EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 2726-2737
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Multipliers of Hilbert Algebras

Aiyared Iampan[1,*], Neelamegarajan Rajesh[2]

[1] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[2] *Department of Mathematics, Rajah Serfoji Government College (Affiliated to Bharathidasan University), Thanjavur-613005, Tamilnadu, India*

Abstract. This paper explores the concept of multipliers in Hilbert algebras, unveiling various intriguing properties. We delve into the intricate relationships between fixed sets, kernels, and near filters of multipliers, shedding light on their interconnected roles within Hilbert algebras. This investigation aims to provide a deeper understanding of algebraic structures and their dynamic interactions.

2020 Mathematics Subject Classifications: 03G25, 43A22

Key Words and Phrases: Hilbert algebra, multiplier, kernel, fixed set, near filter

1. Introduction

The study of multipliers has attracted considerable attention in various branches of algebra, demonstrating its versatility and relevance across different algebraic systems. In 1986, Cirulis [5] was among the pioneers, examining multipliers within the framework of implicative algebras. His work set the stage for further inquiries into how multipliers interact with logical operations in these structures. Subsequently, in 2011, Kim [16] extended the concept to BE-algebras, providing a new perspective emphasising multipliers' role in non-classical algebraic settings. This expansion continued when Chaudhry and Ali [4] introduced multipliers in d-algebras in 2012, shedding light on their function in algebras characterized by a specific set of order and combination rules. In 2013, Kim and Lim [17] brought the concept of multipliers into the realm of BCC-algebras, exploring their implications in algebras defined by bounded commutative cancellative properties. Around the same time, Lee and Kim [18] focused on subtraction algebras, where they defined multipliers that operate within algebraic structures built on the notion of difference rather than addition. In 2014, Khorami and Saeid [15] introduced the concept of multipliers in BL-algebras and conducted an in-depth study on the connections between multipliers and specific mappings, such as closure operators, homomorphisms, and $(\odot, \vee)$-derivations

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5298
Email addresses: aiyared.ia@up.ac.th (A. Iampan), nrajesh_topology@yahoo.co.in (N. Rajesh)

within BL-algebras. A significant leap occurred in 2021 when Iampan [10] introduced a more comprehensive categorization of multipliers within UP-algebras. His work distinguished between left multipliers, right multipliers, anti-left multipliers, and anti-right multipliers, each playing a distinct role in governing the structural dynamics of these algebras. This categorization offered a more nuanced understanding of how multipliers can be applied, expanding their utility across different branches of algebra.

The concept of Hilbert algebras emerged in the early 1950s, introduced by Henkin [9] as a tool for analyzing implications in intuitionistic and other non-classical logic. By the 1960s, the algebraic foundations of these structures gained prominence, particularly through the work of Diego [6], who demonstrated that Hilbert algebras constitute a locally finite variety. Diego's contributions marked a significant milestone, grounding these algebras in formal algebraic theory. Subsequent research further enriched the understanding of Hilbert algebras. Busneag [1, 2] and Jun [13] extended the study by identifying filters that form deductive systems within these algebras. Their work highlighted the logical and algebraic interplay embedded in these structures. Additionally, Dudek [7] introduced a novel perspective by exploring the fuzzification of subalgebras and deductive systems, broadening the scope of Hilbert algebra applications in the study of uncertainty and graded logic.

2. Preliminaries

Understanding the foundational structures within mathematical logic and algebra is pivotal for advancing both theoretical insights and practical applications. Hilbert algebras, named after the eminent mathematician David Hilbert, form a crucial part of this framework. These algebras not only provide a robust structure for exploring logical connectives and implications but also serve as a bridge between logic and algebraic operations. Their unique properties and the relationships they encapsulate are instrumental in various fields, including proof theory, model theory, and lattice theory. By delving into the notion of Hilbert algebras, we can uncover the intricate web of connections that underpin logical deduction and algebraic manipulation, paving the way for deeper mathematical discoveries.

Definition 1. *[6] A Hilbert algebra is a triplet with the formula $\mathcal{H} = (\mathcal{H}, \cdot, 1)$, where $\mathcal{H}$ is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of $\mathcal{H}$ that is true according to the axioms stated below:*

(1) $(\forall x, y \in \mathcal{H})(x \cdot (y \cdot x) = 1)$,

(2) $(\forall x, y, z \in \mathcal{H})((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,

(3) $(\forall x, y \in \mathcal{H})(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

Example 1. *[11] Let* $\mathcal{H} = \{1, \alpha, \beta, \gamma, \epsilon\}$ *with the following Cayley table:*

$\cdot$	1	α	β	γ	ϵ
1	1	α	β	γ	ϵ
α	1	1	β	γ	ϵ
β	1	α	1	γ	γ
γ	1	1	β	1	β
ϵ	1	1	1	1	1

Then $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ *is a Hilbert algebra.*

In [7], the following conclusion was established.

