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Infinitesimal Rigidity Analysis of a Bar-Joint Framework With Connected Braced Triangles

Ghada Matooq Badri

Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia

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Abstract. The aim of this paper is to provide detailed infinitesimal rigidity analysis of a planar infinite bar-joint framework consisting of connected braced triangles. This is achieved in a purely mathematical manner, using the infinitesimal flex condition, as we identify the non trivial infinitesimal flexes of the finite framework consisting of n connected triangles. The result is then generalized to the infinite case leading to the identification of a base for the space of all infinitesimal flexes.

2020 Mathematics Subject Classifications: 52C25, 51M05, 47N50

Key Words and Phrases: Bar-joint framework, infinitesimal rigidity, non trivial flex

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1. Introduction

There has been increasing interest in the analysis of mathematical bar-joint frameworks, both in the finite and infinite sense. The realization of a framework in the Euclidean space enables us to mathematically investigate properties of physical structures and crystalline materials. Applications of rigidity now extend beyond physics and structure engineering finding their way to molecule analysis [8], [13], robotics [26] and much more. Formally, a pair (G, p) in the Euclidean plane $\mathbb{R}^2$ is a mathematical bar-joint framework $\mathcal{G}$ where $G = (V, E)$ represents a simple graph, and $p = (p_1, p_2, p_3, \dots)$ represents a placement of the graph's vertices in $\mathbb{R}^2$, with $p_i \neq p_j$ if (v_i, v_j) is an edge. The line segments $[p_i, p_j]$ connected to the edges of G are the framework edges.

Definition 1. *Let $G = (V, E)$ be a graph with $p \in V$. The number $|E(p)|$ of edges in the graph with the vertex p as an endpoint is the degree of p, or $d(p)$. The minimum degree of G is $\delta(G) := \min\{d(p), p \in V\}$, and the maximum degree is $\Delta(G) := \max\{d(p), p \in V\}$. $G = (V, E)$ is said to be n-regular if all of the vertices have the same degree n.*

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Email address: gmbadri@uqu.edu.sa (G. M. Badri)

Any displacement of the framework that preserves the distances between each pair of framework vertices is called a rigid body motion. This transition, known as a flexing of the structure, occurs when the lengths between vertices that are not connected by an edge change. The outcome is a new configuration that differs from the previous one. In the Euclidean plane $\mathbb{R}^2$, it is evident that rigid body motions result form linear combinations of rotations and translations in either coordinate direction.

Definition 2. *Let $\mathcal{G} = (G, p)$ be a finite framework with $|V| = n$ in $\mathbb{R}^2$. A vector $u = (u_1, \ldots, u_n)$ in the vector space $\mathcal{H}_v(\mathcal{G}) = \bigoplus_{i=1}^{n} \mathbb{R}^2$ is called an infinitesimal flex if and only if the orthogonality relation*

$$\langle p_i - p_j, u_i - u_j \rangle = 0$$

is held for any edge $e = [p_i, p_j]$.

A framework $\mathcal{G}$ is infinitesimally rigid if every infinitesimal flex of $\mathcal{G}$ is trivial, and infinitesimally flexible otherwise. If, however, all pairs of framework vertices, not just those that form edges, satisfy the above condition, then u is considered a trivial infinitesimal flex, or an infinitesimal rigid body motion.

$\mathcal{H}_v(\mathcal{G})$ will be used to represent the vector space containing all velocity vectors allocated to the vertices of the framework. A vector subspace of $\mathcal{H}_v(\mathcal{G})$, which includes the subspace of all infinitesimal rigid motions $\mathcal{H}_{\text{rig}}(\mathcal{G})$, is the space $\mathcal{H}_{\text{fl}}(\mathcal{G})$ of all infinitesimal flexes of $\mathcal{G}$.

Definition 3. *In $\mathbb{R}^2$, let $\mathcal{G} = (G, p)$ be an infinite framework. A vector in $\mathcal{H}_v(\mathcal{G}) = \prod_V \mathbb{R}^2 = \mathbb{R}^2 \bigoplus \mathbb{R}^2 \bigoplus + \ldots$ is an infinitesimal flex of $\mathcal{G}$ for which, just like in the finite case,*

$$\langle u_i - u_j, p_i - p_j \rangle = 0$$

is held for every edge $e = [p_i, p_j]$.

Let $\mathcal{H}_{\text{fl}}(\mathcal{G})$ be the linear space of all infinitesimal flexes, adapting the same notation as in the finite case. This includes the three dimensional space of rigid body motions $\mathcal{H}_{\text{rig}}(\mathcal{G})$ which is spanned by two translations and one rotation. See [5], [11], [14] and [24] for a thorough introduction to bar-joint frameworks.

