EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2349-2360
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Results on Mathai-Haubold Fuzzy Entropy

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Abstract. The concept of entropy, emerged from thermodynamics and statistical mechanics is of fundamental importance in some scientific and technological areas such as communication theory, physics, probability and statistics. Fuzzy entropy is much looked upon concept for measuring fuzzy information. The concept of fuzzy entropy was firstly mentioned by Zadeh way back in 1965 as a measure of fuzziness. In this paper , we introduce Mathai - Haubold fuzzy entropy with the proof of its validity. In addition, the elegant properties are studied of the proposed fuzzy entropy measure.

2020 Mathematics Subject Classifications: 62B10 ,62R07,60G35, 62N05

Key Words and Phrases: Entropy, Generalized Entropy, Fuzzy Sets, Fuzzy Entropy, Shannon Entropy, Renyi Entropy, Generalized Fuzzy Entropy, Mathai-Haubold Entropy

1. Introduction

Theory of information grew from the invention of telegraphs and telephones. To dealt with the problems related to transmission of signal, many researchers contributed in this field. Initially, Harry Nyquist [12],[13] given a formula to calculate the rate of finite bandwidth in noiseless channel. Later on, Hartley [5] established the measure of information. This measure is then modified by Claude Shannon [15], which is known as Shannon's Entropy. Shannon observed that the amount of information sent by a signal is inversely proportional to the size of the message. In probability distribution, entropy takes maximum value when all probabilities are equal. The word entropy is defined as a measure of uncertainty in a probability distribution. The concept of entropy is widely used in many engineering applications such as clustering, image processing, statistical mechanics, finance, neural networks, pattern recognition, in medical images for cells counting, fuzzy clustering, speech recognition etc.

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5304

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2. Properties of Entropy Function

In this section, we have covered some basic concepts like entropy, fuzzy sets and fuzzy entropy which are required for the proposed work.

2.1. Basic Concepts of Entropy

Shannon's entropy for a discrete random variable $Q = \{q_1, q_2, ..., q_n\}$ is defined as

$$D(Q) = -\sum_{k=1}^{n} q_k log q_k \tag{1}$$

where q_k is the probability associated with the event e_k, for $k = 1, 2, ..., n$
Properties of Entropy Function:
a) Continuity: Entropy function $D(Q)$ should be continuous.That means for every independent variable $0 \leq q_k \leq 1$, entropy function must be continuous.
b) Symmetry: Entropy function $D(Q)$ remains unchanged when $q_1, q_2, ..., q_n$ are interchanged with each other.
c) Maximality: Entropy function $D(Q)$ is maximum when all probabilities are equal.
d) Additivity Property: This property of $D(Q)$ states that if a particular event x_n with probability q_n is divided into m mutually exclusive subsets say $e_1, e_2, ..., e_m$ with probabilities $r_1, r_2, ..., r_m$ such that $q_n = r_1 + r_2 + ... + r_m$ then
$D(q_1, q_2, ...q_n, r_1, r_2, ..., r_n) = D(q_1, q_2, ..., q_{n-1}) + q_n D(\frac{r_1}{q_1}, \frac{r_2}{q_2}, ..., \frac{r_n}{q_n})$
After the Shannon's entropy measure, some of the listed generalizations were seen.
a) Renyi entropy [14] of order α

$$D^{\alpha}(Q) = \frac{1}{1-\alpha} log \sum_{k=1}^{n} q_k^{\alpha}, \alpha \neq 1, \alpha > 0 \tag{2}$$

b) Havrda-Charvat [6] entropy of order α

$$D^{\alpha}(Q) = \frac{1}{2^{1-\alpha} - 1} \sum_{k=1}^{n} q_k^{\alpha} - 1, \alpha \neq 1, \alpha > 0 \tag{3}$$

c) Tsallis entropy [16] of order α

$$D^{\alpha}(Q) = \frac{1}{1-\alpha} \sum_{k=1}^{n} q_k^{\alpha} - 1, \alpha \neq 1, \alpha > 0 \tag{4}$$