Lemma 1. *Let* $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ *be a Hilbert algebra. Then*

(1) $(\forall x \in \mathcal{H})(x \cdot x = 1)$,

(2) $(\forall x \in \mathcal{H})(1 \cdot x = x)$,

(3) $(\forall x \in \mathcal{H})(x \cdot 1 = 1)$,

(4) $(\forall x, y, z \in \mathcal{H})(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,

(5) $(\forall x, y, z \in \mathcal{H})((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $\mathcal{H} = (\mathcal{H}, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in \mathcal{H})(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on $\mathcal{H}$ with 1 as the largest element.

Partial order in Hilbert algebra is essential for studying logical structures such as lattice formations and their applications in model theory. It enhances proof systems by verifying entailment and inference theorems, thereby improving logical deduction processes. Additionally, it facilitates the analysis of complex algebraic structures like Boolean algebras, offering insights into their hierarchical relationships. These applications underscore the significance of partial order in comprehending and investigating mathematical and logical frameworks within Hilbert algebras.

Definition 2. *[14] A nonempty subset* $\mathcal{D}$ *of a Hilbert algebra* $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ *is called a subalgebra of* $\mathcal{H}$ *if* $x \cdot y \in \mathcal{D}$ *for all* $x, y \in \mathcal{D}$.

Definition 3. *[3] A nonempty subset* $\mathcal{D}$ *of a Hilbert algebra* $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ *is called an ideal of* $\mathcal{H}$ *if the following conditions hold:*

(1) $1 \in \mathcal{D}$,

(2) $(\forall x, y \in \mathcal{H})(y \in \mathcal{D} \Rightarrow x \cdot y \in \mathcal{D})$,

(3) $(\forall x, y_1, y_2 \in \mathcal{H})(y_1, y_2 \in \mathcal{D} \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in \mathcal{D})$.

Definition 4. *[8] A nonempty subset $\mathcal{D}$ of a Hilbert algebra $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ is called a deductive system (or implication filter or simply filter) of $\mathcal{H}$ if the following conditions hold:*

(1) $1 \in \mathcal{D}$,

(2) $(\forall x, y \in \mathcal{H})(x \cdot y, x \in \mathcal{D} \Rightarrow y \in \mathcal{D})$.

3. Multipliers of Hilbert algebras

In this section, we present the concepts of left multipliers, right multipliers, anti-left multipliers, and anti-right multipliers within Hilbert algebras, along with the notion of near filters. We also examine the intricate relationship between right multipliers and near filters, shedding light on how these elements interact within the algebraic framework.

Henceforth, unless stated otherwise, we will consider $\mathcal{H}$ as a Hilbert algebra denoted by $\mathcal{H} = (\mathcal{H}, \cdot, 1)$.

Definition 5. *A self-map $\mathfrak{m}$ of $\mathcal{H}$ is called*

(1) a left multiplier if $\mathfrak{m}(x \cdot y) = \mathfrak{m}(x) \cdot y$ for all $x, y \in \mathcal{H}$,

(2) a right multiplier if $\mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y)$ for all $x, y \in \mathcal{H}$,

(3) an anti-left multiplier if $\mathfrak{m}(x \cdot y) = y \cdot \mathfrak{m}(x)$ for all $x, y \in \mathcal{H}$,

(4) an anti-right multiplier if $\mathfrak{m}(x \cdot y) = \mathfrak{m}(y) \cdot x$ for all $x, y \in \mathcal{H}$.

Definition 6. *Define a self-map $\mathfrak{i}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ by, for any $x \in \mathcal{H}$, $\mathfrak{i}_{\mathcal{H}}(x) = x$. Then $\mathfrak{i}_{\mathcal{H}}$ is a left multiplier and a right multiplier of $\mathcal{H}$.*

Proposition 1. *Let $\mathfrak{m}$ be a self-map of $\mathcal{H}$. Then the following statements hold:*

(1) $\mathfrak{m}$ is a left multiplier $\mathcal{H}$ if and only if $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$,

(2) if $\mathfrak{m}$ is an anti-left multiplier of $\mathcal{H}$, then $\mathfrak{m}$ is constant,

(3) $\mathfrak{m}$ is an anti-right multiplier of $\mathcal{H}$ if and only if $\mathcal{H} = \{1\}$.

