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Generalizing the Equal Incircles Theorem: Insights from Sangaku Problems

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Abstract. Sangaku problems are traditional Japanese geometrical puzzles, often displayed in religious temples, that have intrigued mathematicians for centuries. This study aims to generalize the Equal Incircles Theorem, extending Angela Drei's proof to N-circles, by applying the trigonometric method alongside foundational mathematical tools, including mathematical induction, Heron's formula, and the telescoping product. A generalized equation for N circles based on the Equal Incircles Theorem is derived through explicit mathematical formulation and characterization. The findings deepen our understanding of geometric relationships, highlight the historical significance of Sangaku problems, and offer potential advancements for future engineering applications, mathematics education, and research in mathematical history.

2020 Mathematics Subject Classifications: 51M04, 01A27

Key Words and Phrases: Sangaku problems, Equal Incircle Theorem, Angela Drei's proof

1. Introduction

Sangaku are wooden tablets inscribed with various geometrical problems and devoted to Shinto shrines and Buddhist temples, as shown in Figure 1, during the Japanese Edo period (1603-1868 CE) this is a book: [2]. The historical significance of this tradition was largely unrecognized by scholars until it was brought to light through the publication Japanese Temple Problems: Sangaku. This source discovered by an article in a collection: [5] notes that Japanese temple geometry often predated the discovery of several well-known Western geometric theorems, such as the Katayamahiko Temple Problem, Meiserinji Temple Problem, and the Equal Incircles Theorem. Subsequently, the geometrical principles underlying Japanese temple architecture were documented in the book Sacred Mathematics: Japanese Temple Geometry this is a book: [4]. Following this, an investigation led by R. J. Hosking [see also [7]] as a technical report, addressed a traditional mathematical approach to a Sangaku problem from Okayama prefecture, as shown in Figure 2.

However, these Sangaku problems still lack modern alternative mathematical methods to gain deeper insights. Recent investigations have reported that these ancient problems have contributed to modern mathematical methods, particularly in the field of discrete geometry. Notable examples include the geometric inversion method [see also [10]] a journal article, Euclidean geometry [see also [6]] a journal article, and so on.

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Figure 1: A Sangaku board hanging under the roof of a temple in Japan.

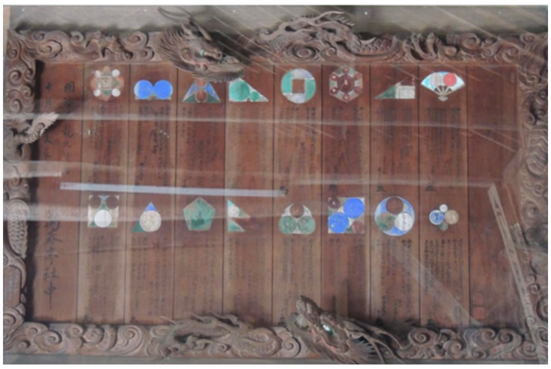

Figure 2: Sangaku Problems at Katayamahiko Shrine.

Equal Incircles Theorem

Let C be a point. Assume points M_i, for $i = 1, 2, \ldots, N$ $(N > 3)$, lie on a line not passing through C. Assume further that the incircles of triangles M_1CM_2, M_2CM_3, $\ldots$, $M_{N-1}CM_N$ all have equal radii. Then the same is true for the triangles M_1CM_3, M_2CM_4, $\ldots$, $M_{N-2}CM_N$, and also for the triangles M_1CM_4, M_2CM_5, $\ldots$, $M_{N-3}CM_N$, and so on.

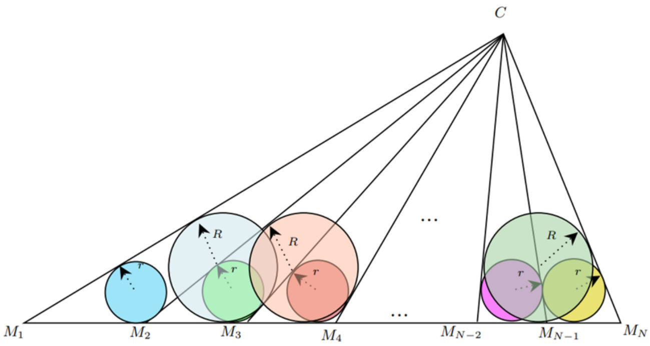

Figure 3: The image depicts the diagram associated with the Equal Incircles Theorem.