d) Mathai-Haubold entropy [11] of order α

$$D^{\alpha}(Q) = \frac{1}{\alpha - 1} \sum_{k=1}^{n} q_k^{2-\alpha} - 1, \alpha \neq 1, -\infty < \alpha < 2 \tag{5}$$

$$D^{\alpha}(Q) = \frac{1}{\alpha - 1} log \sum_{k=1}^{n} q_k^{2-\alpha}, \alpha \neq 1, -\infty < \alpha < 2 \tag{6}$$

As $\alpha \to 1$, above all the equations from (2) to (6) reduces to Shannon's entropy. Hence these are known as generalized entropies of order α.

2.2. Fuzzy Set

The concept of Fuzzy set theory of probability theory was proposed by Lofti Zadeh [18], which achieved a big success in various fields. Zadeh introduced the concept of fuzzy entropy as a measure of uncertainty due to the fuzziness in information. Kapur [8] argued that the fuzzy entropy measures uncertainty due to fuzziness of information, while probabilistic entropy measures uncertainty due to the information available in terms of probability distribution only.
Fuzzy set is an extension of classical set, which is defined as $B = \{(x, \eta_B(x)/x \in X\}$ with the membership function of B as $\eta_B : X \to [0, 1]$. The membership value gives the degree of belongingness of an element $x \in B$. Here the end values 0 and 1 gives no membership and full membership respectively. The membership function $\eta_B(x)$ is defined as follows:

$$\eta_B(x) = \begin{cases} 0 & \text{if } x \notin B \text{ and no ambiguity,} \\ 1 & \text{if } x \in B \text{ and no ambiguity,} \\ 0.5 & \text{if max ambiguity, } x \in B \text{ or not.} \end{cases} \tag{7}$$

Fuzzy set operations are the generalizations of crisp set operations. Some operations on fuzzy sets, which are required for our discussion, are as follows:
a) **Union of fuzzy sets:** Let R, S, T be fuzzy sets of universe of discourse Y, then union operation is defined as:

$$R \cup S = max(\eta_R(x), \eta_S(x)) \tag{8}$$

$$((R \cup S) \cup T) = \{y \in Y, (y, max(max(\eta_R(y), \eta_S(y)), \eta_T(y))\} \tag{9}$$

b) **Intersection of fuzzy sets:** Let R, S, T be fuzzy sets of universe of discourse Y, then intersection operation is defined as:

$$R \cap S = min(\eta_R(x), \eta_S(x)) \tag{10}$$

$$((R \cap S) \cap T) = \{y \in Y, (y, min(min(\eta_R(y), \eta_S(y)), \eta_T(y))\} \tag{11}$$

c) **Complement of Fuzzy set:** Let R be a fuzzy set, complement of R is defined as $\eta_{R^c}(x) = 1 - \eta_R(x)$.

2.3. Fuzzy Entropy

Entropy of a fuzzy set, as the probability measure of fuzzy information is defined by Zadeh [17] and it is given as follows:

$$D(B) = -\sum_{k=1}^{n} \eta_B(x_k) p_k log_d(p_k) \tag{12}$$

where η_B represent the membership function of B and $X= \{x_1, x_2, ..., x_n\}$ be a discrete random variable with probability distribution $\{p_1, p_2, ..., p_n\}$. In (12), when the base value $d = 2$, then the entropy measure is called as bit, for $d = 10$, it is known as Heartley and for $d = e$, it is called as nat. Usually in the communication system the sourcecode is converted to bit and hence we use logarithm to the base 2. In fuzzy set theory each element is associated with the degree of membership, these membership values are lies between 0 and 1 but they are not probabilities as their sum is not equal to 1. Hence Kauffman [9] defined a fuzzy entropy of set B as

