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Some Identities on λ-Analogues of Lah Numbers and Lah-Bell Polynomials

Dae San Kim[1], Taekyun Kim[2,∗], Hyekyung Kim[3], Jongkyum Kwon[4,∗]

[1] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[2] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Mathematics Education, Daegu Catholic University, Gyeongsan 38430, Republic of Korea*
[4] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Republic of Korea*

Abstract. In recent years, some applications of Lah numbers were discovered in the real world problem of telecommunications and optics. The aim of this paper is to study the λ-analogues of Lah numbers and Lah-Bell polynomials which are λ-analogues of the Lah numbers and and Lah-Bell polynomials. Here we note that λ-analogues appear when we replace the falling factorials by the generalized falling factorials in the defining equations. By using generating function method, we study some properties, explicit expressions, generating functions and Dobinski-like formulas for those numbers and polynomials. We also treat the more general λ-analogues of r-Lah numbers and r-extended λ-Lah-Bell polynomials. In addition, we show that the expectations of two random variables, both associated with the Poisson random variable with parameter $\frac{\alpha}{\lambda}$, are equal to the λ-analogue of the Lah-Bell polynomial evaluated at α for one and the r-extended λ-Lah-Bell polynomial evaluated at α for the other.

2020 Mathematics Subject Classifications: 11B73, 11B83

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1. Introduction

The unsigned Lah number $L(n, k)$ counts the number of ways that a set of n elements can be partitioned into k non-empty linearly ordered susbsets, while the Lah-Bell number B_n^L is the number of ways that a set of n elements can be partitioned into nonempty linearly ordered subsets. In recent years, some practical applications of the Lah numbers were found in telecommunications and optics. Indeed, Lah numbers have been used in

∗Corresponding author.
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Email addresses: `dskim@sogang.ac.k` (D. S. Kim), `tkkim@kw.ac.kr` (T. Kim), `hkkim@cu.ac.kr` (H. Kim), `mathkjk26@gnu.ac.kr` (J. Kwon)

steganography for hiding data in images (see [6]). It requires lower complexity of calculation, compared to alternatives DFT (discrete Fourier transform) and DWT (discrete wavelet transform). In addition, the Lah transform naturally arises in the perturbative description of the chromatic dispersion in optics (see [22,23]) and can tremendously speeds up optimization problems.

The study of degenerate versions of many special numbers and polynomials has been done in recent years by some mathematicians for their regained interests (see [2,9,15,16,21] and the references therein). A lot of fascinating results have been discovered. For example, the degenerate Stirling numbers of the first kind and the second kind, which are degenerate versions of the ordinary Stirling numbers of the first kind and the second kind respectively, occur very frequently when one studies degenerate versions of many special numbers and polynomials (see [2,9,15,16,21]). The λ- umbral calculus, which is more convenient when dealing with degenerate versions of Sheffer polynomials, has been uncovered as a natural degenerate version of the usual umbral calculus (see [13]). In addition, the degenerate gamma functions were found as a degenerate version of the ordinary gamma functions (see [14]).

The aim of this paper is to study the λ-analogues of Lah numbers $L_{\lambda}(n,k)$ (see (14)) and Lah-Bell polynomials $B^{L}_{n,\lambda}(x)$ (see (21), (23)) which are λ- analogues of the Lah numbers and and Lah-Bell polynomials. We investigate some properties, explicit expressions, Dobinski-like formulas and generating functions for those numbers and polynomials. We also treat the more general λ-analogues of r-Lah numbers $L_{r,\lambda}(n,k)$ (see (39)) and r-extended λ-Lah-Bell polynomials $LB^{(r)}_{n,\lambda}(x)$ (see (45)). In addition, we show the expectation of one random variable and that of another random variable, both related to the Poisson random variable with parameter $\frac{\alpha}{\lambda}$, are respectively equal to $B^{L}_{n,\lambda}(\alpha)$ and $LB^{(r)}_{n,\lambda}(\alpha)$. Here we note that the degenerate versions arise naturally when we replace the powers of x by the generalized falling factorial polynomials $(x)_{k,\lambda}$ (see (1)) in the defining equations, while the λ-analogues appear when we replace the falling factorials $(x)_k$ by the generalized falling factorials.