Proof. (1) Assume that $\mathfrak{m}$ is a left multiplier of $\mathcal{H}$. Then

$$\begin{aligned} \mathfrak{m}(1) &= \mathfrak{m}(1 \cdot 1) && \text{(Lemma 1 (1))} \\ &= \mathfrak{m}(1) \cdot 1 && \text{(left multiplier)} \\ &= 1. && \text{(Lemma 1 (3))} \end{aligned}$$

Let $x \in \mathcal{H}$. Then

$$\begin{aligned} \mathfrak{m}(x) &= \mathfrak{m}(1 \cdot x) && \text{(Lemma 1 (2))} \\ &= \mathfrak{m}(1) \cdot x && \text{(left multiplier)} \end{aligned}$$

$$
\begin{aligned}
&= 1 \cdot x && (\mathfrak{m}(1) = 1)\\
&= x. && (\text{Lemma 1 (2)})
\end{aligned}
$$

Hence, $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.

The converse is obvious.

(2) Assume that $\mathfrak{m}$ is an anti-left multiplier of $\mathcal{H}$. Let $x \in \mathcal{H}$. Then

$$
\begin{aligned}
\mathfrak{m}(1) &= \mathfrak{m}(x \cdot 1) && (\text{Lemma 1 (3)})\\
&= 1 \cdot \mathfrak{m}(x) && (\text{anti-left multiplier})\\
&= \mathfrak{m}(x). && (\text{Lemma 1 (2)})
\end{aligned}
$$

Hence, $\mathfrak{m}$ is constant.

(3) Assume that $\mathfrak{m}$ is an anti-right multiplier of $\mathcal{H}$. Then

$$
\begin{aligned}
\mathfrak{m}(1) &= \mathfrak{m}(1 \cdot 1) && (\text{Lemma 1 (1)})\\
&= \mathfrak{m}(1) \cdot 1 && (\text{anti-right multiplier})\\
&= 1. && (\text{Lemma 1 (3)})
\end{aligned}
$$

Let $x \in \mathcal{H}$. Then

$$
\begin{aligned}
x &= 1 \cdot x && (\text{Lemma 1 (2)})\\
&= \mathfrak{m}(1) \cdot x && (\mathfrak{m}(1) = 1)\\
&= \mathfrak{m}(x \cdot 1) && (\text{anti-right multiplier})\\
&= \mathfrak{m}(1) && (\text{Lemma 1 (3)})\\
&= 1. && (\mathfrak{m}(1) = 1)
\end{aligned}
$$

Hence, $\mathcal{H} = \{1\}$.

The converse is obvious.

Example 2. *Let $\mathcal{H} = \{1, 2, 3, 4\}$ with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	2	3	1

Then $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ is a Hilbert algebra. We define a self-map $\mathfrak{m} : \mathcal{H} \to \mathcal{H}$ as follows:

$$
\mathfrak{m}(x) = \begin{cases} 1 & \text{if } x = 1, 2 \\ x & \text{if } x = 3, 4. \end{cases}
$$

In this case, $\mathfrak{m}$ serves as a right multiplier of $\mathcal{H}$, while it does not qualify as a left multiplier, an anti-left multiplier, or an anti-right multiplier of $\mathcal{H}$.

Following this, we will focus specifically on right multipliers of $\mathcal{H}$, referring to them simply as multipliers for brevity.

Proposition 2. *A multiplier* $\mathfrak{m}$ *of* $\mathcal{H}$ *is* $\mathfrak{i}_{\mathcal{H}}$ *if and only if* $\mathfrak{m}$ *is a left multiplier of* $\mathcal{H}$*.*

Proof. It is done in Proposition 1 (1).

Example 3. *From Example 1, we define a self-map* $\mathfrak{m} : \mathcal{H} \to \mathcal{H}$ *as follows:* $\mathfrak{m}(1) = 1, \mathfrak{m}(\alpha) = \beta, \mathfrak{m}(\beta) = \gamma, \mathfrak{m}(\epsilon) = \epsilon, \mathfrak{m}(\gamma) = \alpha$*. Then* $\mathfrak{m}$ *is a multiplier of* $\mathcal{H}$*. Since* $\mathfrak{m}^2(\alpha) = \mathfrak{m}(\mathfrak{m}(\alpha)) = \mathfrak{m}(\beta) = \gamma \neq \beta = \mathfrak{m}(\alpha)$*, we have* $\mathfrak{m}^2 \neq \mathfrak{m}$*.*

Example 3 shows that if $\mathfrak{m}$ is a multiplier of $\mathcal{H}$, then $\mathfrak{m}^2 = \mathfrak{m}$, which is not valid in general. The following proposition illustrates the relationship of these conditions.