$$D(B) = -\frac{1}{logn}\sum_{k=1}^{n} \psi_B(x_k) log(\psi_B(x_k)) \tag{13}$$

where $\psi_B(x_k) = \frac{\eta_B(x_k)}{\sum_{k=1}^{n} \eta_B(x_k)}$ is a probability distribution. It means that Fuzzy entropy is nonprobabilistic entropy. A measure of fuzziness $H(B)$ in a fuzzy set should have the following four properties:
a) $D(B) = 0$ if and only if B is a crisp set.
For $\eta_B(x_i) = 0$ or $\eta_B(x_i) = 1$, the value of $D(B)$ is zero.
b) $D(B)$ is maximum if B is most fuzzy set.
If $\eta_B(x_i) = 0.5$ then $D(B)$ takes the maximum value.
c) $D(B^*) > D(B)$ where B^* is a sharpened version of B.
d) $D(B^c)$ $=D(B)$ where B^c is the complement of fuzzy set B
As $\eta_B(x_i)$ and $1 - \eta_B(x_i)$ have same membership value, taking this into account De Luca and Termini [10] introduced a new measure of Fuzzy entropy corresponding to Shannon's entropy

$$D(B) = -\sum_{k=1}^{n} \eta_B(x_k) log(\eta_B(x_k)) + (1 - \eta_B(x_k)) log(1 - \eta_B(x_k)) \tag{14}$$

Equation (14) satisfies all the four properties (a) to (d) , hence it is a valid measure of fuzzy entropy. Later on Bhandari and Pal [2] and J. Kapur [8] suggested the following measure of fuzzy entropy

$$D(B) = -\sum_{i=1}^{n} [\eta_B(x_i) log(\eta_B(x_i))^\alpha + (1 - \eta_B(x_i)) log(1 - \eta_B(x_i))^\alpha] \tag{15}$$

and

$$D(B) = -\sum_{i=1}^{n} (\eta_B(x_i)^\alpha + (1 - \eta_B(x_i))^\alpha - 1) \tag{16}$$

respectively.

In recent years, many researchers [3],[4],[7],[11],[1],[8] etc. have studied and introduced several generalizations of fuzzy entropy measures. The remaining paper is organized as follows. In Section 2, given the basic concepts of entropy function by covering the basic terms like entropy, fuzzy sets and fuzzy entropy, required for the proposed generalization of fuzzy entropy. In Section 3, we have proposed a new parametric generalized fuzzy entropy measure corresponding to [11]. Section 4 provides some more elegant properties of the proposed measure in a number of theorems. Finally some concluding remarks in Section 5.

3. Generalized Fuzzy Entropy of order α

Here we propose a new generalized fuzzy entropy measure of Mathai -Haubold entropy corresponding to [11] and checked it's validity.

Definition 1. *Let B be the fuzzy set defined on $X = \{x_1, x_2, ..., x_n\}$ with the membership values $\eta_B(x_i)$ for $i = 1, 2, ..., n$ then the generalized fuzzy entropy of order α is defined as*

$$M\alpha(B) = \frac{1}{n(2^{\alpha-1} - 1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1], \alpha \neq 1, 0 < \alpha < 2 \quad (17)$$

Theorem 1. *$M\alpha(B)$ is a valid measure of fuzzy entropy.*

Proof. To show $M\alpha(B)$ a valid fuzzy entropy measure.
a) To check $M\alpha(B) = 0$ if and only if B is a crisp set. that is for $\eta_B(x_i) = 0$ or $\eta_B(x_i) = 1$, the value of $M\alpha(B)$ is zero. If $\eta_B(x_i) = 0$ then the equation (16)

$$M\alpha(B) = \frac{1}{n(2^{\alpha-1} - 1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1] \quad (18)$$

is equal to zero.
If $\eta_B(x_i) = 1$ then ,

$$M\alpha(B) = \frac{1}{n(2^{\alpha-1} - 1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1] = 0 \quad (19)$$

which is a minimum. Conversely if ,

$$M\alpha(B) = \frac{1}{n(2^{\alpha-1} - 1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1] = 0 \quad (20)$$

then easily we get $\eta_B(x_i) = 0$ or $\eta_B(x_i) = 1$.
Therefore $M_\alpha(B) = 0$ if and only if when B is a crisp set.
b) To show the extremality condition that is to show $M_\alpha(B)$ is maximum if and only if B is the most fuzzy set. Consider the earlier equation (16),

$$M\alpha(B) = \frac{1}{n(2^{\alpha-1} - 1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1] \quad (21)$$

Table 1: Entropy $M_\alpha(B)$ at different values of α.

$\eta_B(x_i)$	$M_\alpha(B)$			
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1.1$	$\alpha = 1.5$
0	0	0	0	0
0.1	0.3911	0.4148	0.4936	0.6396
0.2	0.6658	0.6839	0.7381	0.8248
0.3	0.8536	0.8628	0.8889	0.9280
0.4	0.9637	0.9661	0.9729	0.9827
0.5	1.0000	1.0000	1.0000	1.0000
0.6	0.9637	0.9661	0.9729	0.9827
0.7	0.8536	0.8628	0.8889	0.9280
0.8	0.6658	0.6839	0.7381	0.8248
0.9	0.3911	0.4148	0.4936	0.6396
1.0	0	0	0	0

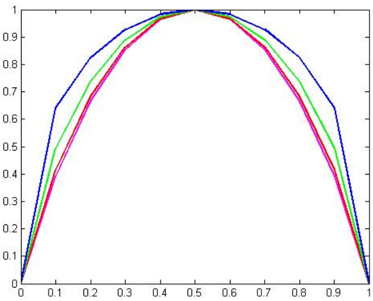

Figure 1: Entropy at different parametric values

differentiating it partially with respect to $\eta_B(x_i)$, we get,

$$\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)} = \frac{2-\alpha}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_B(x_i)^{1-\alpha} - (1-\eta_B(x_i))^{1-\alpha}] \tag{22}$$

Case 1] When $0 \leq \eta_B(x_i) \leq 0.5$ and $\alpha < 0, \alpha \neq 1, \alpha < 2$ then $\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)}$ is positive. (Refer Table 2)

Case 2] When $0.5 \leq \eta_B(x_i) \leq 1$ and $\alpha < 0, \alpha \neq 1, \alpha < 2$ then $\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)}$ is negative.(Refer Table 2)

Case 3] When $\eta_B(x_i) = 0.5$ that is, if B is a most Fuzzy set then

$\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)} = \frac{2-\alpha}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [(0.5)^{1-\alpha} - (1-0.5)^{1-\alpha}]$
Therefore, the value of $\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)}$ becomes zero. Hence $M\alpha(B)$ is an increasing function of $\eta_B(x_i)$ satisfying $0 \leq \eta_B(x_i) \leq 0.5$ and decreasing function for $0.5 \leq \eta_B(x_i) \leq 1$. Also at

$\eta_B(x_i) = 0.5$, the fuzzy entropy function vanishes. Therefore $M\alpha(B)$ is a concave function and has a global maximum at $x = 0.5$. Thus $M\alpha(B)$ is maximum if and only if B is the most fuzzy set.
c) Sharpness : Let B^* be a sharpened version of B , i.e.
(i) If $\eta_B(x_i) < 0.5$, then $\eta_B^*(x_i) \leq \eta_B(x_i)$
(ii) If $\eta_B(x_i) > 0.5$, then $\eta_B^*(x_i) \geq \eta_B(x_i)$
Since $M\alpha(B)$ is increasing function in the interval $0 \leq \eta_B(x_i) \leq 0.5$ and decreasing function in the interval $0.5 \leq \eta_B(x_i) \leq 1$ thus
$\eta_B^*(x_i) \leq \eta_B(x_i) \implies M\alpha^*(B) \leq M\alpha(B)$ in $[0, 0.5]$ and
$\eta_B^*(x_i) \geq \eta_B(x_i) \implies M\alpha^*(B) \geq M\alpha(B)$ in $[0.5, 1]$
Hence $M\alpha^*(B) \leq M\alpha(B)$
d) Symmetry: Since $\eta(x_i)^c = 1 - \eta(x_i)$ hence it is trivial to show that $M\alpha(B^c) = M\alpha(B)$.
Hence all the four properties of fuzzy entropy measure are satisfied by $M\alpha(B)$. Therefore it is a valid fuzzy entropy measure.