In more detail, the outline of this paper is as follows. In Section 1, we recall the generalized falling factorials $(x)_{n,\lambda}$ and the generalized rising factorials $\langle x\rangle_{n,\lambda}$. We remind the reader of the unsigned Lah numbers $L(n,k)$ and Lah-Bell numbers B^{L}_{n}. We recall the λ-analogues of the Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ and the second kind $\left\{ {n \atop k} \right\}_{\lambda}$. We remind the reader of the unsigned λ-Stirling numbers of the first kind $\left[{n \atop k} \right]_{\lambda} = (-1)^{n-k}S_{1,\lambda}(n,k)$, the λ-analogues of r-Stirling numbers of the second $\left\{ {n+r \atop k+r} \right\}_{r,\lambda}$, and the λ-Bell polynomials $\phi_{n,\lambda}(x)$. Section 2 is the main result of this paper. We define the λ-analogues of Lah numbers $L_{\lambda}(n,k)$, and express it as finite sums of products of $\left[{n \atop k} \right]_{\lambda}$ and $\left\{ {n \atop k} \right\}_{\lambda}$ in Theorem 2.1. We define the λ-analogues of Lah-Bell polynomials $B^{L}_{n,\lambda}(x)$ and numbers $B^{L}_{n,\lambda}$, and find explicit formulas for $L_{\lambda}(n,k)$ in Theorem 2.2 and Dobinski-like formulas for $B^{L}_{n,\lambda}(x)$ in Theorem 2.3. In Theorem 2.4, $\phi_{n,\lambda}(x)$ is expressed as a finite sum involving $B^{L}_{n,\lambda}(x)$ and $\left\{ {n \atop k} \right\}_{\lambda}$. As an inversion of this, we also express $B^{L}_{n,\lambda}(x)$ as a finite sum involving $\phi_{n,\lambda}(x)$ and $\left[{n \atop k} \right]_{\lambda}$. We define the λ-analogues of Laguerre polynomials $L^{(\alpha)}_{n,\lambda}(x)$ of order α. Then we show that the convolution of $B^L_{n,\lambda}(x)$ and $L^{(\alpha)}_{n,\lambda}(x)$ is equal to $\langle \alpha+1\rangle_{n,\lambda}$. We define the λ-analogues of r-Lah numbers $L_{r,\lambda}(n,k)$, and find for those numbers an explicit expression in Theorem 2.6 and a recurrence relation in Theorem 2.7. We define the r-extended λ-Lah-Bell polynomials $LB^{(r)}_{n,\lambda}(x)$ and find a Dobinski-like formula for those polynomials in Theorem 2.8. In Theorem 2.9, we express $\left\{ {n+r \atop k+r} \right\}_{r,\lambda}$ as a finite sum of the product of $L_{r,\lambda}(n,k)$ and $\left\{ {n \atop m} \right\}_{-\lambda}$, and conversely $L_{r,\lambda}(n,k)$ as a finite sum of the product of $\left\{ {m+r \atop k+r} \right\}_{r,\lambda}$ and $\left[{n \atop m} \right]_{\lambda}$. Let X be the Poisson distribution with parameter $\frac{\alpha}{\lambda} > 0$. Then, in Section 3, we show that the expectation of the random variable $\langle X\lambda\rangle_{n,\lambda}$ is equal to $B^L_{n,\lambda}(\alpha)$ and that of the random variable $\langle X\lambda + r\rangle_{n,\lambda}$ is equal to $LB^{(r)}_{n,\lambda}(\alpha)$. In the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the generalized falling factorial sequence is given by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1), \quad (\text{see } [1-5, 7-26]). \tag{1}$$

Note that

$$\lim_{\lambda\to 1}(x)_{n,\lambda} = x(x-1)(x-2)\cdots\big(x-(n-1)\big) = (x)_n, \quad (n \geq 1).$$

The generalized rising factorial sequence is defined by

$$\langle x\rangle_{0,\lambda} = 1, \quad \langle x\rangle_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1).$$

Note that

$$\lim_{\lambda\to 1}\langle x\rangle_{n,\lambda} = x(x+1)(x+2)\cdots\big(x+(n-1)\big) = \langle x\rangle_n, \quad (n \geq 1).$$

For $n \geq 0$, the unsigned Lah numbers are defined by

$$\langle x\rangle_n = \sum_{k=0}^{n} L(n,k)(x)_k, \quad (\text{see}[5, 12, 13, 26]). \tag{2}$$

Note that $L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}$, $(n \geq k \geq 1)$.

The Lah-Bell number B^L_n is defined by

$$B^L_n = \sum_{k=0}^{n} L(n,k), \quad (n \geq 0), \quad (\text{see } [12, 13]). \tag{3}$$

The λ-analogues of the Stirling numbers of the first kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k, \quad (n \geq 0). \tag{4}$$

From (4), we get

$$\frac{1}{\lambda^k}\frac{1}{k!}\Big(\log(1+\lambda t)\Big)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\quad (k\geq 0). \tag{5}$$

Note that $\lim_{\lambda\to 1}S_{1,\lambda}(n,k)=S_1(n,k)$ are the ordinary Stirling numbers of the first kind given by

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k,\quad (\text{see } [1-5,7-13,15-21,24-26]). \tag{6}$$

The unsinged λ-Stirling numbers of the first kind are given by

$$(-1)^{n-k}S_{1,\lambda}(n,k)=\begin{bmatrix}n\\k\end{bmatrix}_{\lambda},\quad (n,k\geq 0),$$

and hence we see from (4) and (5) that

$$\langle x\rangle_{n,\lambda}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\lambda}x^k,\quad (n\geq 0),\quad \frac{1}{\lambda^k}\frac{1}{k!}\Big(-\log(1-\lambda t)\Big)^k=\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{\lambda}\frac{t^n}{n!},\quad (k\geq 0). \tag{7}$$

