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Some Properties of $(M, k)-$Quasi Paranormal Operators on Hilbert Spaces

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and let T represent a bounded linear operator on $\mathcal{H}$. In this paper we introduce, a new class of non normal operators, the $(M, k)-$quasi paranormal operator. An operator T is said to be a $(M, k)-$quasi paranormal operator, for a non negative integer k and a real positive number M if it satisfies $\|T^{k+1}x\|^2 \leq M\|T^{k+2}x\| \cdot \|T^k x\|$, for all $x \in \mathcal{H}$. This new class of operators is generalization of some of the non normal operators, such as, the $k-$quasi paranormal and $M-$paranormal operators. We prove the basic properties, the structural and spectral properties and also the matrix representation of this new class of operators.
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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, by ker$T$ and $T(\mathcal{H})$ we denote the null space and the range of T, respectively. The null operator will be denoted by 0 and the identity operator by I. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. By $\sigma(T), r(T), \sigma_a(T)$ we write the spectrum, the spectral radius and the approximate point spectrum of T, respectively.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be: an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$; an unitary operator if $T^*T = TT^* = I$ and positive operator $T \geq 0$, if $\langle Tx, x \rangle \geq 0$, for all $x \in \mathcal{H}$ (see [4], [8]).

One of the attractive areas of research in operator theory is the study of non normal operators. Some of the interesting classes of non normal operators, which have been introduced and studied before are paranormal operator, $M-$paranormal operators, $k-$quasi

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paranormal operators, $M-$quasi paranormal operators, ect. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be: a paranormal operator if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$ (see [2], [5], [6], [15]); a $M-$paranormal operators if $\|Tx\|^2 \leq M\|T^2x\|$ for any unit vector x in $\mathcal{H}$ and for a fixed real positive number M (see [1], [3], [12]); a quasi paranormal operators if $\|T^2x\|^2 \leq \|T^3x\| \cdot \|Tx\|$, for all $x \in \mathcal{H}$ (see [10]); a $k-$quasi paranormal operators if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\| \cdot \|T^kx\|$, for all $x \in \mathcal{H}$ and for a positive integer k (see [7], [13]); a $M-$quasi paranormal operators if $\|T^2x\|^2 \leq M\|T^3x\| \cdot \|Tx\|$, for all $x \in \mathcal{H}$ and for a fixed real positive number M (see [9]).

In the present paper, we introduce a new class of operators $(M, k)-$quasi paranormal as a generalization of these non normal classes of operators. The purpose of this paper is, first to give some properties of this new class of operators, to compare this class with the other non normal classes of operators and also to study the structural and spectral properties of this class of operators.

2. Definition and Some Properties

Definition 1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $(M, k)-$quasi paranormal operator, for a non negative integer k and a real positive number M if it satisfies*

$$\|T^{k+1}x\|^2 \leq M\|T^{k+2}x\| \cdot \|T^kx\|,$$

for all $x \in \mathcal{H}$.

This definition is equivalent to

$$T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0,$$

for all $\lambda > 0$. Similarly as [9, Proposition 2].

Example 1. *On the usual Hilbert space l_2, let T be a weighted shift operator, defined by $T(e_n) = |\alpha_n|e_{n+1}$, where (e_n) is the standard basis and (α_n) is a decreasing weighted sequence. Then, T is a $(M, k)-$quasi paranormal operator if and only if*

$$|\alpha_{n+k}| \leq M|\alpha_{n+k+1}|$$

for every n.

Since T is a weighted shift, its adjoint T^* is also a weighted shift and we have:

$$\begin{aligned}
&T^*(e_n) = |\alpha_{n-1}|e_{n-1},\\
&(T^*T)(e_n) = \alpha_n^2 e_n,\\
&(T^{*2}T^2)(e_n) = \alpha_n^2\alpha_{n+1}^2 e_n, ...\\
&(T^{*(k+2)}T^{k+2})(e_n) = \alpha_n^2\alpha_{n+1}^2...\alpha_{n+k}^2\alpha_{n+k+1}^2 e_n.
\end{aligned}$$

Now, since T is a $(M, k)-$quasi paranormal operator then,

$$M^2T^{*(k+2)}T^{(k+2)} - 2\lambda T^{*(k+1)}T^{(k+1)} + \lambda^2 T^{*k}T^k \geq 0, \textit{for all } \lambda > 0 \Leftrightarrow$$

