overline{\omega_i}\gamma_i(t_j) = \overline{\omega_j}. \tag{57}$$

Using Equation (49) and (57), we have

$$\begin{aligned} \overline{\omega_j} &= \omega_0 + \frac{1}{\Gamma(\mu)} \int_0^{t_j} \left(G_1(z, \Omega(z)) + G_2(z) + \int_0^z \Pi_1(z,s) G_3(\Omega(s)) ds \right) (t-z)^{\mu-1} dz \\ &+ \frac{1}{\Gamma(\mu)} \int_0^{t_j} \left(\int_0^1 \Pi_2(z,s) G_4(\Omega(s)) ds \right) (t-z)^{\mu-1} dz. \end{aligned} \tag{58}$$

Let $\Omega_M(t)$ be the approximate solution of Problem (1)-(2). Then,

$$\Omega_M(t) = \sum_{i=0}^{M-1} \omega_i \gamma_i(t). \tag{59}$$

Using previous argument, we have

$$\begin{aligned} \omega_j &= \omega_0 + \frac{1}{\Gamma(\mu)} \int_0^{t_j} \left(G_1(z, \Omega_M(z)) + G_2(z) + \int_0^z \Pi_1(z,s) G_3(\Omega_M(s)) ds \right) (t-z)^{\mu-1} dz \\ &+ \frac{1}{\Gamma(\mu)} \int_0^{t_j} \left(\int_0^1 \Pi_2(z,s) G_4(\Omega_M(s)) ds \right) (t-z)^{\mu-1} dz. \end{aligned} \tag{60}$$

From Equations (58) and (60), we have

$$\begin{aligned} \mid \overline{\omega_j} - \omega_j \mid &\leq \frac{1}{\Gamma(\mu)} \mid \int_0^{t_j} \left(G_1(z, \Omega(z)) - G_1(z, \Omega_M(z)) \right) (t-z)^{\mu-1} dz \mid \\ &+ \frac{1}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^z \Pi_1(z,s) \left(G_3(\Omega(s)) - G_3(\Omega_M(s)) \right) ds (t-z)^{\mu-1} dz \mid \\ &+ \frac{1}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^1 \Pi_2(z,s) \left(G_4(\Omega(s)) - G_4(\Omega_M(s)) \right) ds (t-z)^{\mu-1} dz \mid . \end{aligned}$$

Thus,

$$\mid \overline{\omega_j} - \omega_j \mid \leq \frac{L_1 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} (t_j - z)^{\mu-1} dz \mid$$

$$
\begin{aligned}
&+ \frac{L_3 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^z \Pi_1(z,s) ds (t_j - z)^{\mu-1} dz \mid \\
&+ \frac{L_4 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^1 \Pi_2(z,s) ds (t_j - z)^{\mu-1} dz \mid . \qquad (61)
\end{aligned}
$$

Since Π_1 and Π_2 are continuous on a compact set $[0,1] \times [0,1]$, then

$$
\begin{aligned}
\mid \overline{\omega_j} - \omega_j \mid &\leq \frac{L_1 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} (t_j - z)^{\mu-1} dz \mid \\
&+ \frac{L_3 Q_1 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^z ds (t_j - z)^{\mu-1} dz \mid \\
&+ \frac{L_4 Q_2 \parallel \Omega - \Omega_M \parallel}{\Gamma(\mu)} \mid \int_0^{t_j} \int_0^1 ds (t_j - z)^{\mu-1} dz \mid \\
&\leq \left(\frac{L_1 {t_j}^{\mu}}{\Gamma(\mu+1)} + \frac{(L_3 Q_1 + L_4 Q_2) {t_j}^{\mu+1}}{\Gamma(\mu+2)} \right) \parallel \Omega - \Omega_M \parallel \\
&\leq \left(\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2)}{\Gamma(\mu+2)} \right) \parallel \Omega - \Omega_M \parallel . \qquad (62)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mid \Omega(t) - \Omega_M(t) \mid &= \parallel \sum_{i=0}^{M-1} (\overline{\omega_i} - \omega_i) \gamma_i(t) + \sum_{j=M}^{\infty} \overline{\omega_j} \gamma_j(t) \parallel \\
&\leq \parallel \sum_{i=0}^{M-1} (\overline{\omega_i} - \omega_i) \gamma_i(t) \parallel + \parallel \sum_{j=M}^{\infty} \overline{\omega_j} \gamma_j(t) \parallel \\
&\leq \sum_{i=0}^{M-1} \left(\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2)}{\Gamma(\mu+2)} \right) \parallel \Omega - \Omega_M \parallel + \parallel \sum_{j=M}^{\infty} \overline{\omega_j} \gamma_j(t) \parallel .
\end{aligned}
$$

Since the series in Equation (55) converges uniformly in $[0,1]$, then for $\epsilon = 1$, there exist positive integer N_1 such that

$$
\parallel \sum_{j=M}^{\infty} \overline{\omega_j} \gamma_j(t) \parallel \leq 1, \quad M \geq N_1, t \in [0,1]. \qquad (63)
$$

Therefore,

$$
\mid \Omega(t) - \Omega_M(t) \mid \leq \left(\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2) M}{\Gamma(\mu+2)} \right) \parallel \Omega - \Omega_M \parallel + 1. \qquad (64)
$$

Take the supreme over $[0,1]$ to get

$$
\parallel \Omega - \Omega_M \parallel \leq \left(\frac{L_1(\mu+1) + (L_3 Q_1 + L_4 Q_2) M}{\Gamma(\mu+2)} \right) \parallel \Omega - \Omega_M \parallel + 1 \qquad (65)
$$

which can simplify as

$$\| \Omega - \Omega_M \| \leq \frac{\Gamma(\mu+2)}{\Gamma(\mu+2) - L_1(\mu+1) + (L_3Q_1 + L_4Q_2)M} \to 0 \tag{66}$$

as M approaches to infinity. Hence, $\{\Omega_M(t)\}_{M=1}^{\infty}$ converges uniformly to the unique solution of Problem (1)-(2) on $[0,1]$.

5. Numerical Results

In this section, we will present three examples to demonstrate the effectiveness of the proposed method.

Example 1. *Consider the following class of integro-differential equations:*

$$\begin{aligned} D^\mu \Omega(t) &= \frac{-t^2e^t}{3}\Omega(t) + \frac{t^{1-\mu}}{\Gamma(2-\mu)} - \frac{t^2}{2} + \int_0^t se^t\Omega(s)ds + \int_0^1 t^2\Omega(s)ds, \\ \Omega(0) &= 0, \end{aligned}$$

where $t \in [0,1]$, and $0 < \mu \leq 1$. Then, the exact solution is $Q(t) = t$. We approximate $\Omega(t)$ as follows

$$\Omega(t) = \sum_{i=0}^{M-1} \omega_j \mu_j(t).$$

Then,

$$\begin{aligned} G_1(t,\Omega(t)) &= \frac{-t^2e^t}{3}\Omega(t), \quad G_2(t) = \frac{t^{1-\mu}}{\Gamma(2-\mu)} - \frac{t^2}{2}, \\ G_3(\Omega(s)) &= \Omega(s), \quad G_4(\Omega(s)) = \Omega(s), \\ \Pi_1(t,s) &= se^t, \quad \Pi_2(t,s) = t^2, \quad \omega_0 = 0. \end{aligned}$$

Direct calculations implies that

$$P_1(\overline{\Omega}) = \overline{\Omega}^T A_1, P_2(\overline{\Omega}) = \overline{\Omega}^T A_2, P_3(\overline{\Omega}) = \overline{\Omega}^T A_3,$$

where

$$\begin{aligned} A_1 &= \frac{r}{2}\begin{pmatrix} f(0) & f(1) & \dots & f(M-1) \\ 3f(0) & 3f(1) & \dots & 3f(M-1) \\ \vdots & \vdots & \dots & \vdots \\ (2M+1)f(0) & (2M+1)f(1) & \dots & (2M+1)f(M-1) \end{pmatrix} \\ A_2 &= \frac{r^2}{3}\begin{pmatrix} 1 & 2^3-1^3 & \dots & M^3-(M-1)^3 \\ 1 & 2^3-1^3 & \dots & M^3-(M-1)^3 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 2^3-1^3 & \dots & M^3-(M-1)^3 \end{pmatrix}, A_3 = \frac{1}{r}\begin{pmatrix} g(0) & 0 & \dots & 0 \\ 0 & g(1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & g(M-1) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} P_4 &= \big(h(0)-h(1) \quad h(1)-h(2) \quad \ldots \quad h(M-1)-h(M)\big) \\ f(j) &= e^{r(j+1)} - e^{rj}, \quad g(j) = \frac{-1}{3}(e^{r(j+1)}(2-2(j+1)r+r^2(j+1)^2) - e^{rj}(2-2rj+r^2j^2)), \\ h(j) &= \frac{1}{r}\left(\frac{(rj)^{2-\mu}}{\Gamma(3-\mu)} - \frac{(rj)^3}{3}\right). \end{aligned}$$

Then,

$$\overline{\Omega}^T\left((A_1+A_2+A_3)O_\mu - I_M\right) = P_4 O_\mu$$

which can be written as

$$\left((A_1+A_2+A_3)O_\mu - I_M\right)^T \overline{\Omega} = (P_4O_\mu)^T. \tag{67}$$

One can see that System (67) is linear system. For $M = 500$, We solve the System (67) in terms of r and we found that

$$\omega_j = \frac{2j+1}{2}r, \quad j = 0, 1, \ldots, M-1.$$

Thus, the approximate solution is given as

$$r\sum_{j=0}^{M-1}\frac{2j+1}{2}\mu_j(t) \to x$$

as M approaches to ∞. Thus, our approximate solution converges to the exact solution. This result can be obtained using Mathematical for linear cases. Let us study a nonlinear problem in the next example.