The λ-analogues of Stirling numbers of the second kind are defined by

$$x^n=\sum_{k=0}^{n}\begin{Bmatrix}n\\k\end{Bmatrix}_{\lambda}(x)_{k,\lambda},\quad (n\geq 0),\quad (\text{see } [10]). \tag{8}$$

From (8), we have

$$\frac{1}{\lambda^k}\frac{1}{k!}\Big(e^{\lambda t}-1\Big)^k=\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}_{\lambda}\frac{t^n}{n!},\quad (\text{see } [10]). \tag{9}$$

Note that $\lim_{\lambda\to 1}\left\{{n \atop k}\right\}_{\lambda}=\left\{{n \atop k}\right\}$ are the ordinary Stirling numbers of the second kind defined by

$$x^n=\sum_{k=0}^{n}\begin{Bmatrix}n\\k\end{Bmatrix}(x)_k,\quad (n\geq 0),\quad (\text{see } [1-5,7-13,15-32]).$$

For $r\in\mathbb{N}\cup\{0\}$, the λ-analogues of r-Stirling numbers of the second kind are given by

$$(x+r)^n=\sum_{k=0}^{n}\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_{r,\lambda}(x)_{k,\lambda},\quad (n\geq 0),\quad (\text{see } [10,11,15]). \tag{10}$$

Thus, by (10), we get

$$\frac{1}{\lambda^k}\frac{1}{k!}\Big(e^{\lambda t}-1\Big)^k e^{rt}=\sum_{n=k}^{\infty}\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_{r,\lambda}\frac{t^n}{n!},\quad (k\geq 0),\quad (\text{see } [10,11]). \tag{11}$$

The λ-Bell polynomials are given by

$$e^{\frac{x}{\lambda}(e^{\lambda t}-1)}=\sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [10])}. \tag{12}$$

Note that

$$\lim_{\lambda\to 1}\phi_{n,\lambda}(x)=\phi_n(x)=\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}x^k$$

are the ordinary Bell polynomials.

Here we note that

$$\phi_{n,\lambda}(x)=\lambda^n e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{k^n}{k!}\Big(\frac{x}{\lambda}\Big)^k,\quad \phi_n(x)=e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k,\quad \phi_{n,\lambda}(x)=\lambda^n\phi_n\Big(\frac{x}{\lambda}\Big). \tag{13}$$

From (9) and (12), we have

$$\phi_{n,\lambda}(x)=\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda}x^k.$$

In particular, for $x=1$, $\phi_{n,\lambda}=\phi_{n,\lambda}(1)$ are called the λ-Bell numbers.

2. Identities on λ-analogues of Lah numbers and Lah-Bell polynomials

In view of (2), we consider the λ-*analogues of Lah numbers* defined by

$$\langle x\rangle_{n,\lambda}=\sum_{k=0}^{n}L_{\lambda}(n,k)(x)_{k,\lambda},\quad (n\ge 0). \tag{14}$$

From (2), we note that $\lim_{\lambda\to 1}L_{\lambda}(n,k)=L(n,k),\quad (n,k\ge 0)$.

By (14), we get

$$\begin{aligned}\sum_{k=0}^{n}L_{\lambda}(n,k)(x)_{k,\lambda}=\langle x\rangle_{n,\lambda}&=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\lambda}x^j=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\lambda}\sum_{k=0}^{j}\begin{Bmatrix} j \\ k \end{Bmatrix}_{\lambda}(x)_{k,\lambda} \\ &=\sum_{k=0}^{n}\bigg(\sum_{j=k}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\lambda}\begin{Bmatrix} j \\ k \end{Bmatrix}_{\lambda}\bigg)(x)_{k,\lambda}.\end{aligned} \tag{15}$$

Theorem 1. *For $n,k\ge 0$, we have*

$$L_{\lambda}(n,k)=\sum_{j=k}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\lambda}\begin{Bmatrix} j \\ k \end{Bmatrix}_{\lambda}.$$

We note from (14) that

$$(x)_{n,\lambda}=(-1)^n\langle -x\rangle_{n,\lambda}=(-1)^n\sum_{k=0}^{n}L_\lambda(n,k)(-x)_{k,\lambda}$$
$$=\sum_{k=0}^{n}(-1)^{n-k}L_\lambda(n,k)\langle x\rangle_{k,\lambda}.$$

For any nonzero $\lambda\in\mathbb{R}$, the λ-exponentials are defined by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{k=0}^{\infty}\frac{(x)_{k,\lambda}}{k!}t^k,\quad (\text{see } [11,15-22]). \tag{16}$$

From (16), we have

$$\begin{aligned}e_\lambda^{-x}(-t)&=\sum_{n=0}^{\infty}(-x)_{n,\lambda}\frac{(-t)^n}{n!}=\sum_{n=0}^{\infty}\langle x\rangle_{n,\lambda}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}L_\lambda(n,k)(x)_{k,\lambda}\right)\frac{t^n}{n!}\\&=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}L_k(n,k)\frac{t^n}{n!}\right)(x)_{k,\lambda}.\end{aligned}\tag{17}$$