Several Sangaku problems from Japan remain unsolved or lack characterization in mathematical theorems that used to explicit mathematical formulations, including the "Equal Incircle Theorem" [see also [3]] as technical reports, which presents a configuration of multiple equal circles inscribed within the same number of sub-triangles, with larger equal circles inscribed within the larger triangles, as shown in Figure 3. Subsequently, Angela Drei's proof provides initial mathematical formulations for two Sangaku problems with equal incircles, employed to characterize the geometrical relationships expressed in terms of trigonometric forms.

So, this paper aims to develop a mathematical framework that generalizes the multiple incircles problem, enabling non-equal radii, by extending from the Equal Incircles Theorem. The study also explores three setups: (1) two tangent circles within a triangle in Section 1; (2) an analysis of the inclusion of incircles in a sectorial triangular configuration in Section 2; (3) the analysis of the inclusion of incircles and ex-circles in a sectorial triangular configuration in Section 3.

2. Main results

This section has formulated an equation that describes generalized adjacent triangles with inscribed circles in the initial part and ex-circles in the remaining part, based on Angela Drei's proof. This formulation is extended to other scenarios, including perpendicular and angle-bisector cases.

Section 1 Two Tangent Circles in a Triangle

Theorem 1 Let a triangle $\triangle AB_1C_1$ with vertices A, B_1, and C_1. Inside this triangle, there are two circles. Let circles O_1 (green) and O_2 (blue) as the incircles of the triangles ΔAB_1B_2 and ΔAC_1B_2 with radii r_1 and r_2 respectively. Circle O_1 is tangent to the sides AB_1, AB_2 and B_1C_1 at points F_1, E_1, and D_1 respectively, and circle O_2 is tangent to the sides AC_1, AB_2, and B_2C_2 at points E_2, F_2 and D_2 respectively by specifying that,

(i) $|B_1C_1| = a$, $|AB_1| = c$, $|C_1A| = b$.

(ii) $|AF_1| = x_2$, $|F_1B_1| = x_1$, $|AE_2| = y_1$, $|D_2B_2| = x_3$, $|E_2C_1| = y_2$

(iii) $\angle AB_1C_1 = B$, $\angle AC_1B_1 = C$, and $\angle B_1AC_1 = A$

(iv) semi-perimeter of AB_1B_2 triangle : $s_1 = x_1 + x_3 + y_1$

(v) semi-perimeter of AC_1B_2 triangle : $s_2 = x_3 + y_1 + y_2$

(vi) semi-perimeter of ABC triangle : $s = \frac{a+b+c}{2} = x_1 + y_3 + y_2 + x_3$

And then,

$$\frac{r_1}{r_2} = \frac{\tan\frac{B}{2} \cdot (s_1 - (x_3 + y_1))}{\tan\frac{C}{2} \cdot (s_2 - (x_3 + y_1))}$$

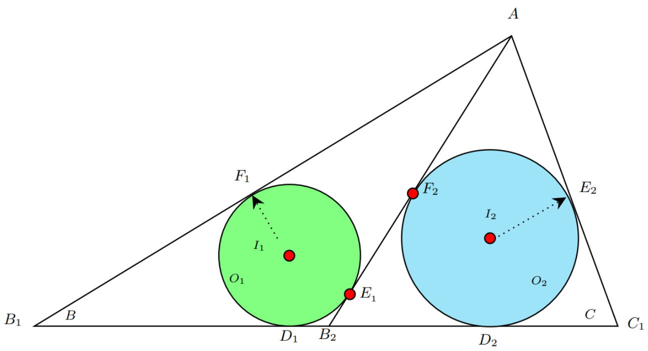

Figure 4: The image depicted the Sangaku Two Circles Problem.