Definition 4. *$R(G, p)$ is the rigidity matrix of the infinite framework $\mathcal{G} = (G, P)$. In $\mathbb{R}^2$, it consists of rows indexed by the framework edges and columns labelled by the vertices but with multiplicity two, i.e., $v_1^x, v_1^y, v_2^x, v_2^y, \ldots$. The entries $x_i - x_j, x_j - x_i, y_i - y_j, y_j - y_i$ occur in the row labelled $e = (v_i, v_j)$ with respective column labels $v_i^x, v_i^y, v_j^x, v_j^y$ and equal zero elsewhere.*

$R(G, p)$ defines a linear transformation from the space $\mathcal{H}_v(\mathcal{G}) = \prod_V \mathbb{R}^2$ to $\mathcal{H}_e(\mathcal{G}) = \prod_E \mathbb{R}$ and it can be seen that a vector u in $\mathcal{H}_v(\mathcal{G})$ is an infinitesimal flex if and only if $R(G, p)u = 0$.

Definition 5. *A countably infinite framework $\mathcal{G} = (G, p)$ is edge vanishing if the sequence $(d_{e_i})_{i=1}^{\infty}$ formed by all bar lengths has no lower bound. $\mathcal{G}$ is edge unbounded if $(d_{e_i})_{i=1}^{\infty}$ has no upper bound. $\mathcal{G}$ is distance-regular if $(d_{e_i})_{i=1}^{\infty}$ is bounded.*

Theorem 1. *[20] For a distance regular framework in $\mathbb{R}^2$ where the degrees of the vertices are uniformly bounded, the rigidity matrix determines a bounded Hilbert space transformation.*

One of the earliest contributions towards rigidity theory is Due to Laman, [16]. Laman's Theorem characterizes the rigidity of a generic framework as purely combinatorial, regardless of its geometry. For three dimensional frameworks, Euler conjectured that "A closed spatial figure allows no changes, as long as it is not ripped apart", see [14]. For convex polyhedra, this conjecture was later answered by Cauchy [9], who proved that "If there is an isometry between the faces of two strictly convex polyhedra which is an isometry on each of the faces, then the two polyhedra are congruent". A corollary of Cauchy's Theorem is that all convex polyhedrons are in fact rigid. The formal identification of a mathematical framework as a set of bars and joints was first introduced by Asimov and Roth [3], [4], [22] where each vertex corresponds to a joint and each bar represents an edge. Finally, Euler's conjecture was proven wrong by Connelly [10], who showed that there exists a polyhedron that is not rigid.

Recent developments in rigidity theory include the formal identification of an infinite bar-joint framework, crystallographic bar-joint frameworks and the rigidity matrix by Owen and Power, [20]. The investigation of various types of flexes admitted by such frameworks was carried out in [18] and [19]. Different forms of rigidity were also introduced, for example, dimensional rigidity [1], global rigidity [12], almost periodic rigidity [6] and bearing rigidity [17]. For more rigidity considerations, one can refer to [2], [15] and [25].

2. The Connected Braced Triangles Finite Bar-Joint Framework

Let $\mathcal{F}_{2\text{tri}}$ be the finite bar-joint framework constructed by connecting two braced triangles, with the planar placement suggested by Figure 1. Although any triangle in the Euclidean plane $\mathbb{R}^2$ is rigid, the presence of a vertex on one edge implies the existence of a non trivial infinitesimal flex of the triangle. In this case, the addition of the internal horizontal edge would prevent any flexing of that edge. Now, let $u_i = (u_i^x, u_i^y)$ be the infinitesimal flex applied to the vertex $p_i = (p_i^x, p_i^y)$. Assuming that all the flexes at the base vertices equal zero, this would prevent any rigid body motion and any flexing of the structure would be non trivial. The following theorem proves that this framework admits a non trivial infinitesimal flex uniquely determined by the velocity applied at vertex p_1.

Theorem 2. *Let $\mathcal{F}_{2tri}$ be the finite framework with two braced triangles, then $\mathcal{F}_{2tri}$ admits a one dimensional space of non trivial infinitesimal flexes.*

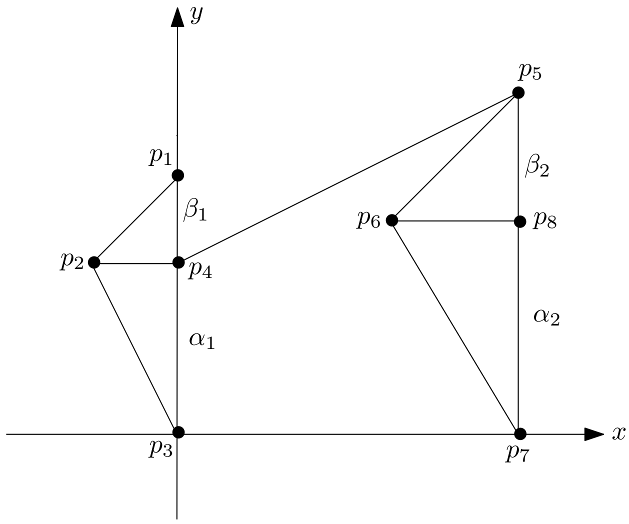

Figure 1: The finite connected braced triangles framework $\mathcal{F}_{2\text{tri}}$

Proof. The proof makes direct use of the infinitesimal flex condition. Let $u = (u_1, \ldots, u_8)$ denote a flex of $\mathcal{F}_{2\text{tri}}$ with the component $u_i = (u_i^x, u_i^y)$ being the velocity applied at vertex p_i. Subtracting appropriate multiples of rigid body motions we can arrange for the vertices p_3 and p_7 to have zero velocity vectors. It follows immediately that vertices p_1, p_4, p_5 and p_8 all have have zero velocity components in the y direction.

Applying the flex condition to the edge $[p_2, p_3]$ we have:

$$\begin{aligned} \langle p_2 - p_3, u_2 \rangle + \langle p_3 - p_2, u_3 \rangle &= 0 \\ (p_2^x - p_3^x)u_2^x + (p_2^y - p_3^y)u_2^y + (p_3^x - p_2^x)u_3^x + (p_3^y - p_2^y)u_3^y &= 0 \\ u_2^x &= \left(\frac{\alpha_1}{\gamma_1}\right) u_2^y \end{aligned}$$

Applying the flex condition to the edge $[p_1, p_2]$:

$$\begin{aligned} \langle p_1 - p_2, u_1 \rangle + \langle p_2 - p_1, u_2 \rangle &= 0 \\ (p_1^x - p_2^x)u_1^x + (p_1^y - p_2^y)u_1^y + (p_2^x - p_1^x)u_2^x + (p_2^y - p_1^y)u_2^y &= 0 \\ \gamma_1 u_1^x - \gamma_1 u_2^x - \beta_1 u_2^y &= 0 \end{aligned}$$

Substituting u_2^x:

$$u_1^x = \left(\frac{\alpha_1 + \beta_1}{\gamma_1}\right) u_2^y$$

The latter expression being equivalent to:

$$u_2^x = \left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right) u_1^x$$

The application of the flex condition to both edges $[p_2, p_4]$ and $[p_4, p_5]$ implies that:

$$u_2^x = u_4^x = u_5^x = \left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right) u_1^x$$

Applying the flex condition to the edge $[p_5, p_6]$:

$$\langle p_5 - p_6, u_5\rangle + \langle p_6 - p_5, u_6\rangle = 0$$
$$(p_5^x - p_6^x)u_5^x + (p_5^y - p_6^y)u_5^y + (p_6^x - p_5^x)u_6^x + (p_6^y - p_5^y)u_6^y = 0$$

Substituting u_6^x it follows that:

$$\begin{aligned}\gamma_2 u_5^x - \alpha_2 u_6^y - \beta_2 u_6^y &= 0\\ \gamma_2 u_5^x &= (\alpha_2+\beta_2)u_6^y\\ u_5^x &= \left(\frac{\alpha_2+\beta_2}{\gamma_2}\right) u_6^y\\ u_5^x &= \left(1+\frac{\beta_2}{\alpha_2}\right) u_6^x\end{aligned}$$

Equivalently,

$$u_6^x = \left(\frac{1}{1+\frac{\beta_2}{\alpha_2}}\right) u_5^x = \left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{1}{1+\frac{\beta_2}{\alpha_2}}\right) u_1^x$$

Applying the flex condition to the edge $[p_6, p_7]$:

$$\langle p_6 - p_7, u_6\rangle + \langle p_7 - p_6, u_7\rangle = 0$$
$$(p_6^x - p_7^x)u_6^x + (p_6^y - p_7^y)u_6^y + (p_7^x - p_6^x)u_7^x + (p_7^y - p_6^y)u_7^y = 0$$

