EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1685-1690
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Generalization of Riesz Type Inequalities for Harmonic Mappings on the Unit Disk

Elver Bajrami

Department of Mathematics, University of Prishtina, Mother Teresa, No. 5, 10000, Prishtina, Kosovo

---

Abstract. In this paper a new generalized norm is defined and Riesz type inequalities for harmonic functions on the unit disk are discussed by applying it. Also sharp constants are obtained for certain special values considered in the reverse case of the standard Riesz inequality.

2020 Mathematics Subject Classifications: 30H05, 30H10, 47B06

Key Words and Phrases: Harmonic functions, Riesz inequality, Isoperimetric inequality

---

1. Introduction and statement of main results

Let $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk and let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ be the unit circle in the complex plane. For $p > 1$ we define the Hardy class h^p as the class of harmonic mappings $f = g + \overline{h}$, where h and g are holomorphic mappings defined on unit disk $\mathbf{U} \subset \mathbf{C}$. Norm in this space is defined as:

$$||f||_p = ||f||_{h^p} = \sup_{0<r<1} M_p(f,r) < \infty,$$

where

$$M_p(f,r) = \left(\int_T |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}.$$

Here σ is probability measure on $\mathbf{T}$. With H^p, we denote the subclass of holomorphic mappings that belong to the class h^p. For the theory of Hardy spaces in the unit disk we refer to [11], [7] and [8].

Based on results of Verbitsky [12], Kalaj in [9] proved these inequalities

$$|||f|||_p \le A_p ||f||_p, \quad ||f||_p \le B_p |||f|||_p,$$

DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5282

Email address: elver.bajrami@uni-pr.edu (E. Bajrami)

where $A_p = \left(1 - |\cos\frac{p}{\pi}|\right)^{-1/2}$, $B_p = \sqrt{2}\cos\frac{\pi}{2p}$ and

$$|||f|||_p = \sup_{0<r<1}\left(\int_{\mathbf{T}}(|g(z)|^2 + |h(z)|^2)^{p/2}d\sigma\right)^{1/p}.$$

These results are used to prove and improve some results about isoperimetric inequalities which are given in [2] and generalized in [6] for a special case. Also, this inequality was generalized in several directions. Let us mention Beckenbach's results: the same inequality holds where in place of $|f|^p$ we have a positive logarithmically subharmonic function. This kind of generalizations can be found in [3]. We refer interested readers to [4], [5] and [9].

The following definition for a generalized norm of this kind is next given: for harmonic mapping $f = g + \bar{h} \in h^p$, $(hg)(0) = 0, q > 1$ we define new norm $|||\cdot|||_{p,q} = |||\cdot|||_{h^p}$ as follows

$$|||f|||_{p,q} = \sup_{0<r<1}\left(\int_{T}(|g(rz)|^q + |h(rz)|^q)^{p/q}d\sigma(z)\right)^{1/p}.$$

Or using $\lim_{r\to 1}$ in last expression we have that

$$|||f|||_{p,q} = \left(\int_{T}(|g(z)|^q + |h(z)|^q)^{p/q}d\sigma(z)\right)^{1/p}.$$

We see that the norm defined in [9] is a special case where $q = 2$. In the same paper, this inequality is proved for the special value (q = 2). In this paper our main aim is to find best constant $A_{p,q}$ in the inequality

$$||f||_p \leq A_{p,q}|||f|||_{p,q}. \tag{1.1}$$

A reverse case of this kind of inequalities is generalized from Melentijević in [10] for a special value of p, t and q. There, these inequalities are analyzed using Riesz's projection operator. Let us say that the Riesz projection operator is not bounded on $L^1(\mathbf{T})$.

2. Main results and strategy of proofs

As the authors of paper did in [9] and [1], in order to prove results we use pluri subharmonic function. First, a lemma is proved, which is applied for the proof of the main theorem.