Proposition 3. *A multiplier* $\mathfrak{m}$ *of* $\mathcal{H}$ *is* $\mathfrak{i}_{\mathcal{H}}$ *if and only if the following statements hold:*

(1) $\mathfrak{m}^2 = \mathfrak{m}$,

(2) $\mathfrak{m}(x \cdot y) = \mathfrak{m}(x) \cdot \mathfrak{m}(y)$ *for all* $x, y \in \mathcal{H}$,

(3) $\mathfrak{m}^2(x) \cdot y = \mathfrak{m}(x) \cdot \mathfrak{m}(y)$ *for all* $x, y \in \mathcal{H}$.

Proof. The condition for necessity is obvious.

Conversely, assume that (1), (2), and (3) hold. Then for $x, y \in \mathcal{H}$, we have $\mathfrak{m}(x \cdot y) = \mathfrak{m}(x) \cdot \mathfrak{m}(y) = \mathfrak{m}^2(x) \cdot y = \mathfrak{m}(x) \cdot y$. Thus, $\mathfrak{m}$ is a left multiplier of $\mathcal{H}$. By Proposition 2, we have $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.

Proposition 4. *Let* $\mathfrak{m}$ *be a multiplier of* $\mathcal{H}$*. Then* $\mathfrak{m}(\mathfrak{m}(x) \cdot x) = 1$ *for all* $x \in \mathcal{H}$*.*

Proof. By Lemma 1 (1), we have $\mathfrak{m}(\mathfrak{m}(x) \cdot x) = \mathfrak{m}(x) \cdot \mathfrak{m}(x) = 1$ for all $x \in \mathcal{H}$.

Definition 7. *A self-map* $\mathfrak{m}$ *of* $\mathcal{H}$ *is said to be regular if* $\mathfrak{m}(1) = 1$*.*

Proposition 5. *Every multiplier of* $\mathcal{H}$ *is regular.*

Proof. Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. Then, by Lemma 1 (3), we have $\mathfrak{m}(1) = \mathfrak{m}(x \cdot 1) = x \cdot \mathfrak{m}(1)$ for all $x \in \mathcal{H}$. By Lemma 1 (1), we have $\mathfrak{m}(1) = \mathfrak{m}(1) \cdot \mathfrak{m}(1) = 1$. Hence, $\mathfrak{m}$ is regular.

Proposition 6. *Let* $\mathfrak{m}$ *be a multiplier of* $\mathcal{H}$*. Then the following statements hold:*

(1) $\mathfrak{m}(1) = 1$,

(2) $x \leq \mathfrak{m}(x)$ *for all* $x \in \mathcal{H}$,

(3) for any $x, y \in \mathcal{H}$*, if* $x \leq y$*, then* $x \leq \mathfrak{m}(y)$,

(4) if $\mathfrak{m}$ *is injective, then* $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.

Proof. (1) It is done in Proposition 5.

(2) Let $x \in \mathcal{H}$. Then, by (1) and Lemma 1 (1), we have $1 = \mathfrak{m}(1) = \mathfrak{m}(x \cdot x) = x \cdot \mathfrak{m}(x)$, that is, $x \leq \mathfrak{m}(x)$.

(3) Let $x, y \in \mathcal{H}$ be such that $x \leq y$. Then $x \cdot y = 1$. By (1), we have $1 = \mathfrak{m}(1) = \mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y)$, that is, $x \leq \mathfrak{m}(y)$.

(4) Assume that $\mathfrak{m}$ is injective. Let $x \in \mathcal{H}$. By (1) and Lemma 1 (1), we have $\mathfrak{m}(\mathfrak{m}(x) \cdot x) = \mathfrak{m}(x) \cdot \mathfrak{m}(x) = 1 = \mathfrak{m}(1)$. Since $\mathfrak{m}$ is injective, $\mathfrak{m}(x) \cdot x = 1$. Then $\mathfrak{m}(x) \leq x$. It follows from (2) and Definition 1 (3) that $\mathfrak{m}(x) = x$, so $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.