Proof. Let us begin to consider the radius of two incircles (green and blue), using the Incircle of a Triangle formulation. We get,

$$r_1 = \frac{[AB_1B_2]}{s_1} \tag{1}$$

$$r_2 = \frac{[AB_2C_1]}{s_2} \tag{2}$$

Divided the Equation (1) by Equation (2). Then,

$$\frac{r_1}{r_2} = \frac{[AB_1B_2]}{[AC_1B_2]} \cdot \frac{s_2}{s_1} \tag{3}$$

From the area-based ratio, we can formulate the area proportion of two sub-triangles ΔAB_1B_2 and ΔAC_1B_2 depending on its bases.

$$\frac{[AB_1B_2]}{[AC_1B_2]} = \frac{(s_1 - x_2)}{(s_2 - y_1)} \tag{4}$$

Substituting Equation (3) into Equation (4) and we arrive at

$$\frac{r_1}{r_2} = \frac{s_2}{s_1} \cdot \frac{(s_1 - x_2)}{(s_2 - y_1)} \tag{5}$$

This equation is straightforward; to derive in terms of bisector-angle trigonometry.

$$\tan(\frac{B}{2}) = \frac{r_1}{s_1 - (y_1 + x_3)}, \tan(\frac{C}{2}) = \frac{r_2}{s_2 - (y_1 + x_3)}$$

Hence,

$$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = \frac{r_1}{r_2} \cdot \frac{(s_2 - (y_1 + x_3))}{(s_1 - (y_1 + x_3))} \Leftrightarrow \frac{r_1}{r_2} = \frac{\tan\frac{B}{2} \cdot (s_1 - (x_3 + y_1))}{\tan\frac{C}{2} \cdot (s_2 - (x_3 + y_1))}$$

Suppose the biggest right triangle, ΔAC_1B_1 , obtained a perpendicular line from vertex A. In that case, this scenario can extend to the scope of mathematical formulation that describes two tangent circles in a triangle with a perpendicular line.

Corollary 1 (Perpendicular triangle scenario)

Consider a triangle $\triangle AB_1C_1$ with vertices A, B_1, and C_1. Inside this triangle, there are three inscribed circles. Let O_1 (green), O_2 (blue), and O_3 (light gray) be the incircles of triangles $\triangle AB_1B_2$, $\triangle AC_1B_2$, and $\triangle AB_1C_1$ with radii r_1, r_2, and r respectively. Circle O_1 is tangent to sides AB_1, AB_2 and B_1B_2 at points F_1 and E_1, and D_1 respectively, and circle O_2 is tangent to sides AC_1, AB_2 and C_1B_2 at points E_2, F_2 and D_2 respectively. Circle O_3 is tangent to sides AB_1, AC_1, and B_1C_1 at points T_1, T_2, and T_3, respectively such that the following conditions hold:

(i) $B_1A \perp C_1A$.

(ii) $AB_2 \perp B_1C_1$.

And then,

$$AB_2 = \sqrt{(s_1 - x_2)(s_2 - y_1)}$$

Proof. We know that the incenter of circle O lies on the line AB_2, with points B_2 and T_1 coinciding in this case. Therefore, AT_1IT_2 forms a square, as all its sides are equal to the radius of the circle O. It follows that

$$b + c = 2r + a$$

Consider the incircles ΔAB_1B_2 and ΔAC_1B_2 can express the connection between incircles radius and side AB_1, influencing useful mathematical formulation after combined for each other.

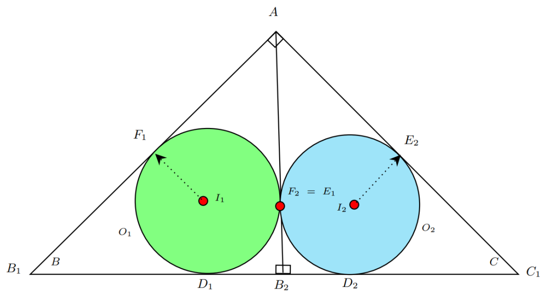

Figure 5: The image depicts the Sangaku Two Circles Problem, illustrating the perpendicular line AB_2.