Table 2: Partial derivative $\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)}$.

$\eta_B(x_i)$	$\frac{\partial M\alpha(B)}{\partial \eta_B(x_i)}$							
	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.9$
0	4.2288	5.1213	6.9242	16.4260	∞	∞	∞	∞
0.1	3.2168	3.2390	3.2384	3.2062	3.1140	2.9168	0.2313	0.7902
0.2	2.3705	2.2903	2.2034	2.0794	1.9061	1.6699	0.1227	0.3504
0.3	1.5650	1.4797	1.3965	1.2877	1.1490	0.9755	0.0692	0.1821
0.4	0.7785	0.7280	0.6804	0.6202	0.5461	0.4566	0.0318	0.0805
0.5	0	0	0	0	0	0	0	0
0.6	-0.7785	-0.7280	-0.6804	-0.6202	-0.5461	-0.4566	-0.0318	-0.0805
0.7	-1.5650	-1.4797	-1.3965	-1.2877	-1.1490	-0.9755	-0.0692	-0.1821
0.8	-2.3705	-2.2903	-2.2034	-2.0794	-1.9061	-1.6699	-0.1227	-0.3504
0.9	-3.2168	-3.2390	-3.2384	-3.2062	-3.1140	-2.9168	-0.2313	-0.7902
1.0	-4.2288	-5.1213	-6.9242	-16.4260	$-\infty$	$-\infty$	$-\infty$	$-\infty$

Example 1. *Let A be the fuzzy set defined as $A = \{(1, 0.2), (2, 0.8), (3, 0.5), (4, 0.7), (5, 0.3)\}$ and $\alpha = 0.2$. Then the value of proposed generalized measure of entropy function $M\alpha(A)$ is*

$M\alpha(A) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1]$
$M\alpha(A) = \frac{1}{5*(2^{0.2-1}-1)} \sum_{i=1}^{5} ([0.2^{1.8} + 0.8^{1.8} - 1] + [0.8^{1.8} + 0.2^{1.8} - 1] + [0.5^{1.8} + 0.5^{1.8} - 1]$

$+[0.7^{1.8} + 0.3^{1.8} - 1] + [0.3^{1.8} + 0.7^{1.8} - 1])$
$= 0.7966084$

Example 2. *Consider a set $X = \{2, 4, 7\}$ and a fuzzy set B on X which is defined as $B = \{(2, 0.4), (4, 0.6), (7, 0.1)\}$. Evaluate entropy function $M\alpha(A)$ by taking $\alpha = 0.6$.*

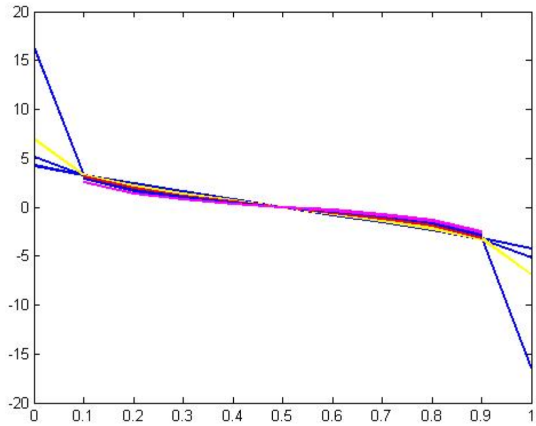

Figure 2: Partial derivative at different parametric values

Solution:
$M\alpha(A) = \frac{1}{3*(2^{0.6-1}-1)} \sum_{i=1}^{3}([0.4^{1.4} + 0.6^{1.4} - 1] + [0.6^{1.4} + 0.4^{1.4} - 1] + [0.1^{1.4} + 0.9^{1.4} - 1])$