Definition 8. *A self-map $\mathfrak{m}$ of $\mathcal{H}$ is said to be nonexpansive if $\mathfrak{m}(x) \leq x$ for all $x \in \mathcal{H}$.*

Example 4. *[12] Let $\mathcal{H} = \{1, \alpha, \beta, \gamma, \epsilon\}$ with the following Cayley table:*

$\cdot$	1	α	β	γ	ϵ
1	1	α	β	γ	ϵ
α	1	1	β	γ	ϵ
β	1	α	1	γ	γ
γ	1	1	β	1	β
ϵ	1	1	1	1	1

Then $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ is a Hilbert algebra. We define a self-map $\mathfrak{m} : \mathcal{H} \to \mathcal{H}$ as follows: $\mathfrak{m}(1) = \beta$, $\mathfrak{m}(\alpha) = \alpha$, $\mathfrak{m}(\beta) = \beta$, $\mathfrak{m}(\gamma) = \gamma$ and $\mathfrak{m}(\epsilon) = \epsilon$. Then $\mathfrak{m}$ is nonexpansive. Since $\mathfrak{m}(1) = \beta \neq 1$, it follows from Proposition 5 that $\mathfrak{m}$ is not a multiplier of $\mathcal{H}$.

Proposition 7. *If $\mathfrak{m}$ is a nonexpansive multiplier of $\mathcal{H}$, then $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.*

Proof. By assumption, Proposition 6 (2), and Definition 1 (3), we have $\mathfrak{m}(x) = x$ for all $x \in \mathcal{H}$. Hence, $\mathfrak{m} = \mathfrak{i}_{\mathcal{H}}$.

Proposition 8. *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two multipliers of $\mathcal{H}$. Then $\mathfrak{m}_1 \circ \mathfrak{m}_2$ is a multiplier of $\mathcal{H}$.*

Proof. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned}(\mathfrak{m}_1 \circ \mathfrak{m}_2)(x \cdot y) &= \mathfrak{m}_1(\mathfrak{m}_2(x \cdot y)) \\ &= \mathfrak{m}_1(x \cdot \mathfrak{m}_2(y)) \\ &= x \cdot \mathfrak{m}_1(\mathfrak{m}_2(y)) \\ &= x \cdot (\mathfrak{m}_1 \circ \mathfrak{m}_2)(y).\end{aligned}$$

Hence, $\mathfrak{m}_1 \circ \mathfrak{m}_2$ is a multiplier of $\mathcal{H}$.

Corollary 1. *Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. Then $\mathfrak{m}^n$ is a multiplier of $\mathcal{H}$ for all positive integer n.*

Let $(\mathcal{H}, \cdot, 1_{\mathcal{H}})$ and $(\mathcal{K}, *, 1_{\mathcal{K}})$ be Hilbert algebras. Then $(\mathcal{H} \times \mathcal{K}, \diamond, (1_{\mathcal{H}}, 1_{\mathcal{K}}))$ is a Hilbert algebra defined by $(a, b) \diamond (c, d) = (a \cdot c, b * d)$ for all $a, c \in \mathcal{H}$ and $b, d \in \mathcal{K}$.

Proposition 9. *Let $\mathcal{H} = (\mathcal{H}, \cdot, 1_{\mathcal{H}})$ and $\mathcal{K} = (\mathcal{K}, *, 1_{\mathcal{K}})$ be Hilbert algebras. Define the self-map $\mathfrak{m}$ of $\mathcal{H} \times \mathcal{K}$ by, for any $(x, y) \in \mathcal{H} \times \mathcal{K}$, $\mathfrak{m}(x, y) = (x, 1_{\mathcal{K}})$. Then $\mathfrak{m}$ is a multiplier of $\mathcal{H} \times \mathcal{K}$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in \mathcal{H} \times \mathcal{K}$. Then

$$\begin{aligned}\mathfrak{m}((x_1, x_2) \diamond (y_1, y_2)) &= \mathfrak{m}(x_1 \cdot y_1, x_2 * y_2)\\ &= (x_1 \cdot y_1, 1_{\mathcal{K}})\\ &= (x_1 \cdot y_1, x_2 * 1_{\mathcal{K}})\\ &= (x_1, x_2) \diamond (y_1, 1_{\mathcal{K}})\\ &= (x_1, x_2) \diamond \mathfrak{m}(y_1, y_2).\end{aligned}$$

Hence, $\mathfrak{m}$ is a multiplier of $\mathcal{H} \times \mathcal{K}$.