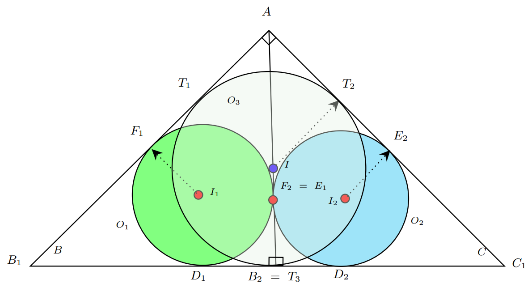

Figure 6: The image depicts the Sangaku Two Circles Problem featuring the inscribed circles O.

$$2(r_1 + r_2) + AB_1 + AC_1 = B_1B_2 + AB_2 + C_1B_2 + AB_2$$

Further simplifying, we get:

$$AB_1 = r_1 + r_2 + r \tag{6}$$

We derived the right-hand side of the main formulation but needed to show that the left-hand side of the equation is equal to the right-hand side, demonstrating the equality between both expressions. According to the similar triangle theorem, and consider the triangle $\Delta AB_1B_2 \sim \Delta AB_2C_1$, which implies the following.

$$\frac{B_1B_2}{AB_1} = \frac{AB_1}{B_2C_1} \implies AB_1^2 = B_1B_2 \cdot B_2C_1 \tag{7}$$

Substituting Equation (6) into Equation (7) and we arrive at

$$AB_2^2 = (B_1B_2) \cdot (C_1B_2) = (x_1 + x_3 + y_1 - x_2) \cdot (x_3 + y_2) = (s_1 - x_2)(s_2 - y_1)$$

Therefore,

$$AB_2 = \sqrt{(s_1 - x_2)(s_2 - y_1)}$$

Consider a triangle ΔAC_1B_1, where a bisector line is drawn from vertex A to the opposite side. Within this configuration, we explored this condition that used to describe two circles that are tangent to each other and tangent to two sides of the triangle with the bisector line.

Corollary 2 (Triangle angle bisector scenario)

Consider a triangle ΔAB_1C_1 with vertices A, B_1, and C_1. Inside this triangle, there are two circles. Let the circles O_1 (green) and O_2 (blue) as the incircles of the triangles ΔAB_1B_2 and ΔAC_1B_2 with radii r_1 and r_2 respectively. Circle O_1 is tangent to the sides AB_1, AB_2 and B_1C_1 at points F_1, E_1, and D_1 respectively, and circle O_2 is tangent to the sides AC_1, AB_2, and B_2C_2 at points E_2, F_2 and D_2 respectively such that the following conditions hold:

(i) $\angle B_1AB_2 = \angle C_1AB_2 = \theta$

(ii) The cevian AB_2 internal bisects the angle $\angle B_1AC_1$

And then,

$$\frac{s_2 - y_1}{s_2 - x_3} = \frac{s_1 - x_2}{s_1 - (x_3 + y_1 - x_2)}$$

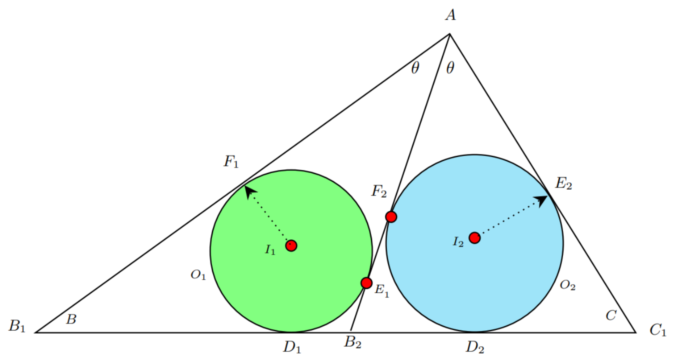

Figure 7: The image depicts the Sangaku Two Circles Problem, illustrating the angle-bisecting scenario.

