EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2235-2245
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

ε-Łukasiewicz Fuzzy UP (BCC)-Subalgebras of UP (BCC)-Algebras

Aiyared Iampan[1,*], Ramasamy Subasini[2], Neelamegarajan Rajesh[3]

[1] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[2] *Department of Mathematics, Pollachi Institute of Engineering and Technology, Pollachi-642205, Tamilnadu, India*
[3] *Department of Mathematics, Rajah Serfoji Government College (affiliated to Bharathidasan University), Thanjavur-613005, