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Riesz Inequality for Harmonic Quasiregular Mappings

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Abstract. In this paper, we generalize the Riesz theorem for harmonic quasiregular mappings for a special case (when $p = 2$) in the unit disc. Our results improve similar results in this field and are proved with milder conditions. Moreover, we prove another variant forms of Riesz inequality for harmonic quasiregular functions.

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Key Words and Phrases: Harmonic mappings, Quasiregular mappings, Riesz theorem

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1. Introduction

Let $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk and let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ be the unit circle in plane. For $p > 1$ we define the Hardy class h^p as the class of harmonic mappings $f = g + \overline{h}$, where h and g are holomorphic mappings defined on unit disk $\mathbf{U} \subset \mathbf{C}$. Norm in this space is defined

$$||f||_p = ||f||_{h^p} = \sup_{0<r<1} M_p(f, r) < \infty,$$

where

$$M_p(f, r) = \left(\int_{\mathbf{T}} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

Here σ is probability measure on $\mathbf{T}$. With $\mathbf{H}^p$ ($\mathcal{H}^p$), we denote the subclass of holomorphic (quasiregular) mappings that belongs to the class h^p. For the theory of Hardy spaces in the unit disk we refer to [12], [4], [5] and [6].

For a given real-valued function u harmonic in $\mathbf{U}$, let v be its harmonic conjugate, normalized by $v(0) = 0$. Then $f = u + iv$ is analytic in $\mathbf{U}$. The following theorem was proved by M. Riesz.

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Theorem A . *([4, Theorem 4.1]) If $u \in h^p$ for some p, $1 < p < \infty$, then its harmonic conjugate v is also of class h^p. Furthermore, there is a constant A_p, depending only on p, such that*

$$M_p(r,v) \leq A_p M_p(r,u), \quad 0 \leq r < 1,$$

for all $u \in h^p$.

Problem of finding the sharp constant A_p is very old and has been calculated for several classes of functions. Pichorides in [10], prove that this constant is $A_p = \cot \frac{\pi}{2\overline{p}}$, where $\overline{p} = \max\left\{p, \frac{p}{p-1}\right\}$.

Later, Verbitsky improved the above inequality in [13] with the following sharp result: if $f = u + iv \in H^p$ and $v(0) = 0$, then

$$\frac{1}{\cos \frac{\pi}{2\overline{p}}} ||v||_p \leq ||f||_p \leq \frac{1}{\sin \frac{\pi}{2\overline{p}}} ||u||_p.$$

Some other results for this constant are obtained by Kalaj in [9], for harmonic functions with constant

$$A_p = \left(1 - |\cos \frac{\pi}{2p}|\right)^{-\frac{1}{2}}.$$

Also, this inequality was generalized in several directions. Let us mention Beckenbach's results: the same inequality holds where in place of $|f|^p$ we have a positive logarithmically subharmonic function. This kind of generalizations can be found in [1]. We refer interested readers to [2], [3] and [9].

Our aim in this paper is to find similar sharp constant but for specific class of function - quasiregular mapping. In case of planar harmonic K-quasiregular mappings it is defined as below.

Given $K \geq 0$, a sense-preserving harmonic function $f = h + \overline{g}$ in $\mathbf{U}$ is said to be K-quasiregular if and only if

$$||\mu||_\infty = \sup \frac{|g'(z)|}{|h'(z)|} \leq k < 1,$$

where $k = \frac{K-1}{K+1}$.

Another definition of quasiregular mappings in domain of $\mathbf{R}^n$ is given in [14] as follows:

Let $A \subset \mathbf{R}^n$ be a domain, and let $n \geq 2$. A mapping $f : A \to \mathbf{R}^n$ is said to be quasiregular, if satisfy next two conditions:

(a) f is an absolutely continuous functions in every line segment parallel to the coordinate axis and there exists partial derivatives which are locally L^n integrable functions on A.

(b) there exists a constant $K \geq 1$ such that, a.e. in A,

$$L_f(x)^n \leq K J_f(x),$$

where $L_f(x)$ is the maximum stretching for f at the point x, i.e.,

$$L_f(x) = \lim_{y \to x} \sup \frac{|f(y) - f(x)|}{|y - x|},$$

and J_f denotes the Jacobian determinant.

The smallest constant $K \leq 1$ in above definition is called the outer dilatation of f and denote by $K_O(f)$. Also for quasiregular function f, the smallest constant $K \leq 1$, for which the inequality

$$J_f(x) \leq K l_f(x)^n,$$

where $l_f(x) = \min\{|f'(x)h| : |h| = 1\}$, holds (in A), is called the inner dilation of f and denoted by $K_I(f)$. The maximal dilation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, then f is said to be K-quasiregular. If f is not quasiregular, we set $K(f) = K_I(f) = K_O(f) = \infty$.