Proposition 10. *Let $\mathcal{H} = (\mathcal{H}, \cdot, 1_{\mathcal{H}})$ and $\mathcal{K} = (\mathcal{K}, *, 1_{\mathcal{K}})$ be Hilbert algebras. Define the self-map $\mathfrak{m}$ of $\mathcal{H} \times \mathcal{K}$ by, for any $(x, y) \in \mathcal{H} \times \mathcal{K}$, $\mathfrak{m}(x, y) = (1_{\mathcal{H}}, y)$. Then $\mathfrak{m}$ is a multiplier of $\mathcal{H} \times \mathcal{K}$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in \mathcal{H} \times \mathcal{K}$. Then

$$\begin{aligned}\mathfrak{m}((x_1, x_2) \diamond (y_1, y_2)) &= \mathfrak{m}(x_1 \cdot y_1, x_2 * y_2)\\ &= (1_{\mathcal{H}}, x_2 * y_2)\\ &= (x_1 \cdot 1_{\mathcal{H}}, x_2 * y_2)\\ &= (x_1, x_2) \diamond (1_{\mathcal{H}}, y_2)\\ &= (x_1, x_2) \diamond \mathfrak{m}(y_1, y_2).\end{aligned}$$

Hence, $\mathfrak{m}$ is a multiplier of $\mathcal{H} \times \mathcal{K}$.

Definition 9. *Let $\mathfrak{m}$ be a self-map of $\mathcal{H}$. Define the fixed set $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$ and the kernel $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$ of $\mathfrak{m}$ by*

$$\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H}) = \{x \in \mathcal{H} : \mathfrak{m}(x) = x\}$$

and

$$\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) = \{x \in \mathcal{H} : \mathfrak{m}(x) = 1\}.$$

Remark 1. *Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. By Proposition 5, we have $\mathfrak{m}(1) = 1$. Then $1 \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$ and $1 \in \mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$.*

Proposition 11. *Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. Then $\mathfrak{m}$ is injective if and only if $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) = \{1\}$.*

Proof. If $\mathfrak{m}$ is injective, it is done in Proposition 6 (4).

Conversely, assume that $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) = \{1\}$. Then $\mathfrak{m}(1) = 1$. Let $a, b \in \mathcal{H}$ be such that $\mathfrak{m}(a) = \mathfrak{m}(b)$. Then

$$\mathfrak{m}(a \cdot b) = a \cdot \mathfrak{m}(b) \qquad \text{(multiplier)}$$

$$
\begin{aligned}
&= a \cdot \mathfrak{m}(a) && (\mathfrak{m}(a) = \mathfrak{m}(b)) \\
&= \mathfrak{m}(a \cdot a) && (\text{multiplier}) \\
&= \mathfrak{m}(1) && (\text{Lemma 1 (1)}) \\
&= 1, && (\mathfrak{m}(1) = 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{m}(b \cdot a) &= b \cdot \mathfrak{m}(a) && (\text{multiplier}) \\
&= b \cdot \mathfrak{m}(b) && (\mathfrak{m}(a) = \mathfrak{m}(b)) \\
&= \mathfrak{m}(b \cdot b) && (\text{multiplier}) \\
&= \mathfrak{m}(1) && (\text{Lemma 1 (1)}) \\
&= 1. && (\mathfrak{m}(1) = 1)
\end{aligned}
$$

Thus, $a \cdot b, b \cdot a \in \mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) = \{1\}$, so $a \cdot b = 1$ and $b \cdot a = 1$. By Definition 1 (3), we have $a = b$. Hence, $\mathfrak{m}$ is injective.

Proposition 12. *Let* $\mathfrak{m}$ *be a multiplier of* $\mathcal{H}$*. If* $x \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$*, then* $x \in \mathrm{Fix}_{\mathfrak{m}^n}(\mathcal{H})$ *for all positive integer* n*.*

Proof. Let n be a positive integer such that $x \in \mathrm{Fix}_{\mathfrak{m}^n}(\mathcal{H})$. Then $\mathfrak{m}^n(x) = x$. Thus,

$$\mathfrak{m}^{n+1}(x) = (\mathfrak{m}^n \circ \mathfrak{m})(x) = \mathfrak{m}^n(\mathfrak{m}(x)) = \mathfrak{m}^n(x) = x.$$

So, $x \in \mathrm{Fix}_{\mathfrak{m}^{n+1}}(\mathcal{H})$. Hence, $x \in \mathrm{Fix}_{\mathfrak{m}^n}(\mathcal{H})$ for all positive integer n.