$M^2\alpha_n^2\alpha_{n+1}^2...\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\lambda\alpha_n^2\alpha_{n+1}^2...\alpha_{n+k}^2 + \lambda^2\alpha_n^2\alpha_{n+1}^2...\alpha_{n+k-1}^2 \geq 0, \textit{for all } \lambda > 0 \Leftrightarrow$
$\alpha_n^2\alpha_{n+1}^2...\alpha_{n+k-1}^2(M^2\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\lambda\alpha_{n+k}^2 + \lambda^2) \geq 0, \textit{for all } \lambda > 0 \Leftrightarrow$
$M^2\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\lambda\alpha_{n+k}^2 + \lambda^2 \geq 0, \textit{for all } \lambda > 0.$

By elementary properties of real quadratic forms, this gives

$$4\alpha_{n+k}^4 - 4M^2\alpha_{n+k}^2\alpha_{n+k+1}^2 \leq 0$$
$$|\alpha_{n+k}| \leq M|\alpha_{n+k+1}|.$$

From definition it is clear that this class of operators is nested with respect to M, i.e., a $(M_1, k)-$quasi paranormal operator is $(M_2, k)-$quasi paranormal operator for $0 < M_1 < M_2$.

From the above definition, the following facts follows: for $M = 1$, a $(1, k)-$quasi paranormal operator is a $k-$quasi paranormal operator; for $k = 1$, a $(M, 1)-$quasi paranormal operator is a $M-$quasi paranormal operator; for $k = 0$, a $(M, 0)-$quasi paranormal operator is a $M-$paranormal operator for any unit vector x in $\mathcal{H}$; for $M = 1, k = 0$, a $(1, 0)-$quasi paranormal operator is a paranormal operator for any unit vector x in $\mathcal{H}$; for $M = 1, k = 1$ a $(1, 1)-$quasi paranormal operator is a quasi paranormal operator.

Important properties of this new class of operators are shown in the following theorems.

Theorem 1. *The class of $(M, k)-$quasi paranormal operators is closed under scalar multiplication.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M, k)-$quasi paranormal operator, and let α be any complex scalar. For all $x \in \mathcal{H}$ we have

$$\begin{aligned} \|(\alpha T)^{k+1}x\|^2 & \\ &= |\alpha|^{2k+2}\|T^{k+1}x\|^2 \leq |\alpha|^{2k+2}M(\|T^{k+2}x\| \cdot \|T^k x\|) \\ &= M\|(\alpha T)^{k+2}x\| \cdot \|(\alpha T)^k x\|. \end{aligned}$$

Then, αT is also $(M, k)-$quasi paranormal operator.

□

Theorem 2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M, k)-$quasi paranormal operator and let $S \in \mathcal{L}(\mathcal{H})$ be an isometric operator. If T double commutes with S, then TS is a $(M, k)-$quasi paranormal operator.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M, k)-$quasi paranormal operator. Let be S an isometric operator and let be $B = TS$. Since operator T double commutes with operator S we have $TS = ST, S^*T = TS^*$ and $S^*S = I$. Now,

$$B^{*k}(M^2B^{*2}B^2 - 2\lambda B^*B + \lambda^2)B^k$$

$$
\begin{aligned}
&= (TS)^{*k}(M^2(TS)^{*2}(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2)(TS)^k \\
&= S^*T^*S^*T^*...S^*T^*(M^2S^*T^*S^*T^*TSTS - 2\lambda S^*T^*TS + \lambda^2)TSTS...TS \\
&= S^{*k}T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^kS^k \\
&= (TS)^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)(TS)^k \geq 0, \text{for all } \lambda > 0,
\end{aligned}
$$

so TS is a $(M,k)-$quasi paranormal operator.

□

Theorem 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator. If $S \in \mathcal{L}(\mathcal{H})$ is unitarily equivalent to operator T, then S is a $(M,k)-$quasi paranormal operator.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator.
Since operator S is unitarly equivalent to operator T, then there exists an unitary operator U such that $S = U^*TU$. Since T is a $(M,k)-$quasi paranormal operator then

$$T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0.$$

Hence,

$$
\begin{aligned}
&S^{*k}(M^2S^{*2}S^2 - 2\lambda S^*S + \lambda^2)S^k \\
&= (U^*TU)^{*k}(M^2(U^*TU)^{*2}(U^*TU)^2 - 2\lambda(U^*TU)^*(U^*TU) + \lambda^2)(U^*TU)^k \\
&= U^{*k}T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^kU^k \geq 0, \text{for all } \lambda > 0,
\end{aligned}
$$

so S is a $(M,k)-$quasi paranormal operator.