For a function $f \in \mathbf{U}$, the norm in class $\mathcal{H}^p$ is provided by:

$$||f||_p = ||f||_{\mathcal{H}^p} = \sup_{0<r<1} M_p(r, f).$$

For a more detailed observation of quasiregular mappings we refer to [8], [11] and [14].

Similar results about Riesz theorem in class of quasiregular functions, also are proved by J. Li and J. Zhu, in [7], but there are used additional conditions about functions u and v. They generalize this theorem for planar harmonic K-quasiregular mappings ($1 < p \leq 2$) provided that the real part does not vanish at the unit disk. Moreover, they extended this results for invariant harmonic quasiconformal mappings in the unit ball assuming that the first coordinate in non-vanishing.

In this paper, using different methods, we find similar constant as in [7], but without restriction about conditions of functions u and v. These results are given with next theorem.

Theorem 1. *If $f = u + iv$ is harmonic K-quasiregular, with $\Re f(0) = 0$, then*

$$\|u\|_2 \leq K\|v\|_2,$$

and the constant K is sharp.

The following results easily follows from Theorem 1.

Corollary 1. *Let $f = h + \overline{g}$ be K-quasiregular in $\mathbf{U}$. If $u \in \mathcal{H}^2$, then its harmonic conjugate v is also of class $\mathcal{H}^2$.*

Based on Theorem 1 and Corollary 1, we can prove next theorem.

Theorem 2. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}$ be harmonic K-quasiregular, with $g(0) = 0$. If $f \in \mathcal{H}^2$, then following sharp inequality holds*

$$||f||_2 \leq c_k ||h||_2,$$

where $c_k = 1 + k^2$.

Let we say that this results coincide with results in Theorem 2 on [7], in case of $n = 2$. Next theorem generalize previous theorem.

Theorem 3. *Let* $f(z) = h(z)+\overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}$ *be harmonic* K*-quasiregular, with* $g(0) = 0$ *. If* $f \in \mathcal{H}^n$ *and* $n > 2$*, then following inequality holds*

$$||f||_n \leq c(k,n)||h||_n,$$

where $c(k,n) = (2(1+k^2))^{n/2}$.

2. Proof of main results

Proof. [Proof of Theorem 1] Let

$$f = h + \bar{g} = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} b_j \bar{z}^j.$$

We can assume that both of following integrals converge. If no, then we take the dilatation $f(rz)$ for $r < 1$.

If we integrate in the unit circle

$$\int_{\mathbb{T}} |g'(z)|^2 |dz| \leq k^2 \int_{\mathbb{U}} |h'(z)|^2 |dz|$$

we get

$$\sum_{j=1} j^2 |b_j|^2 \leq k^2 \sum_{j=1} j^2 |a_j|^2.$$

Here $k = \frac{K-1}{K+1}$.

Let $u = \Re(g+h)$ and $v = \Re(i(h-g))$. Then

$$4\frac{1}{\pi}\int_{\mathbb{U}} |u|^2 dxdy = 4(\Re a_0)^2 + \sum_{j=1}^{\infty}(|a_j|^2 + |b_j|^2 + 2\Re(a_k\overline{b_j}))$$

and

$$4\frac{1}{\pi}\int_{\mathbb{U}} |v|^2 dxdy = 4(\Im a_0)^2 + \sum_{j=1}^{\infty}(|a_j|^2 + |b_j|^2 - 2\Re(a_k\overline{b_j})).$$

Without loss of generality we assume that $a_0 = 0$, because $\Re(a_0) = 0$ by assumption. We will find the best constant c_k in the inequality

$$\frac{1}{\pi}\int_{\mathbb{U}} |u|^2 dxdy = \sum_{j=0}^{\infty}(|a_j|^2 + |b_j|^2 + 2|a_j||b_j|) \leq c_k \frac{1}{\pi}\int_{\mathbb{U}} |u|^2 dxdy$$
$$= \sum_{j=0}^{\infty}(|a_j|^2 + |b_j|^2 - 2|a_j||b_j|)$$

under the condition

$$\sum_{j=1} j^2|b_j|^2 \le k^2 \sum_{j=1} j^2|a_j|^2.$$

Let

$$W(a,b) = \frac{\sum_{j=0}^{\infty}(|a_j|^2 + |b_j|^2 + 2|a_j||b_j|)}{\sum_{j=0}^{\infty}(|a_j|^2 + |b_j|^2 - 2|a_j||b_j|)}.$$