Therefore:

$$\begin{aligned}u_6^y &= \left(\frac{\gamma_2}{\alpha_2}\right) u_6^x\\ u_6^y &= \left(\frac{\gamma_2}{\alpha_2}\right)\left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{1}{1+\frac{\beta_2}{\alpha_2}}\right) u_1^x\\ u_6^y &= \left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{\gamma_2}{\alpha_2+\beta_2}\right) u_1^x\end{aligned}$$

Finally, applying the flex condition to the edge $[p_6, p_8]$:

$$\begin{aligned} \langle p_6 - p_8, u_6\rangle + \langle p_8 - p_6, u_7\rangle = 0 \\ (p_6^x - p_8^x)u_6^x + (p_6^y - p_8^y)u_6^y + (p_8^x - p_6^x)u_8^x + (p_8^y - p_6^y)u_8^y = 0 \\ u_8^x = u_6^x \end{aligned}$$

The argument above, together with the appropriate substitutions we can identify the non trivial infinitesimal flex $u = (u_1, u_2, \ldots, u_8)$ of $\mathcal{F}_{2\text{tri}}$ with the velocity components:

$$\begin{aligned} u_1 &= (u_1^x, 0) \\ u_2 &= \left(\left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)u_1^x, \left(\frac{\gamma_1}{\alpha_1+\beta_1}\right)u_1^x\right) \\ u_3 &= (0,0) \\ u_4 &= \left(\left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)u_1^x, 0\right) \\ u_5 &= (0,0) \\ u_6 &= \left(\left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{1}{1+\frac{\beta_2}{\alpha_2}}\right)u_1^x, \left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{\gamma_2}{\alpha_2+\beta_2}\right)u_1^x\right) \\ u_7 &= (0,0) \\ u_8 &= \left(\left(\frac{1}{1+\frac{\beta_1}{\alpha_1}}\right)\left(\frac{1}{1+\frac{\beta_2}{\alpha_2}}\right)u_1^x, 0\right) \end{aligned}$$

Clearly, u is uniquely determined by the initial velocity applied at u_1 and the conclusion follows.

Corollary 1. *Let $\mathcal{H}_{fl}(\mathcal{F}_{2tri})$ denote the linear space of all infinitesimal flexes of $\mathcal{F}_{2tri}$. Then* $\dim(\mathcal{H}_{fl}(\mathcal{F}_{2tri})) = 4$.

We now consider the finite framework constructed by connecting three braced triangles, $\mathcal{F}_{3\text{tri}}$, suggested by Figure 2. In addition to the flex determined by the velocity applied at p_1, the following theorem identifies a new infinitesimal flex of this framework.

Theorem 3. *Let $\mathcal{F}_{3tri}$ be the finite framework with three braced triangles. Let $u = (u_1, u_2, \ldots, u_{12})$ be the flex of $\mathcal{F}_{3tri}$ with zero velocity components at all the vertices of the first and third triangle. Then $\mathcal{F}_{3tri}$ admits a non trivial infinitesimal flex uniquely determined by u_7^y.*

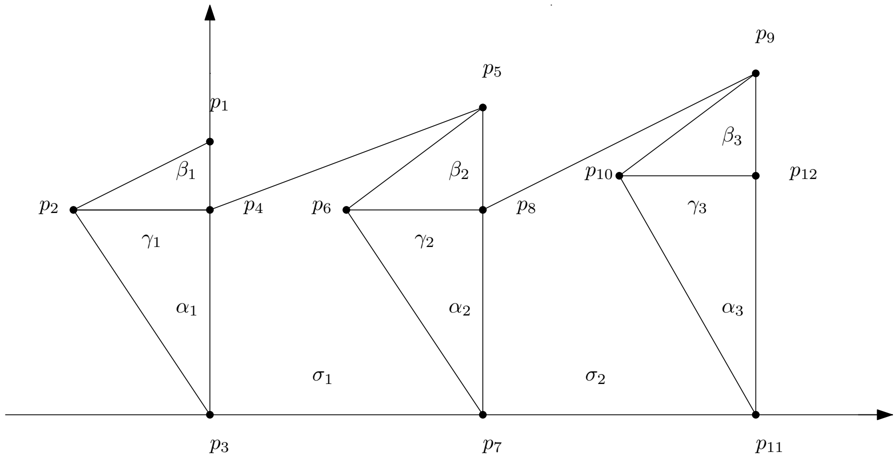

Figure 2: The finite connected braced triangles framework $\mathcal{F}_{3\text{tri}}$

Proof. With the first and third framework vertices all having zero velocity components, $u_1 = u_2 = u_3 = u_4 = u_9 = u_{10} = u_{11} = u_{12} = (0,0)$. From which it immediately follows that u_7 only admits a velocity component in the y direction, the velocity in the x direction, u_7^x, being zero.