Proposition 13. *If* $\mathfrak{m}$ *is a multiplier of* $\mathcal{H}$*, then* $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$ *is a subalgebra of* $\mathcal{H}$*.*

Proof. By Proposition 6 (1), we have $\mathfrak{m}(1) = 1$ and so $1 \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$. Let $x, y \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$. Then $\mathfrak{m}(x) = x$ and $\mathfrak{m}(y) = y$, so $\mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y) = x \cdot y$. Thus, $x \cdot y \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$. Hence, $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$ is a subalgebra of $\mathcal{H}$.

Proposition 14. *If* $\mathfrak{m}$ *is a multiplier of* $\mathcal{H}$*, then* $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$ *is a subalgebra of* $\mathcal{H}$*.*

Proof. By Proposition 6 (1), we have $\mathfrak{m}(1) = 1$ and so $1 \in \mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) \neq \emptyset$. Let $x, y \in \mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$. Then $\mathfrak{m}(x) = 1$ and $\mathfrak{m}(y) = 1$, it follows from Lemma 1 (3) that $\mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y) = x \cdot 1 = 1$. Thus, $x \cdot y \in \mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$. Hence, $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$ is a subalgebra of $\mathcal{H}$.

Definition 10. *A nonempty subset* $\mathcal{S}$ *of a Hilbert algebra* $\mathcal{H} = (\mathcal{H}, \cdot, 1)$ *is called a near filter of* $\mathcal{H}$ *if it satisfies the following properties:*

(1) $1 \in \mathcal{S}$,

(2) $(\forall x, y \in \mathcal{H})(y \in \mathcal{S} \Rightarrow x \cdot y \in \mathcal{S})$.

We see that near filters are generalizations of ideals, and subalgebras are generalizations of near filters.

Theorem 1. *Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. Then the following statements hold:*

(1) if $\mathcal{S}$ is a near filter of $\mathcal{H}$, then $\mathfrak{m}(\mathcal{S})$ is a near filter of $\mathcal{H}$,

(2) if $\mathcal{S}$ is a near filter of $\mathcal{H}$, then $\mathfrak{m}^{-1}(\mathcal{S})$ is a near filter of $\mathcal{H}$.

Proof. (1) Assume that $\mathcal{S}$ is a near filter of $\mathcal{H}$. Since $1 \in \mathcal{S}$ and $\mathfrak{m}(1) = 1$, we have $1 = \mathfrak{m}(1) \in \mathfrak{m}(\mathcal{S})$. Let $x \in \mathcal{H}$ and $y \in \mathfrak{m}(\mathcal{S})$. Then $y = \mathfrak{m}(s)$ for some $s \in \mathcal{S}$, so $x \cdot y = x \cdot \mathfrak{m}(s) = \mathfrak{m}(x \cdot s) \in \mathfrak{m}(\mathcal{S})$ since $x \cdot s \in \mathcal{S}$. Hence, $\mathfrak{m}(\mathcal{S})$ is a near filter of $\mathcal{H}$.

(2) Assume that $\mathcal{S}$ is a near filter of $\mathcal{H}$. Since $\mathfrak{m}(1) = 1 \in \mathcal{S}$, we have $1 \in \mathfrak{m}^{-1}(\mathcal{S})$. Let $x \in \mathcal{H}$ and $y \in \mathfrak{m}^{-1}(\mathcal{S})$. Then $\mathfrak{m}(y) \in \mathcal{S}$, so $\mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y) \in \mathcal{S}$. Thus, $x \cdot y \in \mathfrak{m}^{-1}(\mathcal{S})$. Hence, $\mathfrak{m}^{-1}(\mathcal{S})$ is a near filter of $\mathcal{H}$.

Theorem 2. *Let $\mathfrak{m}$ be a multiplier of $\mathcal{H}$. Then the following statements hold:*

(1) $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$ *is a near filter of $\mathcal{H}$,*

(2) $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$ *is a near filter of $\mathcal{H}$,*

(3) $\mathrm{Im}(\mathfrak{m})$ *is a near filter of $\mathcal{H}$.*

Proof. (1) Since $\{1\}$ is a near filter of $\mathcal{H}$, it follows from Theorem 1 (2) that $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H}) = \mathfrak{m}^{-1}(\{1\})$ is a near filter of $\mathcal{H}$.