Example 2. *Consider the following class of integro-differential equations:*

$$\begin{aligned} D^\mu\Omega(t) &= \Omega^2(t) + G_2(t) + \int_0^t (s^\mu+1)(t^\mu+1)\Omega^2(s)ds + \int_0^1 (s^\mu+1)t^\mu\Omega^2(s)ds, \\ \Omega(0) &= 1, \end{aligned}$$

where $t \in [0,1]$, $0 < \mu \le 1$, and Then,

$$\begin{aligned} G_2(t) &= -\frac{(5\mu+4)t^{3\mu+2}}{3\mu^2+5\mu+2} - \frac{3(3\mu+2)t^{5\mu+3}}{20\mu^2+27\mu+9} - \frac{(5\mu+4)(\mu(2\mu+3)(6\mu+23)+21)t^\mu}{(\mu+1)(3\mu+2)(4\mu+3)(5\mu+3)} \\ &- \frac{(\mu+2)t^{\mu+1}}{\mu+1} - \frac{(2\mu+t+3)t^{2\mu+1}}{\mu+1} - \frac{(3\mu+4)t^{4\mu+2}}{3\mu+2} - \frac{t^{4\mu+3}}{4\mu+3} - \frac{t^{6\mu+3}}{5\mu+3} \\ &+ \frac{\Gamma(2\mu+2)t^{\mu+1}}{\Gamma(\mu+2)} - t - 1. \end{aligned}$$

First, let's examine the influence of the fractional order derivative on the behavior of the solution. Figures 1 shows the approximate solutions Ω for different values of μ =

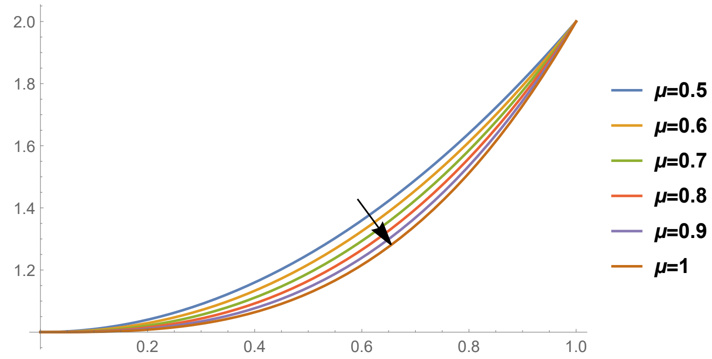

Figure 1: The approximate solution Ω for $\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

$0.5, 0.6, 0.7, 0.8, 0.9, 1$ *with* $r = 0.005$. *Since the exact solution is not given, we assess the accuracy of our approximation by computing the* L_2*-truncation errors, defined as*

$$\epsilon(\mu) = \left(\int_0^1 \left(D^\mu \Omega(t) - \Omega^2(t) - G_2(t) + \int_0^t (s^\mu + 1)(t^\mu + 1)\Omega^2(s)ds - \int_0^1 (s^\mu + 1)t^\mu \Omega^2(s)ds\right) dt\right)^{\frac{1}{2}}.$$

The errors are presented in Table 1.

Table 1: The L_2-error for $\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1.$.

μ	$\epsilon(\mu)$
0.5	1.99×10^{-12}
0.6	1.98×10^{-12}
0.7	1.74×10^{-12}
0.8	1.66×10^{-12}
9	1.23×10^{-12}
1	1.10×10^{-12}

Example 3. *Consider the following class of integro-differential equations:*

$$\begin{aligned} D^\mu \Omega(t) &= \Omega^3(t) + G_2(t) + \int_0^t (s+1)(t+1)\Omega^3(s)ds + \int_0^1 (s^\mu + 1)t^3 \Omega^2(s)ds, \\ \Omega(0) &= 0, \end{aligned}$$