On the other hand, by (16), we get

$$\begin{aligned}e_\lambda^{-x}(-t)&=(1-\lambda t)^{-\frac{x}{\lambda}}=\left(\frac{1}{1-\lambda t}\right)^{\frac{x}{\lambda}}=\left(1+\frac{\lambda t}{1-\lambda t}\right)^{\frac{x}{\lambda}}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k(x)_{k,\lambda}.\end{aligned}\tag{18}$$

By (17) and (18), we get

$$\frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k=\frac{1}{\lambda^k}\frac{1}{k!}\left(\frac{1}{1-\lambda t}-1\right)^k=\sum_{n=k}^{\infty}L_\lambda(n,k)\frac{t^n}{n!},\quad (k\geq 0). \tag{19}$$

The left hand side of (19) can be written as

$$\begin{aligned}\frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k&=\frac{1}{\lambda^k}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1-\lambda t)^{-\frac{\lambda l}{\lambda}}\\&=\sum_{n=0}^{\infty}\left(\frac{1}{\lambda^k}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\langle \lambda l\rangle_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{20}$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2. *For $n, k \geq 0$, with $n \geq k$, we have*

$$L_\lambda(n,k) = \frac{1}{\lambda^k}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\langle \lambda l\rangle_{n,\lambda}.$$

Note that

$$L(n,k) = \lim_{\lambda\to 1} L_\lambda(n,k) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\langle l\rangle_n.$$

In view of (3), we define the λ-*analogues of Lah-Bell numbers* as

$$B_{n,\lambda}^L = \sum_{k=0}^{n} L_\lambda(n,k), \quad (n \geq 0). \tag{21}$$

From (19) and (21), we can easily derive the following equation:

$$e^{\frac{t}{1-\lambda t}} = \sum_{n=0}^{\infty} B_{n,\lambda}^L \frac{t^n}{n!}. \tag{22}$$

Now, we define the λ-*analogues of Lah-Bell polynomial*s as

$$B_{n,\lambda}^L(x) = \sum_{k=0}^{n} L_\lambda(n,k)x^k, \quad (n \geq 0). \tag{23}$$

Note that $B_{n,\lambda}^L(1) = B_{n,\lambda}^L$, $(n \geq 0)$. By (22) and (23), we get

$$e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t}-1)} = \sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!}. \tag{24}$$

From (24), we have

$$\begin{aligned} e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t}-1)} &= e^{-\frac{x}{\lambda}} e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t})} = e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{1}{\lambda^k}\left(\frac{1}{1-\lambda t}\right)^k \\ &= e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{x^k}{\lambda^k k!}\sum_{n=0}^{\infty}\langle\lambda k\rangle_{n,\lambda}\frac{t^n}{n!} \\ &= e^{-\frac{x}{\lambda}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{\lambda^k k!}\langle\lambda k\rangle_{n,\lambda}x^k\right)\frac{t^n}{n!}. \end{aligned} \tag{25}$$

Therefore, by comparing the coefficients on both sides of (25), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$B_{n,\lambda}^L(x) = e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\frac{\langle\lambda k\rangle_{n,\lambda}}{k!}x^k.$$

We observe that

$$B_n^L(x) = \lim_{\lambda \to 1} B_{n,\lambda}^L(x) = e^{-x} \sum_{k=0}^{\infty} \frac{\langle k \rangle_n}{k!} x^k,$$

where $B_n^L(x)$ are the ordinary Lah-Bell polynomials given by

$$B_n^L(x) = \sum_{k=0}^{n} L(n,k) x^k, \quad (n \geq 0).$$

Replacing t by $\frac{1}{\lambda}(1 - e^{-\lambda t})$ in (24) and from (9), we get

$$\begin{aligned} e^{\frac{x}{\lambda}(e^{\lambda t}-1)} &= \sum_{k=0}^{\infty} B_{k,\lambda}^L(x) \frac{1}{k!} \frac{1}{\lambda^k} \left(1 - e^{-\lambda t}\right)^k \\ &= \sum_{k=0}^{\infty} B_{k,\lambda}^L(x) \sum_{n=k}^{\infty} {n \brace k}_{\lambda} (-1)^{n-k} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} B_{k,\lambda}^L(x) {n \brace k}_{\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

From (12) and (26), we have

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} (-1)^{n-k} {n \brace k}_{\lambda} B_{k,\lambda}^L(x). \tag{27}$$

Replacing t by $\frac{1}{\lambda} \log\left(\frac{1}{1-\lambda t}\right)$ in (12) and from (7), we get

$$\begin{aligned} e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t}-1)} &= \sum_{k=0}^{\infty} \phi_{k,\lambda}(x) \frac{1}{k!} \left(\frac{1}{\lambda} \log\left(\frac{1}{1-\lambda t} \right) \right)^k \\ &= \sum_{k=0}^{\infty} \phi_{k,\lambda}(x) (-1)^k \frac{1}{k!} \left(\frac{\log(1-\lambda t)}{\lambda} \right)^k \\ &= \sum_{k=0}^{\infty} \phi_{n,\lambda}(x) \sum_{n=k}^{\infty} {n \brack k}_{\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \phi_{k,\lambda}(x) {n \brack k}_{\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{28}$$