Lemma 1. *Let $p > 2$ and $q > 1$. Then for complex numbers $z = |z|e^{it}$ and $w = |w|e^{is}$ we have*

$$|z + \overline{w}| \leq C_{p,q}(|z|^q + |w|^q)^{p/q} - D_{p,q}|zw|^{\frac{p}{2}}\cos\frac{(\pi - |t+s|)p}{2}.$$

where $C_{p,q}$ and $D_{p,q}$ are defined as below

$$C_{p,q} = 2^{p-\frac{p}{q}}\sin^p\frac{\pi}{2p},$$

and

$$D_{p,q} = 2^{\frac{3p}{2}-\frac{p}{q}}\sin^p\frac{\pi}{2p}\cot\frac{\pi}{2p}.$$

Based on homogeneity of expression (as in proof of Lemma 2 in [1]), we can assume that $|z| = r < 1$ and $w = 1$. So, rather than proving the last lemma, we will present and prove the next lemma.

Lemma 2. *Let $p > 2$ and $q > 1$. Then this sharp inequality hold*

$$(1 + r^2 + 2r\cos t)^{p/2} \leq 2^p \sin^p \frac{\pi}{2p}(\frac{1 + r^q}{2})^{p/q} - 2^{\frac{3p}{2}-\frac{p}{q}} r^{\frac{p}{2}} \sin^p \frac{\pi}{2p} \cot \frac{\pi}{2p} \cos \frac{(\pi - |t + s|)p}{2}$$

for $0 \leq r \leq 1, 0 \leq t \leq \pi$.

Proof. Define

$$\begin{aligned} P(r,t) &= (1 + r^2 + 2r\cos t)^{p/2} - 2^p \sin^p \tfrac{\pi}{2p} \left(\tfrac{1+r^q}{2}\right)^{p/q} \\ &+2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \tfrac{\pi}{2p} \cot \tfrac{\pi}{2p} r^{\frac{p}{2}} \cos \tfrac{(\pi-|t+s|)p}{2} \end{aligned} \tag{2.1}$$

We must prove that $P(r,t) \leq 0$. Calculate partial derivative

$$\frac{2}{p}\frac{\partial P(r,t)}{\partial r} = 2(r + \cos t)(1 + 2r\cos t + r^2)^{\frac{p}{2}-1} - 2^p \sin^p \frac{\pi}{2p} r^{q-1} \left(\frac{1 + r^q}{2}\right)^{\frac{p}{q}-1} + 2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \frac{\pi}{2p} \cot \frac{\pi}{2p} r^{\frac{p}{2}-1} \cos \frac{(\pi - |t|)p}{2}$$

and

$$\frac{\partial P(r,t)}{\partial t} = \frac{p}{2}(1 + 2r\cos t + r^2)^{\frac{p}{2}-1}(-2r\sin t) + 2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \frac{\pi}{2p} r^{\frac{p}{2}} \cot \frac{\pi}{2p} \sin \frac{(\pi - t)p}{2}.$$

Using equality $\frac{\partial P}{\partial t} = 0$, from the last equation we have

$$(1 + 2r\cos t + r^2)^{\frac{p}{2}-1} = \frac{2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \frac{\pi}{2p} r^{\frac{p}{2}} \cot \frac{\pi}{2p} \sin \frac{(\pi-t)p}{2}}{pr\sin t} \tag{2.2}$$

and substitute in equation $\frac{2}{p}\frac{\partial P}{\partial r} = 0$ to get

$$2(r + \cos t)\frac{2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \frac{\pi}{2p} r^{\frac{p}{2}} \cot \frac{\pi}{2p} \sin \frac{(\pi-t)p}{2}}{pr\sin t} - 2^p \sin^p \frac{\pi}{2p} r^{q-1} \left(\frac{1 + r^q}{2}\right)^{\frac{p}{q}-1}$$

$$+2^{\frac{3p}{2}-\frac{p}{q}} \sin^p \frac{\pi}{2p} \cot \frac{\pi}{2p} r^{\frac{p}{2}-1} \cos \frac{(\pi - t)p}{2} = 0$$

or

$$\left(\frac{1+r^q}{2}\right)^{\frac{p}{q}-1} = 2^{\frac{p}{2}-\frac{p}{q}} \cot\frac{\pi}{2p} r^{\frac{p}{2}-q}\left((r+\cos t)\frac{\sin\frac{(\pi-t)p}{2}}{p\sin t} - \cos\frac{(\pi-t)p}{2}\right). \tag{2.3}$$