Applying the flex condition to the edge $[p_4, p_5]$ we have:

$$\begin{aligned}\langle p_4 - p_5, u_4\rangle + \langle p_5 - p_4, u_5\rangle &= 0\\ (p_4^x - p_5^x)u_4^x + (p_4^y - p_5^y)u_4^y + (p_5^x - p_4^x)u_5^x + (p_5^y - p_4^y)u_5^y &= 0\\ \sigma_1 u_5^x + (\alpha_2 + \beta_2 - \alpha_1)u_5^y &= 0\end{aligned}$$

Which implies:

$$u_5^x = \left(\frac{\alpha_1 - (\alpha_2 + \beta_2)}{\sigma_1}\right) u_5^y$$

Applying the flex condition to the edge $[p_7, p_8]$:

$$\begin{aligned}\langle p_7 - p_8, u_8\rangle + \langle p_8 - p_7, u_8\rangle &= 0\\ (p_7^x - p_8^x)u_7^x + (p_7^y - p_8^y)u_7^y + (p_8^x - p_7^x)u_8^x + (p_8^y - p_7^y)u_8^y &= 0\\ u_7^y &= u_8^y\end{aligned}$$

Applying the flex condition to the edge $[p_5, p_8]$:

$$\begin{aligned}\langle p_5 - p_8, u_5\rangle + \langle p_8 - p_5, u_8\rangle &= 0\\ (p_5^x - p_8^x)u_5^x + (p_5^y - p_8^y)u_5^y + (p_8^x - p_5^x)u_8^x + (p_8^y - p_5^y)u_8^y &= 0\\ u_5^y &= u_8^y\end{aligned}$$

Applying the flex condition to the edge $[p_8, p_9]$:

$$\begin{aligned}\langle p_8 - p_9, u_8\rangle + \langle p_9 - p_8, u_9\rangle = 0\\ (p_8^x - p_9^x)u_8^x + (p_8^y - p_9^y)u_8^y + (p_9^x - p_8^x)u_9^x + (p_9^y - p_8^y)u_9^y = 0\\ (-\sigma_1)u_8^x + (\alpha_2 - (\alpha_3 + \beta_3))u_8^y = 0\end{aligned}$$

And it follows that

$$u_8^x = \left(\frac{\alpha_2 - (\alpha_3 + \beta_3)}{\sigma_2}\right) u_8^y$$

Applying the flex condition to the edge $[p_6, p_7]$:

$$\begin{aligned}\langle p_6 - p_7, u_6\rangle + \langle p_7 - p_6, u_7\rangle = 0\\ (p_6^x - p_7^x)u_6^x + (p_6^y - p_7^y)u_6^y + (p_7^x - p_6^x)u_7^x + (p_7^y - p_6^y)u_7^y = 0\\ (-\gamma_2)u_6^x = \alpha_2 u_6^y\\ u_6^y = \left(\frac{\gamma_2}{\alpha_2}\right) u_6^x\end{aligned}$$

Applying the flex condition to the edge $[p_6, p_8]$:

$$\begin{aligned}\langle p_6 - p_8, u_6\rangle + \langle p_8 - p_6, u_8\rangle = 0\\ (p_6^x - p_8^x)u_6^x + (p_6^y - p_8^y)u_6^y + (p_8^x - p_6^x)u_8^x + (p_8^y - p_6^y)u_8^y = 0\\ u_6^x = u_8^x\end{aligned}$$

Finally, applying the flex condition to the edge $[p_5, p_6]$:

$$\begin{aligned}\langle p_5 - p_6, u_5\rangle + \langle p_6 - p_5, u_6\rangle = 0\\ (p_5^x - p_6^x)u_5^x + (p_5^y - p_6^y)u_5^y + (p_6^x - p_5^x)u_6^x + (p_6^y - p_5^y)u_6^y = 0\end{aligned}$$

From which it follows that:

$$\gamma_2 u_5^x + \beta_2 u_5^y = \gamma_2 u_6^x + \beta_2 u_6^y$$

Substituting u_6^y:

$$\begin{aligned}\gamma_2 u_5^x + \beta_2 u_5^y = \gamma_2 u_6^x + \beta_2 \left(\frac{\gamma_2}{\alpha_2}\right) u_6^x\\ u_5^x = \left(1 + \frac{\beta_2}{\alpha_2}\right) u_6^x - \left(\frac{\beta_2}{\gamma_2}\right) u_5^y\end{aligned}$$