Thus, by (24) and (28), we get

$$B_{n,\lambda}^L(x) = \sum_{k=0}^{n} \phi_{k,\lambda}(x) {n \brack k}_{\lambda}, \quad (n \geq 0). \tag{29}$$

Therefore, by (27) and (29), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n}(-1)^{n-k}\begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda} B_{n,\lambda}^{L}(x), \quad \text{and} \quad B_{n,\lambda}^{L}(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{\lambda}\phi_{n,\lambda}(x).$$

It is well known that the Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α, $(\alpha > -1)$, are given by

$$(1-t)^{-\alpha-1}e^{x(\frac{t}{t-1})} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{30}$$

Now, we consider the λ-*analogues of Laguerre polynomials* $L_{n,\lambda}^{(\alpha)}(x)$ of order α, $(\alpha > -1)$, which are given by

$$(1-\lambda t)^{-\frac{\alpha+1}{\lambda}}e^{x(\frac{t}{\lambda t-1})} = \sum_{n=0}^{\infty} L_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{31}$$

Note that

$$\lim_{\lambda\to 1} L_{n,\lambda}^{(\alpha)}(x) = L_n^{(\alpha)}(x), \quad (n \geq 0).$$

From (31), we have

$$\begin{aligned}(1-\lambda t)^{-\frac{\alpha+1}{\lambda}} &= e^{x(\frac{t}{1-\lambda t})}\sum_{k=0}^{\infty} L_{k,\lambda}^{(\alpha)}(x)\frac{t^k}{k!} \\ &= \sum_{m=0}^{\infty} B_{m,\lambda}^{L}(x)\frac{t^m}{m!}\sum_{k=0}^{\infty} L_{k,\lambda}^{(\alpha)}(x)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}^{L}(x)L_{n-m,\lambda}^{(\alpha)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{32}$$

On the other hand, by binomial expansion, we get

$$(1-\lambda t)^{-\frac{\alpha+1}{\lambda}} = \sum_{n=0}^{\infty}\langle\alpha+1\rangle_{n,\lambda}\frac{t^n}{n!}. \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\langle\alpha+1\rangle_{n,\lambda} = \sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}^{L}(x)L_{n-m,\lambda}^{(\alpha)}(x). \tag{34}$$

We observe that

$$\langle\alpha+1\rangle_{n,\lambda} = \sum_{k=0}^{n} L_{\lambda}(n,k)(\alpha+1)_{k,\lambda} = \sum_{k=0}^{n} L_{\lambda}(n,k)\sum_{j=0}^{k}\binom{k}{j}(\alpha)_{j,\lambda}(1)_{k-j,\lambda} \tag{35}$$

$$= \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{k}{j} L_\lambda(n,k)(\alpha)_{j,\lambda}(1)_{k-j,\lambda}.$$

Hence, by (34) and (35), we get

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{k}{m} L_\lambda(n,k)(\alpha)_{m,\lambda}(1)_{k-m,\lambda} = \sum_{m=0}^{n} \binom{n}{m} B^L_{m,\lambda}(x) L^{(\alpha)}_{n-m,\lambda}(x). \tag{36}$$

Now, we consider the *bivariate* λ*-Lah-Bell polynomials* given by

$$\left(1 + y\frac{t}{1-\lambda t}\right)^x = \sum_{n=0}^{\infty} B^L_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{37}$$

Thus, by (37) and (19), we get

$$\begin{aligned} \left(1 + y\frac{t}{1-\lambda t}\right)^x &= \sum_{k=0}^{\infty} \binom{x}{k} y^k \left(\frac{t}{1-\lambda t}\right)^k = \sum_{k=0}^{\infty} (x)_k y^k \frac{1}{k!}\left(\frac{t}{1-\lambda t}\right)^k \\ &= \sum_{k=0}^{\infty} (x)_k y^k \sum_{n=k}^{\infty} L_\lambda(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} L_\lambda(n,k)(x)_k y^k\right)\frac{t^n}{n!}. \end{aligned} \tag{38}$$

By (37) and (38), we get

$$B^L_{n,\lambda}(x,y) = \sum_{k=0}^{n} L_\lambda(n,k)(x)_k y^k, \quad (n \geq 0).$$

Replacing y by $\frac{y}{x}$ and letting $x \to \infty$, we see that $B^L_{n,\lambda}(y) = \lim_{x\to\infty} B^L_{n,\lambda}(x, \frac{y}{x})$. For $r \in \mathbb{N} \cup \{0\}$, we define the λ*-analogues of* r*-Lah numbers* by

$$\langle x + r\rangle_{n,\lambda} = \sum_{k=0}^{n} L_{r,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0). \tag{39}$$

From (39), we note that

$$\begin{aligned} e_\lambda^{-(x+r)}(-t) &= \sum_{n=0}^{\infty} \langle x+r\rangle_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L_{r,\lambda}(n,k)(x)_{k,\lambda}\right)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} L_{r,\lambda}(n,k)\frac{t^n}{n!}\right)(x)_{k,\lambda}. \end{aligned} \tag{40}$$