$= 0.77720$

4. Some properties of $M\alpha(B)$ of order α

The proposed generalized measure of fuzzy entropy of order α has the following properties:

Theorem 2. *For any two fuzzy sets R and S of universe of discourse X,*
$M\alpha(R \cup S) + M\alpha(R \cap S) = M\alpha(R) + M\alpha(S)$

Proof. Let us divide the set X into two sets as :
$X_+ = \{x/x \in X, \eta_A(x_i) \geq \eta_B(x_i)\}$
$X_- = \{x/x \in X, \eta_A(x_i) < \eta_B(x_i)\}$
where $\eta_R(x_i)$ and $\eta_S(x_i)$ are the fuzzy membership values of R and S respectively. Therefore,
$M\alpha(R \cup S) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_{R\cup S}(x_i)^{2-\alpha} + (1 - \eta_{R\cup S}(x_i))^{2-\alpha} - 1]$
using X_+ the value of $M\alpha(R \cup S)$ becomes
$M\alpha(R \cup S) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_R(x_i)^{2-\alpha} + (1 - \eta_R(x_i))^{2-\alpha} - 1]$
also
$M\alpha(R \cap S) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_{R\cup S}(x_i)^{2-\alpha} + (1 - \eta_{R\cup S}(x_i))^{2-\alpha} - 1]$
using X_- we get,
$M\alpha(R \cap S) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_R(x_i)^{2-\alpha} + (1 - \eta_S(x_i))^{2-\alpha} - 1]$
$M\alpha(R \cup S) + M\alpha(R \cap S)$
$= \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_{R\cup S}(x_i)^{2-\alpha} + (1 - \eta_{R\cup S}(x_i))^{2-\alpha} - 1] +$
$\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n}[\eta_{R\cap S}(x_i)^{2-\alpha} + (1 - \eta_{R\cap S}(x_i))^{2-\alpha} - 1]$

$=M_\alpha(R) + M\alpha(S)$
Hence the result is proved.

Corollary 1. *For any fuzzy set R in a universe of discourse X and R^c be the complement of fuzzy set then*
$M_\alpha(R) = M_\alpha(R^c) = M_\alpha(R \cup R^c) = M_\alpha(R \cap R^c)$

Proof. The proof is trivially follows from Theorem 2.

Theorem 3. *For a fuzzy set R, S, T of set X, $M_\alpha(B)$ satisfies the following properties:*
(i) $M_\alpha((R \cup S) \cup T) = M_\alpha(R \cup (S \cup T))$
(ii) $M_\alpha((R \cap S) \cap T) = M_\alpha(R \cap (S \cap T))$
(iii) $M_\alpha(R \cup S) = M_\alpha(R^c \cap S^c)^c$
(iv) $M_\alpha(R \cap S) = M_\alpha(R^c \cup S^c)^c$

Proof. Let $X_p = \{x/x \in X, \eta_R(x_i) \geq \eta_S(x_i) \geq \eta_T(x_i)\}$ and

$X_q = \{x/x \in X, \eta_R(x_i) < \eta_S(x_i) < \eta_T(x_i)\}$

(i) To show $M_\alpha((R \cup S) \cup T) = M_\alpha(R \cup (S \cup T))$

Consider the left hand side:

$M_\alpha(R \cup S)=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{R \cup S}(x_i)^{2-\alpha} + (1 - \eta_{R \cup S}(x_i))^{2-\alpha} - 1]$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_R(x_i)^{2-\alpha} + (1 - \eta_R(x_i))^{2-\alpha} - 1]$