Proof. According to the angle-bisector theorem in the triangle ΔAB_1C_1 so that

$$\frac{x_1 + x_2}{y_1 + y_2} = \frac{x_1 + x_3 + y_1 - x_2}{x_3 + y_2}$$

Therefore,

$$\frac{s_2 - y_1}{s_2 - x_3} = \frac{s_1 - x_2}{s_1 - (x_3 + y_1 - x_2)}$$

The next section is inspired by the American Invitational Mathematics Examination 2018 Problem 13 [see also [1]] as a technical report. This problem is the best sample to point out the extended scope of this study. However, it obtained a new mathematical formulation that can describe the minimum area of a triangle, which contains two incircles centered in the base of the triangle. The relation can be expressed in the form of trigonometry.

Corollary 3 (Area minimum on bisecting triangle angle scenario)

Consider a triangle $\triangle AB_1C_1$ with vertices A, B_1, and C_1. Inside this triangle, there are two circles. Denote the circles O_1 (green) and O_2 (blue) as the incircles of the triangles ΔAB_1B_2 and ΔAC_1B_2 with radii r_1 and r_2 respectively. Circle O_1 is tangent to the sides AB_1, AB_2 and B_1C_1 at points F_1, E_1, and D_1 respectively, and circle O_2 is tangent to the sides AC_1, AB_2, and B_2C_2 at points E_2, F_2 and D_2 respectively such that the following conditions hold:

(i) The cevian AB_2 internal bisects the angle $\angle B_1AC_1$ and the angle $\angle I_1AI_2$.

And then, the minimum area of the triangle AI_1I_2 is:

$$\frac{(s-a)(s-b)(s-c)}{a}$$

Proof. Assume that the angle $\measuredangle AB_2B_1 = \delta$, and that we have drawn BI_1 and CI_2, which bisect the angles $\angle AB_1C_1$ and $\angle AC_1B_1$ respectively.

$$\measuredangle AB_1I_1 = \measuredangle I_1B_1C_1 = \frac{B}{2}, \quad \measuredangle AC_1I_2 = \measuredangle I_2C_1B_1 = \frac{C}{2}, \quad \measuredangle AI_1B_1 = 90^\circ + \frac{\delta}{2},$$

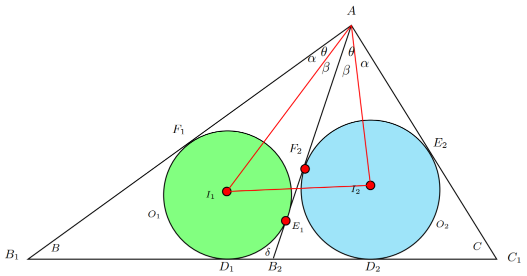

Figure 8: This image presents the line AB_2 internally bisecting the angles $\angle B_1AC_1$ and $\angle I_1AI_2$.

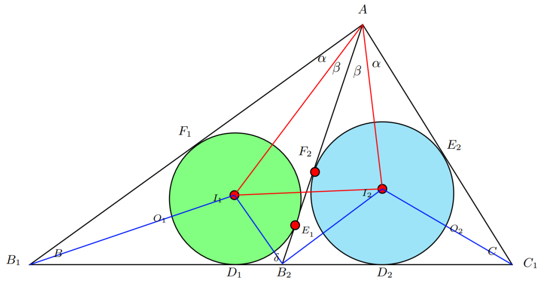

Figure 9: This image illustrates multiple lines bisecting angles within this configuration.

$$\measuredangle AB_2C_1 = 180^\circ - \delta, \quad \measuredangle AI_2C_1 = 180^\circ - \frac{\delta}{2}, \quad \measuredangle I_1AI_2 = \alpha + \beta$$

Next, using the sine law is important for deriving relationships between the sides of the triangle and the relevant angles in this configuration:

$$AI_1 = \frac{\sin\left(\frac{B}{2}\right)}{\sin\left(90^\circ + \frac{\delta}{2}\right)} \cdot c \tag{5}$$

$$AI_2 = \frac{\sin\left(\frac{C}{2}\right)}{\sin\left(180^\circ - \frac{\delta}{2}\right)} \cdot b \tag{6}$$

Substituting the Equation (5) and (6) into the Area of Triangle Trigonometry formulation.