where $t \in [0, 1]$, $0 < \mu \le 1$, *and Then,*

$$\begin{aligned} G_2(t) &= \Gamma(\mu+1) - \frac{\left(9\mu^2 + 9\mu + 3\mu t(t+1)^2 + t(t+2)(t+1) + 2\right) t^{3\mu}}{9\mu(\mu+1) + 2} \\ &- \frac{3\left(16\mu^2 + 20\mu + 4\mu t(t+1)^2 + t(2t+3)(t+1) + 6\right) t^{4\mu+1}}{2(2\mu+1)(4\mu+3)} \\ &- \frac{3\left(25\mu^2 + 35\mu + 5\mu t(t+1)^2 + t(3t+4)(t+1) + 12\right) t^{5\mu+2}}{(5\mu+3)(5\mu+4)} \end{aligned}$$

$$- \frac{\left(36\mu^2+54\mu+6\mu t(t+1)^2+t(4t+5)(t+1)+20\right)t^{6\mu+3}}{2(3\mu+2)(6\mu+5)}$$
$$- \frac{(7\mu+4)(\mu(7\mu(18\mu+31)+120)+21)t^3}{(2\mu+1)(3\mu+1)(3\mu+2)(4\mu+3)(5\mu+3)} + \frac{\Gamma(2\mu+2)t^{\mu+1}}{\Gamma(\mu+2)}.$$

First, let's examine the influence of the fractional order derivative on the behavior of the solution. Figures 2 shows the approximate solutions Ω *for different values of* $\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$ *with* $r = 0.005$*. Since the exact solution is not given, we assess the*

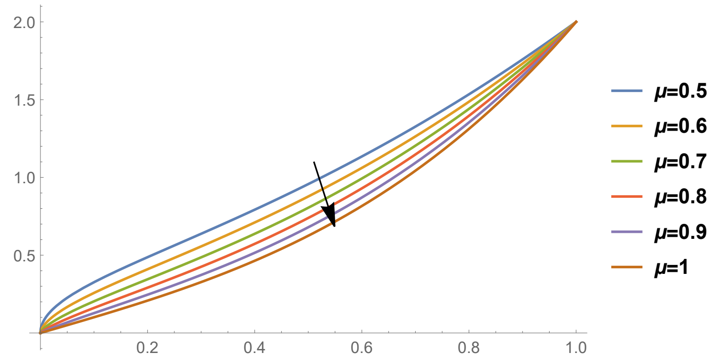

Figure 2: The approximate solution Ω for $\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

accuracy of our approximation by computing the L_2*-truncation errors, defined as*

$$\epsilon(\mu) = \left(\int_0^1 \left(D^\mu\Omega(t) - \Omega^2(t) - G_2(t) + \int_0^t (s^\mu+1)(t^\mu+1)\Omega^2(s)ds - \int_0^1 (s^\mu+1)t^\mu\Omega^2(s)ds\right)dt\right)^{\frac{1}{2}}.$$

The errors are presented in Table 2.

Table 2: The L_2-error for $\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

μ	$\epsilon(\mu)$
0.5	2.34×10^{-12}
0.6	2.21×10^{-12}
0.7	2.01×10^{-12}
0.8	1.98×10^{-12}
9	1.95×10^{-12}
1	1.82×10^{-12}

6. Conclusion

The operational matrix method is a useful approach for solving fractional Volterra-Fredholm integro-differential equations. It involves converting the differential system into a system of algebraic equations to determine the coefficients of the approximating solution.

Typically, these coefficients are computed by establishing operational matrices corresponding to integral, derivative, and product operators. The Block Pulse functions, upon which our approximation basis, possess three key properties—disjointness, orthogonality, and completeness—that facilitate the computations. In this paper, we investigate fractional Volterra-Fredholm integro-differential equations, a fundamental problem in various fields such as control theory, biology, and particle dynamics in physics. We develop a numerical method based on the operational matrix method to solve these equations, proving the existence and uniqueness of the exact solution. Additionally, we demonstrate the uniform convergence of numerical solutions to the exact solution and present several numerical examples illustrating the method's efficiency. In one example, for a linear problem, we observe convergence of the approximate solution to the exact solution as the number of Block Pulse functions increases. In nonlinear cases, where exact solutions are unavailable, we compute the L_2-local truncation error, which is on the order of 10^{-12}. We also examine the influence of the fractional derivative on solution profiles through graph sketches. Theoretical and numerical results affirm the accuracy and applicability of our proposed method to nonlinear problems in science.

Concluding this paper, we highlight the following observations:

(i) Example 1 demonstrates that in linear cases, the approximate solution converges to the exact solution with increasing numbers of Block Pulse functions, yielding solutions in closed form.

(ii) Example 2 illustrates the decreasing influence of the fractional order on solution profiles as the fractional derivative increases. The L_2-truncation error is of order 10^{-12}, as shown in Table 1 and Figure 1.

(iii) Similar results are obtained in Example 3, as presented in Table 2 and Figure 2.

(iv) Our findings suggest that our proposed method is promising and applicable across diverse models in physics and engineering, even in the presence of significant nonlinearity.

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