On the other hand, by binomial expansion, we get

$$e_\lambda^{-(x+r)}(-t) = \left(\frac{1}{1-\lambda t}\right)^{\frac{r}{\lambda}} (1-\lambda t)^{-\frac{x}{\lambda}} = \left(\frac{1}{1-\lambda t}\right)^{\frac{r}{\lambda}} \left(1 + \frac{\lambda t}{1-\lambda t}\right)^{\frac{x}{\lambda}} \tag{41}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t}{1-\lambda t} \right)^k \left(\frac{1}{1-\lambda t} \right)^{\frac{r}{\lambda}} (x)_{k,\lambda}.$$

By (40) and (41), we get

$$\frac{1}{k!} \left(\frac{t}{1-\lambda t} \right)^k \left(\frac{1}{1-\lambda t} \right)^{\frac{r}{\lambda}} = \sum_{n=k}^{\infty} L_{r,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0). \tag{42}$$

The left hand side of (42) can be written as

$$\begin{aligned} \frac{1}{k!} \left(\frac{t}{1-\lambda t} \right)^k \left(\frac{1}{1-\lambda t} \right)^{\frac{r}{\lambda}} &= \frac{1}{\lambda^k} \frac{1}{k!} \left(\frac{1}{1-\lambda t} - 1 \right)^k \left(\frac{1}{1-\lambda t} \right)^{\frac{r}{\lambda}} \\ &= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{1}{\lambda^k} \frac{1}{k!} \left(\frac{1}{1-\lambda t} \right)^{\frac{r+l\lambda}{\lambda}} \\ &= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{1}{\lambda^k k!} \sum_{n=0}^{\infty} \langle r+l\lambda \rangle_{n,\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{\lambda^k k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \langle r+l\lambda \rangle_{n,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{43}$$

Therefore, by (42) and (43), we obtain the following theorem.

Theorem 6. *For $n, k \geq 0$, with $n \geq k$, we have*

$$L_{r,\lambda}(n,k) = \frac{1}{\lambda^k} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \langle r+l\lambda \rangle_{n,\lambda}.$$

From (39), we note that

$$\begin{aligned} \sum_{k=0}^{n+1} L_{r,\lambda}(n+1,k)(x)_{k,\lambda} &= \langle x+r \rangle_{n+1,\lambda} = \langle x+r \rangle_{n,\lambda}(x+r+n\lambda) \\ &= \sum_{k=0}^{n} L_{r,\lambda}(x)_{k,\lambda}\big(x - k\lambda + r + (n+k)\lambda\big) \\ &= \sum_{k=0}^{n} L_{r,\lambda}(n,k)(x)_{k+1,\lambda} + \sum_{k=0}^{n} L_{r,\lambda}(n,k)\big(r+(n+k)\lambda\big)(x)_{k,\lambda} \\ &= \sum_{k=0}^{n+1} \Big(L_{r,\lambda}(n,k-1) + \big(r+(n+k)\lambda\big) L_{r,\lambda}(n,k) \Big)(x)_{k,\lambda}. \end{aligned} \tag{44}$$

Comparing the coefficients on both sides of (44), we obtain the following theorem.

Theorem 7. *For $n, k \in \mathbb{N}$, with $n \geq k$, we have*

$$L_{r,\lambda}(n+1,k) = L_{r,\lambda}(n,k-1) + \big(r+(n+k)\lambda\big)L_{r,\lambda}(n,k).$$

Now, we consider the *r-extended λ-Lah-Bell polynomials* defined by

$$LB_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} L_{r,\lambda}(n,k)x^k, \quad (n \geq 0). \tag{45}$$

Thus, by (42) and (45), we easily get

$$e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t}-1)}\left(\frac{1}{1-\lambda t}\right)^{\frac{r}{\lambda}} = \sum_{n=0}^{\infty} LB_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{46}$$

In particular, for $x = 1$, $LB_{n,\lambda}^{(r)} = LB_{n,\lambda}^{(r)}(1)$ are called the *r-extended λ-Lah-Bell numbers*. The left hand side of (46) can be written as

$$\begin{aligned} e^{\frac{x}{\lambda}(\frac{1}{1-\lambda t}-1)}\left(\frac{1}{1-\lambda t}\right)^{\frac{r}{\lambda}} &= e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{x^k}{\lambda^k!k!}\left(\frac{1}{1-\lambda t}\right)^{\frac{\lambda k+r}{\lambda}} \\ &= e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{x^k}{\lambda^k k!}\sum_{n=0}^{\infty}\langle \lambda k + r\rangle_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{\langle \lambda k + r\rangle_{n,\lambda}}{\lambda^k k!}x^k\right)\frac{t^n}{n!}. \end{aligned} \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, we have*

$$LB_{n,\lambda}^{(r)}(x) = e^{-\frac{x}{\lambda}}\sum_{k=0}^{\infty}\frac{\langle \lambda k + r\rangle_{n,\lambda}}{k!\lambda^k}x^k.$$