We need to find the maximum of expression W under the condition

$$H(a,b) = \sum_{j=1} j^2|b_j|^2 - k^2 \sum_{j=1} j^2|a_j|^2 = 0.$$

It is equivalent of finding the maximum of expression $M = \frac{U}{V}$, where

$$U = \sum_{j=0}^{\infty} |a_j|^2, \quad V = \sum_{j=0}^{\infty} |b_j|^2$$

under the condition $H(a,b) = 0$. The Lagrangian is $L = M - \lambda H$. Assume without loss of generality that $a_j \ge 0$ and $b_j \ge 0$. Also we have $a_0 = 0$. Then $L_{a_j} = 0$ and $L_{b_j} = 0$ imply that

$$\frac{b_j}{U} = \lambda j^2 b_j, \quad \frac{a_j V}{U^2} = k^2 \lambda j^2 a_j, \quad j \ge 1.$$

λ cannot be zero, because in that case $a = 0$ and $b = 0$. If $\lambda \ne 0$, then there exists j_0 so that $a_{j_0} \ne 0$ and $b_{j_0} \ne 0$, $a_j = b_j = 0$ for $j \ne j_0$ and $\frac{1}{U} = \frac{k^2 V}{U^2}$. In this case we get $M = k^2$. This implies that $W \le \frac{1+k^2+2k}{1+k^2-2k} = K^2$.

In order to prove Theorem 2, we will need next result.

Lemma 1. *Let f be analytic function with condition $f(0) = 0$. Then*

$$\int_{\mathbf{U}} \Re(f(z))|dz| = 0.$$

Proof. Since $f(0) = 0$, this function can be expressed as $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Integrating last expression in unit circle, we get

$$\int_{\mathbf{U}} f(z)|dz| = \sum_{k=1}^{\infty} a_k \int_0^1 r^k dr \int_0^{2\pi} e^{ikt} dt.$$

Since the integral $\int_0^{2\pi} e^{ikt} dt = 0$, for each $k \in \mathbb{N}$, we get

$$\int_{\mathbf{U}} f(z)|dz| = 0.$$

Similarly $\int_{\mathbf{U}} \overline{f(z)}|dz| = 0$. Now, identity $\Re(f(z)) = \frac{f(z)+\overline{f(z)}}{2}$, follows that $\int_{\mathbf{U}} \Re(f(z))|dz| = 0$.

Proof. [Proof of Theorem 2] Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}$, then

$$\int_{\mathbf{U}} |f(z)|^2 |dz| = \int_{\mathbf{U}} |h(z)|^2 |dz| + \int_{\mathbf{U}} |g(z)|^2 |dz| + 2 \int_{\mathbf{U}} \Re(h(z)g(z))|dz|.$$

As in the proof of Theorem 1, using Cauchy-Schwartz inequality, Lemma 1, we get

$$\begin{aligned} \int_{\mathbf{U}} |f(z)|^2 |dz| &\leq (1+k^2) \int_{\mathbf{T}} |h(z)|^2 |dz| + 2 \int_{\mathbf{T}} \Re(h(z)g(z))|dz| \\ &= c_k \int_{\mathbf{T}} |h(z)|^2 |dz|. \end{aligned}$$

Which give required results.

Proof. [Proof of Theorem 3] Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}$ be harmonic K-quasiregular, with $g(0) = 0$. We have

$$\int_{\mathbf{U}} |f(z)|^n = \int_{\mathbf{U}} (|g + \overline{h}|^2)^{\frac{n}{2}} = \int_{\mathbf{U}} (|g|^2 + |h|^2 + 2\Re(gh))^{\frac{n}{2}}.$$

Using Theorem 2 and inequality $\Re(gh) \leq |hg| \leq \frac{1}{2}(|h|^2 + |g|^2)$, we get

$$\int_{\mathbf{U}} |f(z)|^n \leq \int_{\mathbf{U}} (2(|h|^2 + |g|^2))^{\frac{n}{2}} \leq (2(1+k^2))^{\frac{n}{2}} \int_{\mathbf{U}} |h|^n.$$

3. Conclusion

In this paper we generalize the Riesz theorem for harmonic quasiregular mappings for a special case in the unit disc. This result is given thought Theorem 1. In order to proving this result we use Lagrange multipliers. Probably this method can be used to prove also for other cases, to make a generalization of Riesz theorem. Moreover, thought Theorem 2 and 3, we prove another variant forms of Riesz inequality for harmonic quasiregular functions.

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