□

Theorem 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator. If A is a closed T invariant subset of $\mathcal{H}$, then, the restriction $T_{|A}$ is a $(M,k)-$quasi paranormal operator.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator.

$$
\begin{aligned}
&\|(T\mid_A)^{k+1}u\|^2 \\
&\quad = \|T^{k+1}u\|^2 \leq M(\|T^{k+2}u\| \cdot \|T^ku\|) \\
&\quad = M(\|(T\mid_A)^{k+2}u\| \cdot \|(T\mid_A)^ku\|).
\end{aligned}
$$

This implies that $T\mid_A$ is a $(M,k)-$quasi paranormal operator.

□

Theorem 5. *If $T \in \mathcal{L}(\mathcal{H})$ is a invertible $(M,k)-$quasi paranormal operator then T^{-1} is also $(M,k)-$quasi paranormal operator.*

Proof. Since T is a $(M,k)-$quasi paranormal operator, for a non negative integer k and a fixed real positive number M, we have

$$\|T^{k+1}x\|^2 \leq M\|T^{k+2}x\| \cdot \|T^k x\|,$$

for all $x \in \mathcal{H}$. Then,

$$\frac{\|T^{k+1}x\|}{\|T^{k+2}x\|} \leq \frac{M\|T^k x\|}{\|T^{k+1}x\|}$$

for each vector $x \in \mathcal{H}$. Now replacing x by $T^{-2k-2}x$, we have

$$\begin{aligned}
&\frac{\|T^{k+1}T^{-2k-2}x\|}{\|T^{k+2}T^{-2k-2}x\|} \leq \frac{M\|T^k T^{-2k-2}x\|}{\|T^{k+1}T^{-2k-2}x\|}\\
&\frac{\|T^{-k-1}x\|}{\|T^{-k}x\|} \leq \frac{M\|T^{-k-2}x\|}{\|T^{-k-1}x\|}\\
&\|T^{-(k+1)}x\|^2 \leq M\|T^{-(k+2)}x\| \cdot \|T^{-k}x\|
\end{aligned}$$

for each vector $x \in \mathcal{H}$. This shows that T^{-1} is a $(M,k)-$quasi paranormal operator.

□

Theorem 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator. If T^k has dense range, then T is a $M-$paranormal operator.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a $(M,k)-$quasi paranormal operator and let suppose that T^k has dense range, $\overline{T^k(\mathcal{H})} = \mathcal{H}$. Let $x \in \mathcal{H}$, then there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ in $\mathcal{H}$ such that $T^k(x_n) \to x$, $n \to +\infty$.

Since T is a $(M,k)-$quasi paranormal operator, we have

$$\begin{aligned}
&\langle (M^2T^{*(k+2)}T^{(k+2)} - 2\lambda T^{*(k+1)}T^{(k+1)} + \lambda^2 T^{*k}T^k)x_n | x_n\rangle \geq 0, \text{for all } \lambda > 0;\\
&\langle (T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k)x_n, x_n\rangle \geq 0, \text{for all } \lambda > 0;\\
&\langle (M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k x_n, T^k x_n\rangle \geq 0, \text{for all } \lambda > 0.
\end{aligned}$$

By the continuity of the inner product, we have

$$\langle (M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)x, x\rangle \geq 0, \text{for } x \in \mathcal{H}, \text{for all } \lambda > 0.$$

Therefore T is a $M-$paranormal operator.

□

Theorem 7. *Let T be a $(M,k)-$quasi paranormal operator. Then the tensor product $T \otimes I$ and $I \otimes T$ are both $(M,k)-$quasi paranormal operators.*

Proof. Since, T is $(M,k)-$quasi paranormal operator, then we have:

$$T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0,$$

for all $\lambda > 0$. Now, from the properties of tensor product (see [11], [14]) we have :

$$\begin{aligned}
&(T \otimes I)^{*k}[M^2(T \otimes I)^{*2}(T \otimes I)^2 - 2\lambda(T \otimes I)^*(T \otimes I) + \lambda^2](T \otimes I)^k \\
&= (T^{*k} \otimes I)[M^2(T^{*2}T^2 \otimes I) - 2\lambda(T^*T \otimes I) + \lambda^2](T^k \otimes I) \\
&= [T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k] \otimes I \geq 0.
\end{aligned}$$

Therefore, $T \otimes I$ is $(M,k)-$quasi paranormal operator.