(2) Since $\mathfrak{m}(1) = 1$, we have $1 \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$. Let $x \in \mathcal{H}$ and $y \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$. Then $\mathfrak{m}(y) = y$, so $\mathfrak{m}(x \cdot y) = x \cdot \mathfrak{m}(y) = x \cdot y$. Thus, $x \cdot y \in \mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$. Hence, $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$ is a near filter of $\mathcal{H}$.

(3) Since $\mathcal{H}$ is a near filter of $\mathcal{H}$, it follows from Theorem 1 (1) that $\mathrm{Im}(\mathfrak{m}) = \mathfrak{m}(\mathcal{H})$ is a near filter of $\mathcal{H}$.

Since subalgebras are generalizations of near filters, we can replace a near filter with a subalgebra in Theorem 2.

4. Conclusion

In this study, we extensively investigated the four key notions of left multipliers, right multipliers, anti-left multipliers, and anti-right multipliers within the context of Hilbert algebras. Our findings revealed that the identity function is the only valid left multiplier, the anti-left multiplier remains constant, and the anti-right multiplier exists solely if the algebra is a singleton. We further identified the conditions under which a right multiplier becomes the identity function, as detailed in Proposition 3. Moreover, we developed a specific right multiplier for the Cartesian products of Hilbert algebras. Lastly, we demonstrated that the sets $\mathrm{Ker}_{\mathfrak{m}}(\mathcal{H})$, $\mathrm{Fix}_{\mathfrak{m}}(\mathcal{H})$, and $\mathrm{Im}(\mathfrak{m})$ associated with a right multiplier $\mathfrak{m}$ on a Hilbert algebra $\mathcal{H}$ form a near filter.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

References

[1] D. Busneag. A note on deductive systems of a Hilbert algebra. *Kobe J. Math.*, 2:29–35, 1985.

[2] D. Busneag. Hilbert algebras of fractions and maximal Hilbert algebras of quotients. *Kobe J. Math.*, 5:161–172, 1988.

[3] I. Chajda and R. Halaš. Congruences and ideals in Hilbert algebras. *Kyungpook Math. J.*, 39(2):429–432, 1999.

[4] M. A. Chaudhry and F. Ali. Multipliers in d-algebras. *World Appl. Sci. J.*, 18(11):1649–1653, 2012.

[5] J. Cirulis. Multipliers in implicative algebras. *Bull. Sect. Logic*, 15(4):152–157, 1986.

[6] A. Diego. Sur les algébres de Hilbert. *Collection de Logique Math. Ser. A (Ed. Hermann, Paris)*, 21:1–52, 1966.

[7] W. A. Dudek. On fuzzification in Hilbert algebras. *Contrib. Gen. Algebra*, 11:77–83, 1999.

[8] W. A. Dudek. On ideals in Hilbert algebras. *Acta Universitatis Palackianae Olomuciensis Fac. rer. nat. ser. Math.*, 38:31–34, 1999.

[9] L. Henkin. An algebraic characterization of quantifiers. *Fund. Math.*, 37:63–74, 1950.

[10] A. Iampan. Multipliers and near UP-filters of UP-algebras. *J. Discrete Math. Sci. Cryptography*, 24(3):667–680, 2021.

[11] A. Iampan, P. Jayaraman, S. D. Sudha, and N. Rajesh. Interval-valued neutrosophic ideals of Hilbert algebras. *Int. J. Neutrosophic Sci.*, 18(4):223–237, 2022.

[12] A. Iampan, N. Rajesh, and B. Brundha. Neutrosophic set theory applied to Hilbert algebras. *Int. J. Neutrosophic Sci.*, 21(4):84–93, 2023.

[13] Y. B. Jun. Deductive systems of Hilbert algebras. *Math. Japon.*, 43:51–54, 1996.

[14] Y. B. Jun, J. W. Nam, and S. M. Hong. A note on Hilbert algebras. *Pusan Kyongnam Math. J.*, 10(2):279–285, 1994.

[15] R. T. Khorami and A. B. Saeid. Multiplier in BL-algebras. *Iran. J. Sci.*, 38(2):95–103, 2014.

[16] K. H. Kim. Multipliers in BE-algebras. *Int. Math. Forum*, 6(17):815–820, 2011.

[17] K. H. Kim and H. J. Lim. On multipliers of BCC-algebras. *Honam Math. J.*, 35(2):201–210, 2013.

[18] S. D. Lee and K. H. Kim. A note on multipliers of subtraction algebras. *Hacet. J. Math. Stat.*, 42(2):165–171, 2013.