Now, if we substitute expressions from (2.2) and (2.3) in (2.1) and avoid the factor $2^{\frac{3p}{2}-\frac{p}{q}}\sin^p\frac{\pi}{2p} > 0$, we get

$$\begin{aligned}P(r,t) = {} & \frac{r^{\frac{p}{2}}\cot\frac{\pi}{2p}}{pr\sin t} \\ & \Bigg((1+r^2+2r\cos t)\sin\frac{(\pi-t)p}{2} \\ & -\frac{1+r^q}{2r^{q-1}}(r\sin\frac{(\pi-t)p}{2}+\sin(t-\frac{tp}{2})) + r\cos\frac{(\pi-t)p}{2}\sin t\Bigg).\end{aligned}$$

After some transformation we obtain

$$P(r,t) = \frac{r^{\frac{p}{2}}\cot\frac{\pi}{2p}}{pr\sin t}\left(\sin(t-\frac{tp}{2})\left(2r+\frac{1+r^q}{2r^{q-1}}\right)+\sin\frac{tp}{2}\left(1+r^2-\frac{1+r^q}{2r^q}\right)\right).$$

Finally, for such r, t and p, we have

$$\left(\frac{3r^q+1}{2r^{q-1}}\right)\sin(t-\frac{tp}{2})+\left(\frac{r^q(1+2r^2)-1}{2r^q}\right)\sin\frac{tp}{2}\le 0$$

and

$$\frac{r^{\frac{p}{2}}\cot\frac{\pi}{2p}}{pr\sin t}\ge 0.$$

which give $P(r,t)\le 0$ and prove the lemma.

At finish, we analyze inequality $P(r,t)\le 0$ at the boundary points.

For $r=0$ our inequality is transformed as

$$P(0,t) = 1-2^p\sin^p\frac{\pi}{2p}\left(\frac{1}{2}\right)^{\frac{p}{q}} \le 1-2^{\frac{p}{2}}\sin^p\frac{\pi}{2p}\le 0$$

and hold for this case.

For $r=1$ we have

$$P(1,t) = 2^p\sin^p\frac{\pi}{2p}\left(\frac{\cos^p\frac{t}{2}}{\sin^p\frac{\pi}{2p}} - 1 + 2^{\frac{p}{2}-\frac{p}{q}}\cot\frac{\pi}{2p}\cos\frac{p(\pi-t)}{2}\right)\le 0$$

With substitute of variable $t=\pi-2y$ and using derivative on variable y, this inequality also hold.

For $t = 0$ our inequality has a form

$$P(r,0) = (1-r)^p - 2^p \sin^p \frac{\pi}{2p}\left(\frac{1+r^2}{2}\right)^{\frac{p}{q}}.$$

As in the case of $r = 0$, we can transform and prove inequality

$$P(r,0) = (1+r^2)\left(1 - 2^{\frac{p}{2}} \sin^p \frac{\pi}{2}\right) \le 0.$$

For $t = \pi$ we see that

$$P(r,\pi) = 2^p r^{\frac{p}{2}} \sin^p \frac{\pi}{2p}\left(\frac{\left(\frac{1+r^2}{r} - 2\right)^{\frac{p}{2}}}{\sin^p \frac{\pi}{2p}} + \cot\frac{\pi}{2p} - \left(\frac{1+r^2}{2r}\right)^{\frac{p}{2}}\right).$$

As in the [9], using transformation $a = \frac{1+r^2}{2r} \ge 0$ we get $R(a) = \frac{(2a-2)^{\frac{p}{2}}}{\sin^p \frac{\pi}{2}} + \cot\frac{\pi}{2p} - a^{\frac{p}{2}}$. It is easy to see that $R(a)$ is increasing, and finally $P(r,\pi) \le 0$.