$$[AI_1I_2] = \frac{1}{2} \cdot AI_1 \cdot AI_2 \cdot \sin\left(\frac{A}{2}\right) = \frac{1}{2} \cdot b \cdot c \cdot \frac{\sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right)}{\sin\left(90^\circ + \frac{\delta}{2}\right) \cdot \sin\left(180^\circ - \frac{\delta}{2}\right)}$$

And then,

$$[AI_1I_2] = \frac{b \cdot c \cdot \sin(\frac{A}{2}) \cdot \sin(\frac{B}{2}) \cdot \sin(\frac{C}{2})}{\sin\delta}$$

noting that $1 \leq \frac{1}{\sin\delta}$ and $\sin(\frac{A}{2}) = \pm\sqrt{\frac{1-\cos A}{2}}$. We deduce the minimum area of triangle AI_1I_2 from the following inequality establishes a lower bound.

$$[AI_1I_2] \geq b \cdot c \cdot \left(\frac{\sqrt{(1-\cos A)(1-\cos B)(1-\cos C)}}{8}\right) \tag{7}$$

We applied the Law of Cosines to the concyclic angles. This is an important point because it reduces the trigonometric calculations involving cosines to a relationship between the sides of the triangle.

$$1 - \cos A = \frac{(a-b+c)(a+b-c)}{2bc}, \tag{8}$$

$$1 - \cos B = \frac{(b+c-a)(b+a-c)}{2ca}, \tag{9}$$

$$1 - \cos C = \frac{(b+c-a)(a+c-b)}{2ab}. \tag{10}$$

Substituting Equations (8), (9), and (10) into Equation (7). Therefore, the minimum area of the triangle AI_1I_2 is:

$$\frac{(s-a)(s-b)(s-c)}{a}$$

Section 2 Generalization of Angela Drei's Proof-Inspired Analysis of Inclusion In- in a Sectorial Triangular Configuration

Theorem 2 Let a triangle $\triangle AB_1B_{n+1}$ with rays AB_u from vertex A. For $u \in \{1, \ldots, n\}$, suppose that O_u is the u-th inscribed circle with radii r_u in the triangle AB_uB_{u+1}. It holds that O_u is touch to AB_u, AB_{u+1}, and B_uB_{u+1} at F_u, E_u and D_u respectively. Given the lengths AE_u, and AF_u be x_u, and y_u respectively, where s_u is the semi-parameter of triangle AB_uB_{u+1}. Then,

$$\frac{r_1}{r_n} = \frac{s_n}{s_1} \cdot \prod_{i=2}^{n} \left(\frac{s_{i-1} - x_i}{s_i - y_{i-1}} \right)$$

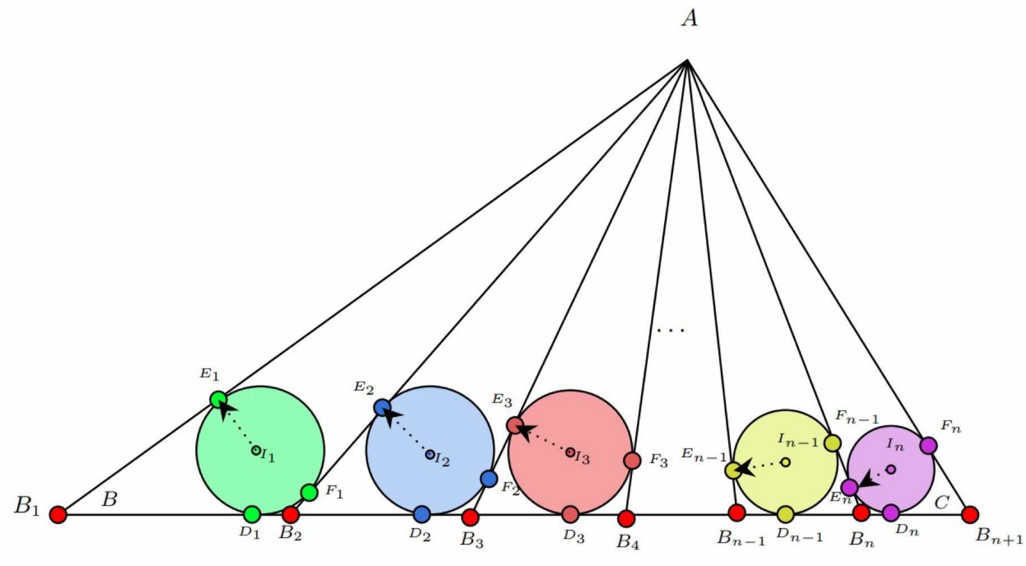

Figure 10: The image illustrates the approach to the Generalized Sangaku Problem for N-circles.