Substituting $u_6^x = u_8^x$ and $u_8^y = u_5^y$:

$$\begin{aligned} u_5^x &= \left(1+\frac{\beta_2}{\alpha_2}\right)\left(\frac{\alpha_2+(\alpha_3+\beta_3)}{\sigma_2}\right)u_8^y - \left(\frac{\beta_2}{\gamma_2}\right)u_5^y \\ &= \left[\left(1+\frac{\beta_2}{\alpha_2}\right)\left(\frac{\alpha_2+(\alpha_3+\beta_3)}{\sigma_2}\right) - \frac{\beta_2}{\gamma_2}\right]u_5^y \end{aligned}$$

With the appropriate substitutions we arrive at an infinitesimal flex of $\mathcal{F}_{3\text{tri}}$ with the velocity components:

$$\begin{aligned} u_5 &= \left(\left[\left(1+\frac{\beta_2}{\alpha_2}\right)\left(\frac{\alpha_2+(\alpha_3+\beta_3)}{\sigma_2}\right) - \frac{\beta_2}{\gamma_2}\right]u_7^y, u_7^y\right) \\ u_6 &= \left(\left(\frac{\alpha_2-(\alpha_3+\beta_3)}{\sigma_2}\right)u_7^y, \left(\frac{\gamma_2}{\alpha_2}\right)\left(\frac{\alpha_2-(\alpha_3+\beta_3)}{\sigma_2}\right)u_7^y\right) \\ u_7 &= (0, u_7^y) \\ u_8 &= \left(\left(\frac{\alpha_2-(\alpha_3+\beta_3)}{\sigma_2}\right)u_7^y, u_7^y\right) \end{aligned}$$

uniquely determined by u_7^y and with zero velocities elsewhere.

Corollary 2. *Let $\mathcal{H}_{fl}(\mathcal{F}_{3tri})$ denote the linear space of all infinitesimal flexes of $\mathcal{F}_{3tri}$. Then* $\dim(\mathcal{H}_{fl}(\mathcal{F}_{3tri})) = 5$.

Corollary 3. *Let $\mathcal{H}_{fl}(\mathcal{F}_{ntri})$ denote the linear space of all infinitesimal flexes of the finite framework with n connected braced triangles. Then*

$$\dim(\mathcal{H}_{fl}(\mathcal{F}_{ntri})) = n + 2.$$

3. The Connected Braced Triangles Infinite Bar-Joint Framework

Let $\mathcal{G}_{\text{tri}}$ be the infinite strip bar-joint framework in $\mathbb{R}^2$ constructed by joining copies of the connecting braced triangles finite framework along in the positive x coordinate direction, as suggested by Figure 3.

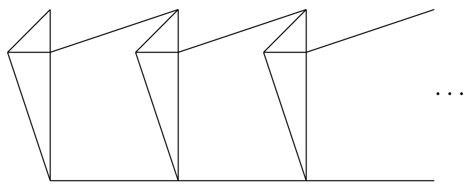

Figure 3: The infinite connected braced triangles framework $\mathcal{G}_{\text{tri}}$

This section is dedicated to the identification of a base for the space of all infinitesimal flexes of the infinite connected braced triangles framework. In fact, such bases do exist for infinite frameworks and this was thoroughly investigated in [7]. For more considerations of infinite strip frameworks see [20], [21] and [23].

Similar to the finite case, with the base vertices having zero velocity components, the action implied by the velocity at vertex p_1 does in fact extend to the rest of the framework in the infinite case.

Theorem 4. *Let $\mathcal{G}_{tri}$ be the connected braced triangles infinite bar-joint framework in $\mathbb{R}^2$. Let u be the non trivial infinitesimal flex of $\mathcal{G}_{tri}$, with the velocity component $u_{n,i}$ applied at vertex $p_{n,i}$, and with $u_{n,3} = (0,0)$ for all $n \in \mathbb{N}$. Then:*

$$u_{n,1} = \left(\prod_{i=1}^{n-1} \left(\frac{1}{1+\frac{\beta_i}{\alpha_i}} \right) u_{1,1}^{x}, 0 \right)$$