The main outcome of this paper is given in the next theorem.

Theorem 1. *Let $1 < p, q < \infty$ and assume that $f = g + \bar{h} \in h^p$ is a harmonic mapping on the unit disk with $\Re(g(0)h(0)) \le 0$. Then we have the following sharp inequality*

$$\|f\|_{h^p} \le 2^{1-\frac{1}{q}} \max\{\sin\frac{\pi}{2p}, \cos\frac{\pi}{2p}\}\left(\int_T (|g|^q + |h|^q)^{p/q}\right)^{1/p}.$$

Proof. Applying Lemma 2.1, Lemma 2.2 and integrating over $\mathbf{T}$, $0 < r < 1$ and letting $r \to 1^-$ we get

$$\int_{\mathbf{T}} |g(z) + \overline{h(z)}|^p \le 2^{p-\frac{p}{q}} \sin^p \frac{\pi}{2p} \int_{\mathbf{T}} (|z|^q + |w|^q)^{p/q} - D_{p,q} T(z),$$

in case of $p > 2$. For $1 < p < 2$

$$\int_{\mathbf{T}} |g(z) + \overline{h(z)}|^p \le 2^{p-\frac{p}{q}} \cos^p \frac{\pi}{2p} \int_{\mathbf{T}} (|z|^q + |w|^q)^{p/q} - D_{p,q} T(z).$$

Since $\int_{\mathbf{T}} T(z) \ge 0$ (because of subharmonicity). We conclude that

$$\|f\|_{h^p} \le 2^{1-\frac{1}{q}} \max\{\sin\frac{\pi}{2p}, \cos\frac{\pi}{2p}\}\left(\int_T (|g|^q + |h|^q)^{p/q}\right)^{1/p}.$$

Finally, we see that constant in the norm (1.1) has a form

$$A_{p,q} = 2^{1-\frac{1}{q}} \max\{\sin\frac{\pi}{2p}, \cos\frac{\pi}{2p}\}$$

which coincide with result in [9], for a special case when $q = 2$.

Acknowledgements

I would like to express my gratitude to the anonymous referees for their helpful comments that have improved the quality of the paper.

References

[1] I. E. Verbitsky B. Hollenbeck. Best constants for the Riesz projection. *J. Funct. Anal.*, 175(2):370–392, 2000.

[2] E. Bajrami. Improvement of isoperimetric type inequality for harmonic functions in the case $p = 4$. *Indagationes Mathematicae*, 28(2):383–389, 2017.

[3] E. F. Beckenbach. On a theorem of Fejér and Riesz. *J. London Math. Soc.*, 13:82–86, 1938.

[4] M. Stein C. Efferman. H^p spaces of several variables. *Acta Math.*, 129:137–193, 1972.

[5] T. Tao S. Wainger C. Efferman, A. Lonescu. Analysis and its applications: the mathematical work of elias stein. *Bull. Amer. Math. Soc. New Ser.*, 57:523–594, 2020.

[6] E.Bajrami D. Kalaj. On some Riesz and Carleman Type Inequalities for Harmonic Functions in the Unit Disk. *Comput. Methods Funct. Theory*, 18:295–305, 2018.

[7] P. L. Duren. Theory of H^p spaces. *Pure and Applied Mathematics*, 38:xii+25, 1970.

[8] J. B. Garnett. *Bounded Analytic Functions.* Springer, New York, 2007.

[9] D. Kalaj. On Riesz type inequalities for harmonic mappings in the unit disk. *Trans. Amer. Math. Soc.*, 372(6):4031–4051, 2019.

[10] P. Melentijević. Hollenbeck-Verbitsky Conjecture on best constant inequalities for analytic and co-analytic projections. *arxiv:2203.14364v1.*

[11] W. Ramey S. Axler, P. Bourdon. *Harmonic function theory.* Springer Verlag, New York, 1992.

[12] I. E. Verbitsky. Estimate of the norm of a function in hardy space in terms of a norms of its real and imaginary parts. *Linear operators. Mat. Issled.*, 54:16–20, 164–165, 1980.