Proof. Consider the statement $P(n)$ defined as:

$$P(n) = \frac{r_{n-1}}{r_n} = \frac{s_n}{s_{n-1}} \cdot \left(\frac{s_{n-1} - x_n}{s_n - y_{n-1}} \right) \tag{11}$$

where it is known from Equation (11) that $P(n)$ holds true for all positive integers n. Substituting the values from $n = 2$ to $n = n$ into the equation, we obtain:

$$\prod_{i=2}^{n} P(i) = \left(\frac{r_1}{r_2}\right)\left(\frac{r_2}{r_3}\right)\cdots\left(\frac{r_{n-1}}{r_n}\right) = \left(\frac{s_2}{s_1}\right)\left(\frac{s_3}{s_2}\right)\cdots\left(\frac{s_n}{s_{n-1}}\right) \cdot \prod_{i=2}^{n} \left(\frac{s_{i-1} - x_i}{s_i - y_{i-1}} \right)$$

Further simplifying. Therefore,

$$\frac{r_1}{r_n} = \frac{s_n}{s_1} \cdot \prod_{i=2}^{n} \left(\frac{s_{i-1} - x_i}{s_i - y_{i-1}} \right)$$

Remark 1. This is a remark about the telescoping product from Equation (X), which presented the inscribed radii fractions is shown, generalized to the independent variables j and k for all $j, k \in \{2, \ldots, n\}$. Therefore,

$$\frac{r_k}{r_{j+1}} = \frac{s_{j+1}}{s_k} \cdot \prod_{o=k}^{j+1} \left(\frac{s_o - x_{o+1}}{s_{o+1} - y_o} \right).$$

Section 3 Generalization of Angera Drei's Proof-Inspired Analysis of Inclusion In- and Ex-Circles in a Sectorial Triangular Configuration

Theorem 3 Let a triangle $\triangle AB_1B_{n+1}$ with extended rays AB_u from vertex A. For $u \in \{1, \ldots, n\}$, suppose that O_u is the u-th inscribed circle with radii r_u in the triangle AB_uB_{u+1}, and contains O'_u is the escribed circle with radii R_u that are tangent to the common base B_nB_{n+1} by supposing the angle $\angle AB_iB_{i+1}$ is set to B_{i-1}. Then,

$$\prod_{i=1}^{n} \frac{r_i}{R_i} = \tan\left(\frac{B}{2}\right) \cdot \tan\left(\frac{C}{2}\right)$$