Similarly, $I \otimes T$ is $(M,k)-$quasi paranormal operator.

□

Theorem 8. *If $T \in L(H)$ is a regular $(M,k)-$quasi paranormal operator, then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{1}{\sqrt{M}\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\| \cdot \|T^{k-1}\|}} \leq |\lambda| \leq \|T\|\}.$$

Proof. Suppose that T is a regular $(M,k)-$quasi paranormal operator. For every unit vector x in Hilbert space $\mathcal{H}$, we have:

$$\begin{aligned}
&\|x\|^2 = \|T^{-k-1} \cdot T^{k+1}x\|^2 \\
&\leq \|T^{-k-1}\|^2 \cdot \|T^{k+1}x\|^2 \\
&\leq \|T^{-k-1}\|^2 \cdot M \cdot \|T^{k+2}x\| \cdot \|T^kx\| \\
&\leq M \cdot \|T^{-k-1}\|^2 \cdot \|T^{k+1}\| \cdot \|Tx\| \cdot \|T^{k-1}\| \cdot \|Tx\| \\
&= M \cdot \|T^{-k-1}\|^2 \cdot \|T^{k+1}\| \cdot \|T^{k-1}\| \cdot \|Tx\|^2.
\end{aligned}$$

So,

$$1 \leq M \cdot \|T^{-k-1}\|^2 \cdot \|T^{k+1}\| \cdot \|T^{k-1}\| \cdot \|Tx\|^2,$$

where we have

$$\|Tx\| \geq \frac{1}{\sqrt{M}\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\| \cdot \|T^{k-1}\|}}.$$

Now, assume that $\lambda \in \sigma_a(T)$, then there exists a sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to +\infty$.

From the last inequation we have:

$$\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda| \cdot \|x_n\| \geq \frac{1}{\sqrt{M}\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\| \cdot \|T^{k-1}\|}} - |\lambda|.$$

Now, when $n \to +\infty$ we have

$$|\lambda| \geq \frac{1}{\sqrt{M}\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\| \cdot \|T^{k-1}\|}}.$$

So, we have

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{1}{\sqrt{M}\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\| \cdot \|T^{k-1}\|}} \leq |\lambda| \leq \|T\|\}.$$

□

3. Matrix Representation of $(M, k)-$quasi paranormal operators

In this section we represent some results for the matrix representation of $(M, k)-$quasi paranormal operators.

Theorem 9. *Let $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is a $M-$paranormal operator, then T is a $(M, k)-$quasi paranormal operator.

Proof. Let $D = M^2A^{*2}A^2 - 2\lambda A^*A + \lambda^2$. Similarly as [9, Proposition 9] we have:

$$\begin{aligned}
T^* &= \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}, \\
T^{*(k+2)} &= \begin{pmatrix} A^{*(k+2)} & 0 \\ B^*A^{*(k+1)} & 0 \end{pmatrix}, \\
T^{(k+2)} &= \begin{pmatrix} A^{(k+2)} & A^{(k+1)}B \\ 0 & 0 \end{pmatrix}, \\
T^{*(k+2)}T^{(k+2)} &= \begin{pmatrix} A^{*(k+2)}A^{(k+2)} & A^{*(k+2)}A^{(k+1)}B \\ B^*A^{*(k+1)}A^{(k+2)} & B^*A^{*(k+1)}A^{(k+1)}B \end{pmatrix}.
\end{aligned}$$

After some calculations, we have:

$$\begin{aligned}
&T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \\
=&M^2T^{*(k+2)}T^{(k+2)} - 2\lambda T^{*(k+1)}T^{(k+1)} + \lambda^2T^{*k}T^k \\
=&\begin{pmatrix} A^{*k}DA^k & A^{*k}DA^{(k-1)}B \\ B^*A^{*(k-1)}DA^k & B^*A^{*(k-1)}DA^{(k-1)}B \end{pmatrix}
\end{aligned}$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (T^{*(k+2)}T^{(k+2)} - 2\lambda T^{*(k+1)}T^{(k+1)} + \lambda^2 T^{*k}T^k)u, u\rangle \\
&= \langle A^{*k}DA^k x, x\rangle + \langle A^{*k}DA^{(k-1)}By, x\rangle \\
&+ \langle B^*A^{*(k-1)}DA^k x, y\rangle + \langle B^*A^{*(k-1)}DA^{(k-1)}By, y\rangle \\
&= \langle DA^k x, A^k x\rangle + \langle DA^{(k-1)}By, A^k x\rangle \\
&+ \langle DA^k x, A^{(k-1)}By\rangle + \langle DA^{(k-1)}By, A^{(k-1)}By\rangle \\
&= \langle D(A^k x + A^{(k-1)}By), (A^k x + A^{(k-1)}By)\rangle \geq 0
\end{aligned}$$

because A is a $M-$paranormal operator then, $D = A^{*2}A^2 - 2A^*A + I \geq 0$, so this proves the result.