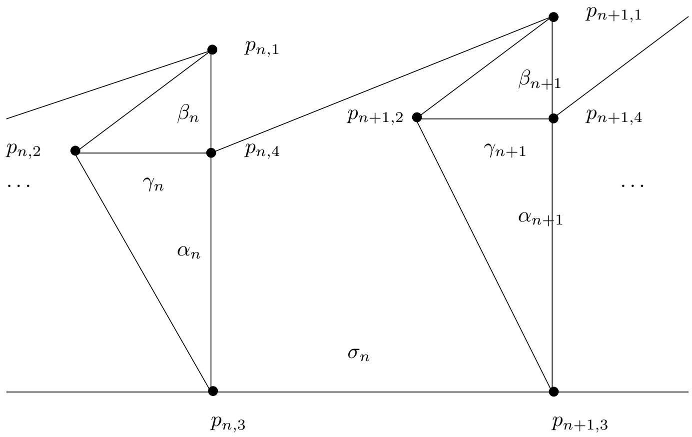

Figure 4: Labelling of the infinite connected braced triangles framework $\mathcal{G}_{\text{tri}}$

Proof. The proof proceeds by Mathematical Induction. The case $n = 2$ follows immediately from the proof of Theorem 2. We now assume that this is true for n, that is

$$u_{n,1} = \left(\prod_{i=1}^{n-1} \left(\frac{1}{1+\frac{\beta_i}{\alpha_i}} \right) u_{1,1}^{x}, 0 \right).$$

The proof for $n+1$ makes direct use of the flex condition. It is obvious that each flex $u_{n,1}$ has a zero velocity component in the y direction since $u_{n,3} = (0,0)$ for all n. To find the velocity component in the x direction we first start by applying the flex condition to the edge $[p_{n,2}, p_{n,3}]$:

$$\begin{aligned}\langle p_{n,2} - p_{n,3}, u_{n,2}\rangle + \langle p_{n,3} - p_{n,2}, u_{n,3}\rangle = 0\\ (p^x_{n,2} - p^x_{n,3})u^x_{n,2} + (p^y_{n,2} - p^y_{n,3})u^y_{n,2} + (p^x_{n,3} - p^x_{n,2})u^x_{n,3} + (p^y_{n,3} - p^y_{n,2})u^y_{n,3} = 0\\ -\gamma_n u^x_{n,2} + \alpha_n u^y_{n,2} = 0\end{aligned}$$

From which it follows that:

$$u^y_{n,2} = \left(\frac{\gamma_n}{\alpha_n}\right) u^x_{n,2}$$

Applying the flex condition to the edge $[p_{n,1}, p_{n,2}]$:

$$\begin{aligned}\langle p_{n,1} - p_{n,2}, u_{n,1}\rangle + \langle p_{n,2} - p_{n,1}, u_{n,2}\rangle = 0\\ (p^x_{n,1} - p^x_{n,2})u^x_{n,1} + (p^y_{n,1} - p^y_{n,2})u^y_{n,1} + (p^x_{n,2} - p^x_{n,1})u^x_{n,2} + (p^y_{n,2} - p^y_{n,1})u^y_{n,2} = 0\\ \gamma_n u^x_{n,1} - \gamma_n u^x_{n,2} - \beta_n u^y_{n,2} = 0\end{aligned}$$

Substituting $u^y_{n,2}$:

$$\gamma_n u^x_{n,1} - \gamma_n u^x_{n,2} - \beta_n \left(\frac{\gamma_n}{\alpha_n}\right) u^x_{n,2} = 0$$

$$u^x_{n,2} = \left(\frac{1}{1 + \frac{\beta_n}{\alpha_n}}\right) u^x_{n,1}$$

Applying the flex condition to the edge $[p_{n,2}, p_{n,4}]$:

$$\begin{aligned}\langle p_{n,2} - p_{n,4}, u_{n,2}\rangle + \langle p_{n,4} - p_{n,2}, u_{n,4}\rangle = 0\\ (p^x_{n,2} - p^x_{n,4})u^x_{n,2} + (p^y_{n,2} - p^y_{n,4})u^y_{n,2} + (p^x_{n,4} - p^x_{n,2})u^x_{n,4} + (p^y_{n,4} - p^y_{n,2})u^y_{n,4} = 0\\ -\gamma_n u^x_{n,2} + \gamma_n u^x_{n,4} = 0\end{aligned}$$

Hence,

$$u^x_{n,4} = u^x_{n,2} = \left(\frac{1}{1 + \frac{\beta_n}{\alpha_n}}\right) u^x_{n,1}$$

Applying the flex condition to the edge $[p_{n,4}, p_{n+1,1}]$:

$$\begin{aligned}\langle p_{n,4} - p_{n+1,1}, u_{n,4}\rangle + \langle p_{n+1,1} - p_{n,4}, u_{n+1,1}\rangle = 0\\ (p^x_{n,4} - p^x_{n+1,1})u^x_{n,4} + (p^y_{n,4} - p^y_{n+1,1})u^y_{n,4} + (p^x_{n+1,1} - p^x_{n,4})u^x_{n+1,1} + (p^y_{n+1,1} - p^y_{n,4})u^y_{n+1,1} = 0\\ u^x_{n+1,1} = u^x_{n,4}\end{aligned}$$