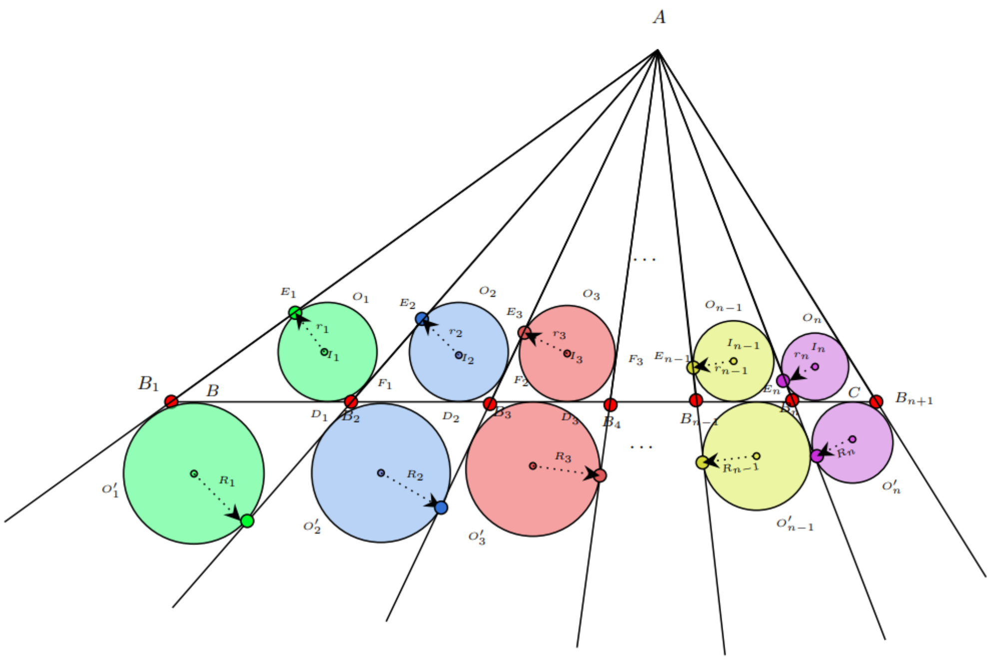

Figure 11: The image illustrates the approach to the Generalized Sangaku Problem for N-inscribe and escribed circles.

Proof 1. By trigonometric formulations for bisecting angles,

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Applying Heron's formula, the expression can be written as follows the relationship between any consecutive angles B is given by:

$$\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{B_1}{2}\right) = \sqrt{\frac{(s-B_1B_2)(s-AB_1)(s-AB_1)(s-AB_2)}{s^2(s-AB_2)(s-B_1B_2)}} = \frac{r_1}{R_1} \tag{12}$$

Similarly, this mathematical formulation allows us to repeat angle terms for each substitution in Equation (12):

$$\begin{aligned}
\frac{r_2}{R_2} &= \tan\left(\frac{180^\circ - B_1}{2}\right)\cdot\tan\left(\frac{B_2}{2}\right),\\
\frac{r_3}{R_3} &= \tan\left(\frac{180^\circ - B_2}{2}\right)\cdot\tan\left(\frac{B_3}{2}\right),\\
&\vdots\\
\frac{r_n}{R_n} &= \tan\left(\frac{180^\circ - B_{n-1}}{2}\right)\cdot\tan\left(\frac{C}{2}\right).
\end{aligned}$$

Upon multiplying these previous n equations, the final product is derived as follows:

$$\prod_{i=1}^{n}\frac{r_i}{R_i} = \tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)\cdot\prod_{i=1}^{n-1}\tan\left(\frac{180^\circ - B_i}{2}\right)$$

and simplifying using the tangent identity, we deduce:

$$\prod_{i=1}^{n}\frac{r_i}{R_i} = \tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)$$

3. Discussions and Conclusions

By extending the Equal Incircles Theorem, this paper aims to develop a mathematical framework that generalizes the multiple incircles problem, allowing for non-equal radii. This study enhances the understanding of the Equal Incircles Theorem based on the generalized Sangaku problem.

In Section 1, this paper revealed the relationship between the fraction of sub-circles in terms of trigonometry for both initial angles (B, C). Although the numerator and denominator are similar in form, they differ in the initial angles and the semi-perimeter of the triangle involving two sub-circles. When generalized to N sub-circles within the same triangle, separated by cevians AB_2, the result revealed that the fraction between any two sub-inscribed circles radii forms a telescoping product, as described in Section 2.

Also, this paper explores other generalized cases, illustrating the relationship between N sub-inscribed circles and sub-escribed circles, as shown in Figure 9. Our findings reveal that this formulation differs significantly from other sections, as the product of their relation provides meaningful interpretations. These relationships are directly tied to the radius of the largest circumcircle within the same triangle.

On the other hand, the study considers the scenario of perpendicular triangles, as discussed in Corollary 1, presenting the sum of the radii of all circles in these configurations. This case guides us toward understanding the perpendicular scenario. At the same time, our findings also consider the angle-bisector case, a general concept frequently encountered in problem-solving across many sources.

This finding demonstrates a novel approach with applications in many fields, such as mathematical education, particularly through the development of new geometry teaching materials [see also [9]] as a technical report. Additionally, it offers potential for the progression of optimization methods [[8]] is a journal article and research in mathematical history.

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