$M_\alpha((R \cup S) \cup T)$
$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{R \cup T}(x_i)^{2-\alpha} + (1 - \eta_{R \cup T}(x_i))^{2-\alpha} - 1]$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_R(x_i)^{2-\alpha} + (1 - \eta_R(x_i))^{2-\alpha} - 1]$

Now consider the right hand side:

$M_\alpha(R \cup (S \cup T))$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{(R \cup (S \cup T))}(x_i)^{2-\alpha} + (1 - \eta_{(R \cup (S \cup T))}(x_i))^{2-\alpha} - 1]$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_R(x_i)^{2-\alpha} + (1 - \eta_R(x_i))^{2-\alpha} - 1]$

Hence $M_\alpha((R \cup S) \cup T) = M_\alpha(R \cup (S \cup T))$

(ii) Similarly , associativity property holds for intersection also.

(iii) To show that $M_\alpha(R \cup S) = M_\alpha(R^c \cap S^c)^c$

Consider $M_\alpha(R^c \cap S^c)$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{R^c \cup S^c}(x_i)^{2-\alpha} + (1 - \eta_{R^c \cup S^c}(x_i))^{2-\alpha} - 1]$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{R \cap S}(x_i)^{2-\alpha} + (1 - \eta_{R \cap S}(x_i))^{2-\alpha} - 1]$

Now $M_\alpha(R^c \cap S^c)^c$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta^c_{R \cap S}(x_i)^{2-\alpha} + (1 - \eta^c_{R \cap S}(x_i))^{2-\alpha} - 1]$

$=\frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_{R \cup S}(x_i)^{2-\alpha} + (1 - \eta_{R \cup S}(x_i))^{2-\alpha} - 1]$

$=M_\alpha(R \cup S)$

Hence proved. Exactly in the similar way property (iv) can be proved.

Theorem 4. *$M_\alpha(B)$ attains the maximum when B is most fuzzy set and attains minimum when B is least fuzzy set and it is independent of order α.*

Proof. In Theorem no 1, it was already proved that $M_\alpha(B)$ is maximum if and only if $\eta_B(x_i) = 0.5$ that means B is most fuzzy set and minimum when B is a crisp set. Now to prove that both these results are independent of α. Let B is most fuzzy set therefore put $\mu_B(x_i) = 0.5$ in the following equation.

$M\alpha(B) = \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [\eta_B(x_i)^{2-\alpha} + (1 - \eta_B(x_i))^{2-\alpha} - 1]$

$= \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [0.5^{2-\alpha} + (1 - 0.5)^{2-\alpha} - 1]$

$= \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [0.5^{2-\alpha} + (0.5)^{2-\alpha} - 1]$

$= \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [2\frac{1}{2}^{2-\alpha} - 1]$

$= \frac{1}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [2^{\alpha-1} - 1]$

$= \frac{[2^{\alpha-1}-1]}{n(2^{\alpha-1}-1)} \sum_{i=1}^{n} [1]$

$= 1.$

which is independent of α.

On the other hand when B is least fuzzy set.That is B is a crisp set then $\eta_B(x_i) = 0$ or $\eta_B(x_i) = 1$ then $M\alpha(B) = 0$ which is again independent of α.

Hence the theorem is proved.

5. Conclusion

In this paper, we have reviewed the concept of entropy in information theory for discrete random variable and studied several generalizations of Shannon entropy. A brief introduction about fuzzy sets and a journey from entropy to fuzzy entropy is discussed. Numerical examples are provided for understanding the concept of proposed fuzzy entropy measure. We have proposed a new parametric generalized fuzzy entropy measure of Mathai-Haubold entropy and given the proof of validation. The particular cases have been discussed in detail along with some of the properties of this fuzzy entropy measure. For the future study, we will propose a new parametric generalizations of parametric fuzzy entropy, a new divergence measures, total ambiguity and fuzzy improvement information measures.

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