Finally, using the induction hypothesis:

$$\begin{aligned}
u^x_{n+1,1} &= u^x_{n,4} \\
&= \left(\frac{1}{1+\frac{\beta_n}{\alpha_n}}\right) u^x_{n,1} \\
&= \left(\frac{1}{1+\frac{\beta_n}{\alpha_n}}\right)\left(\prod_{i=1}^{n-1}\left(\frac{1}{1+\frac{\beta_i}{\alpha_i}}\right)\right) u^x_{1,1} \\
&= \prod_{i=1}^{n}\left(\frac{1}{1+\frac{\beta_i}{\alpha_i}}\right) u^x_{1,1}
\end{aligned}$$

and the result follows.

Definition 6. *Let $\mathcal{G} = (G, p)$ be an infinite framework in $\mathbb{R}^2$. An infinitesimal flex $u = (u_n)$ of $\mathcal{G}$ is called local if $u_n = (0, 0)$ for all but finitely many values of n.*

Corollary 4. *Let $\mathcal{G}_{tri}$ be the connected braced triangles infinite bar-joint framework in $\mathbb{R}^2$. If the vertices of all but one triangle have zero velocity components, then $\mathcal{G}_{tri}$ admits the local flex identified in Theorem 3.*

Theorem 5. *Let $\mathcal{G}_{tri}$ be the connected braced triangles infinite bar-joint framework in $\mathbb{R}^2$. Then $\mathcal{B} = \{w^x, w^y, r\} \bigcup \{v\} \bigcup \{u_n : n \in \mathbb{N}\}$ is a base for $\mathcal{H}_{fl}(\mathcal{G}_{tri})$, where w^x, w^y, r are the three planar rigid body motions, v is the non trivial flex in Theorem 4 and u_n is the local flex with non zero velocity components at one triangle and zero velocities elsewhere.*

Proof. The proof proceeds with an exhaustion argument where appropriate multiples are subtracted from an arbitrary flex until we achieve zero flexing of the structure. Let $s = (s_n) = \left((s_{n,i})_{i=1}^4\right)_{n=1}^\infty$ be an arbitrary flex of $\mathcal{G}_{\mathrm{tri}}$. Subtracting $s^x_{1,3}w^x + s^y_{1,3}w^y$ from the flex s one can arrange for vertex $p_{1,3}$ to have a zero velocity. Subtracting $s^y_{2,3}r$, vertex $p_{2,3}$ now has a zero velocity implying that any further flexing of $\mathcal{G}_{\mathrm{tri}}$ would be non trivial. Proceeding in the same manner, subtracting $s^x_{1,1}v$ results in a zero velocity at vertex $p_{1,1}$ from which it follows that vertex $p_{2,1}$ is also assigned a zero velocity. To this end, vertices $p_{1,1}, p_{1,2}, \ldots, p_{2,3}, p_{2,4}$ of the first two triangles are all fixed. Finally, successive subtraction of $s^y_{n,3}u_n$, assigns zero velocities to the points $p_{n,3}$. This results in every vertex of $\mathcal{G}_{\mathrm{tri}}$ admitting a zero velocity and the conclusion follows.

4. Conclusion

In this paper we determined the non trivial flexes of the planar finite frameworks with n connected braced triangles. This lead to the identification of a bases for the spaces of all infinitesimal flexes, $\mathcal{H}_{\mathrm{fl}}(\mathcal{F}_{\mathrm{ntri}})$. The nature of the non trivial base elements emphasises how the flexibility of a framework is impacted by its unique geometry. This knowledge can be taken forward to provide detailed rigidity analysis of infinite frameworks by understanding the rigidity of their finite subframeworks placed in Euclidean spaces. In this specific case, the infinite framework admits similar flexing properties to those of the finite frameworks. Additionally, we found that the action of the non trivial infinitesimal flex with zero velocities at the points $p_{n,3}$ extends to the rest of the framework and is uniquely determined by the x component of the velocity initially applied at vertex $p_{1,1}$.

Furthermore, such analysis can be applied to crystal frameworks for example, a class of bar-joint frameworks known for their translational symmetry. This also can be used to identify special types of flexes such as strictly periodic, phase periodic and supercell periodic flexes.

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