For $r \geq 0$, the *λ-analogues of r-Stirling numbers of the second kind* are defined by

$$\sum_{n=k}^{\infty}\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!} = \frac{1}{\lambda^k}\frac{1}{k!}\big(e^{\lambda t}-1\big)^k e^{rt}, \quad (k \geq 0), \quad \text{(see [13])}. \tag{48}$$

Replacing t by $\frac{1}{\lambda}(1-e^{-\lambda t})$ in (42) and using (9), we get

$$\begin{aligned} \frac{1}{k!}\frac{1}{\lambda^k}\big(e^{\lambda t}-1\big)^k e^{rt} &= \sum_{m=k}^{\infty} L_{r,\lambda}(m,k)\frac{1}{m!}\left(\frac{e^{-\lambda t}-1}{-\lambda}\right)^m \\ &= \sum_{m=k}^{\infty} L_{r,\lambda}(m,k)\sum_{n=m}^{\infty}\left\{\begin{matrix} n \\ m \end{matrix}\right\}_{-\lambda}\frac{t^n}{n!} \end{aligned} \tag{49}$$

$$= \sum_{n=k}^{\infty} \left(\sum_{m=k}^{n} L_{r,\lambda}(m,k) \left\{ {n \atop m} \right\}_{-\lambda} \right) \frac{t^n}{n!}.$$

Thus, by (48) and (49), we have

$$\left\{ {n+r \atop k+r} \right\}_{r,\lambda} = \sum_{m=k}^{n} L_{r,\lambda}(m,k) \left\{ {n \atop m} \right\}_{-\lambda}, \quad (k \geq 0). \tag{50}$$

Replacing t by $\frac{1}{\lambda} \log \left(\frac{1}{1-\lambda t} \right)$ in (48) and using (7), we see that

$$\begin{aligned} \frac{1}{\lambda^k} \frac{1}{k!} \left(\frac{1}{1-\lambda t} - 1 \right)^k \left(\frac{1}{1-\lambda t} \right)^{\frac{r}{\lambda}} &= \sum_{m=k}^{\infty} \left\{ {m+r \atop k+r} \right\}_{r,\lambda} \frac{1}{m!} \left(\frac{1}{\lambda} \log \left(\frac{1}{1-\lambda t} \right) \right)^m \\ &= \sum_{m=k}^{\infty} \left\{ {m+r \atop k+r} \right\}_{r,\lambda} \sum_{n=m}^{\infty} \left[{n \atop m} \right]_{\lambda} \frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty} \left(\sum_{m=k}^{n} \left\{ {m+r \atop k+r} \right\}_{r,\lambda} \left[{n \atop m} \right]_{\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{51}$$

By (42) and (51), we get

$$L_{r,\lambda}(n,k) = \sum_{m=k}^{n} \left\{ {m+r \atop k+r} \right\}_{r,\lambda} \left[{n \atop m} \right]_{\lambda}, \quad (k \geq 0). \tag{52}$$

Therefore, by (50) and (52), we obtain the following theorem.

Theorem 9. *For $n, k \geq 0$ with $n \geq k$, we have*

$$\left\{ {n+r \atop k+r} \right\}_{r,\lambda} = \sum_{m=k}^{n} L_{r,\lambda}(n,k) \left\{ {n \atop m} \right\}_{-\lambda},$$

and

$$L_{r,\lambda}(n,k) = \sum_{m=k}^{n} \left\{ {m+r \atop k+r} \right\}_{r,\lambda} \left[{n \atop m} \right]_{\lambda}.$$

3. Further Remarks

A Poisson random variable indicates how many events occured within a given period of time. A random variable X, taking on one of the variables $0, 1, 2, \ldots$, is said to be the Poisson random variable with parameter $\alpha > 0$ if the probability mass function of X is given by

$$p(i) = P\{X = i\} = e^{-\alpha} \frac{\alpha^i}{i!}, \quad \text{(see [16, 25])}. \tag{53}$$

Let f be a real valued function, and let X be a random variable. Then we define

$$E[f(X)] = \sum_{i=0}^{\infty} f(i)p(i), \quad \text{(see [25])}. \tag{54}$$

For $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$, assume that X is the Poisson random variable with parameter $\frac{\alpha}{\lambda}(> 0)$. Then we note from (54) that

$$\begin{aligned} E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{X}\Big] &= \sum_{i=0}^{\infty}\Big(\frac{1}{1-\lambda t}\Big)^{i} p(i) \\ &= \sum_{i=0}^{\infty}\Big(\frac{1}{1-\lambda t}\Big)^{i}\frac{1}{i!}\Big(\frac{\alpha}{\lambda}\Big)^{i} e^{-\frac{\alpha}{\lambda}} = e^{\frac{1}{\lambda}\frac{\alpha}{1-\lambda t}} e^{-\frac{\alpha}{\lambda}} \\ &= e^{\frac{\alpha}{\lambda}(\frac{1}{1-\lambda t}-1)} = \sum_{n=0}^{\infty} B_{n,\lambda}^{L}(\alpha)\frac{t^n}{n!}. \end{aligned} \tag{55}$$

On the other hand, by binomial expansion, we get

$$E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{X}\Big] = E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{\frac{\lambda X}{\lambda}}\Big] = \sum_{n=0}^{\infty} E\Big[\langle X\lambda\rangle_{n,\lambda}\Big]\frac{t^n}{n!}. \tag{56}$$