□

Theorem 10. *If T^k does not have a dense range, then the following statements are equivalent:*

(i) Operator T is a $(M,k)-$quasi paranormal operator, for a non negative integer k;

(ii) $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where A is a $M-$paranormal operator on $\overline{T^k(\mathcal{H})}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. (1) ⇒ (2) Similarly as [9, Proposition 10].

(2) ⇒ (1) Suppose that $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where A is a $M-$paranormal operator on $\overline{T^k(\mathcal{H})}$, and $C^k = 0$.

A simple calculation shows that:

$$\begin{aligned}
T^* &= \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}, \\
T^*T &= \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B + C^*C \end{pmatrix}, \\
T^{*2}T^2 &= \begin{pmatrix} A^{*2}A^2 & A^{*2}AB + A^{*2}BC \\ B^*A^*A^2 + C^*B^*A^2 & |AB + BC|^2 + |C^2|^2 \end{pmatrix}, \\
T^{*k} &= \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* & 0 \end{pmatrix}, \\
T^k &= \begin{pmatrix} A^k & (\sum_{j=0}^{k-1} A^j BC^{k-1-j}) \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

Then, we have

$$T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k$$

$$= \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} D & A^{*2}AB + A^{*2}BC \\ B^*A^*A^2 + C^*B^*A^2 - 2\lambda B^*A & |AB+BC|^2 + |C^2|^2 - 2\lambda(B^*B + C^*C) + \lambda^2 \end{pmatrix}$$
$$\times \begin{pmatrix} A^k & \sum_{j=0}^{k-1} A^j BC^{k-1-j} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A^{*k}DA^k & A^{*k}D\sum_{j=0}^{k-1} A^j BC^{k-1-j} \\ (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* DA^k & (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* D \sum_{j=0}^{k-1} A^j BC^{k-1-j} \end{pmatrix},$$

where $D = M^2A^{*2}A^2 - 2\lambda A^*A + \lambda^2$. Let $v = x \oplus y$ be a vector in $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where $x \in \overline{T^k(\mathcal{H})}$ and $y \in \ker T^{*k}$. Then,

$$\begin{aligned}
&\left\langle T^{*k}(M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k v, v \right\rangle \\
&= \left\langle A^{*k}DA^k x, x \right\rangle \\
&+ \left\langle A^{*k}D\sum_{j=0}^{k-1} A^j BC^{k-1-j} y, x \right\rangle \\
&+ \left\langle (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* DA^k x, y \right\rangle \\
&+ \left\langle (\sum_{j=0}^{k-1} A^j BC^{k-1-j})^* D \sum_{j=0}^{k-1} A^j BC^{k-1-j} y, y \right\rangle \\
&= \left\langle D(A^k x + \sum_{j=0}^{k-1} A^j BC^{k-1-j} y), A^k x + \sum_{j=0}^{k-1} A^j BC^{k-1-j} y \right\rangle.
\end{aligned}$$

Since A is a $M-$paranormal operator we have that $D = M^2A^{*2}A^2 - 2A^*A + I \geq 0$. Therefore,

$$\left\langle T^{*k}(M^2T^{*2}T^2 - 2T^*T + I)T^k v, v \right\rangle \geq 0$$

for all $v \in \mathcal{H}$. Hence,

$$T^{*k}(M^2T^{*2}T^2 - 2T^*T + I)T^k \geq 0$$

So we have that T is a $(M,k)-$quasi paranormal operator.

□

4. Conclusion

In this paper we have introduced a new class of operators in Hilbert spaces, which we named the $(M,k)-$quasi paranormal operators. First we have proved some basic properties and also the structural and spectral properties of this class of operators. We also have given the relations of with new class of operators with other non normal classes of operators in Hilbert spaces. We also have given an example that support the theoretical approach. Our future work will be focused on studying the conditions under which composition operators and weighted composition operators on $L^2(\mu)$ spaces become $M-$quasi paranormal and $(M,k)-$quasi paranormal operators, in terms of Radon–Nikodym derivative h_m.

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