Hence, by (55) and (56), we get

$$E\Big[\langle X\lambda\rangle_{n,\lambda}\Big] = B_{n,\lambda}^{L}(\alpha), \quad (n \geq 0). \tag{57}$$

For $r \geq 0$, from (46) and (55), we observe that

$$\begin{aligned} E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{X+\frac{r}{\lambda}}\Big] &= E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{X}\Big]\Big(\frac{1}{1-\lambda t}\Big)^{\frac{r}{\lambda}} \\ &= e^{\frac{\alpha}{\lambda}(\frac{1}{1-\lambda t}-1)}\Big(\frac{1}{1-\lambda t}\Big)^{\frac{r}{\lambda}} = \sum_{n=0}^{\infty} LB_{n,\lambda}^{(r)}(\alpha)\frac{t^n}{n!}. \end{aligned} \tag{58}$$

On the other hand, by binomial expansion, we get

$$E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{X+\frac{r}{\lambda}}\Big] = E\Big[\Big(\frac{1}{1-\lambda t}\Big)^{\frac{\lambda X+r}{\lambda}}\Big] = \sum_{n=0}^{\infty} E[\langle\lambda X + r\rangle_{n,\lambda}]\frac{t^n}{n!}. \tag{59}$$

Thus, by (58) and (59), we get

$$LB_{n,\lambda}^{(r)}(\alpha) = E\big[\langle\lambda X + r\rangle_{n,\lambda}\big], \quad (n \geq 0). \tag{60}$$

From (4), (39), and (60), we note that

$$LB_{n,\lambda}^{(r)}(\alpha) = E\big[\langle X\lambda + r\rangle_{n,\lambda}\big] = \sum_{k=0}^{n} L_{r,\lambda}(n,k)E[(\lambda X)_{k,\lambda}]. \tag{61}$$

$$= \sum_{k=0}^{n} L_{r,\lambda}(n,k) \sum_{j=0}^{k} S_{1,\lambda}(k,j)\lambda^j E[X^j].$$

From (13), we have

$$E[X^j] = \sum_{k=0}^{\infty} k^j p(k) = \sum_{k=0}^{\infty} \frac{(\frac{\alpha}{\lambda})^k}{k!} e^{-\frac{\alpha}{\lambda}} k^j \tag{62}$$
$$= e^{-\frac{\alpha}{\lambda}} \sum_{k=0}^{\infty} \frac{k^j}{k!} (\frac{\alpha}{\lambda})^k = \phi_j(\frac{\alpha}{\lambda}).$$

Hence, by (13), (61), and (62), we get

$$LB_{n,\lambda}^{(r)}(\alpha) = \sum_{j=0}^{n} \sum_{k=j}^{n} L_{r,\lambda}(n,k) S_{1,\lambda}(k,j)\phi_{j,\lambda}(\alpha), \quad (n \geq 0). \tag{63}$$

We obtain the following theorem from (57), (60), and (63).

Theorem 10. *Assume that X is the Poisson random variable with parameter $\frac{\alpha}{\lambda}(> 0)$, for λ with $0 < \lambda < 1$.*

$$E\Big[\langle X\lambda\rangle_{n,\lambda}\Big] = B_{n,\lambda}^{L}(\alpha), \quad LB_{n,\lambda}^{(r)}(\alpha) = E\big[\langle \lambda X + r\rangle_{n,\lambda}\big]$$
$$= \sum_{j=0}^{n} \sum_{k=j}^{n} L_{r,\lambda}(n,k) S_{1,\lambda}(k,j)\phi_{j,\lambda}(\alpha), \quad (n \geq 0).$$

4. Conclusion

The degenerate versions arise when we replace the powers of x by the generalized falling factorial polynomials $(x)_{k,\lambda}$ in the defining equations, whereas the λ-analogues appear when we replace the falling factorials $(x)_k$ by the generalized falling factorials.

In this paper, as λ- analogues of the Lah numbers and Lah-Bell polynomials, we studied the λ-analogues of Lah numbers $L_\lambda(n,k)$ and Lah-Bell polynomials $B_{n,\lambda}^{L}(x)$. For those numbers and polynomials, we investigated some properties, explicit expressions, generating functions and Dobinski-like formulas. We also considered the more general λ-analogues of r-Lah numbers $L_{r,\lambda}(n,k)$ and r-extended λ-Lah-Bell polynomials $LB_{n,\lambda}^{(r)}(x)$ and similar results to $L_\lambda(n,k)$ and $B_{n,\lambda}^{L}(x)$ were derived. In addition, we showed the expectation of one random variable and that of another random variable, both related to the Poisson random variable with parameter $\frac{\alpha}{\lambda}$, are respectively equal to $B_{n,\lambda}^{L}(\alpha)$ and $LB_{n,\lambda}^{(r)}(\alpha)$.

As one of our future research projects, we would like to continue to explore λ-analogues of some special numbers and polynomials and their applications to physics, science and engineering as well as to mathematics.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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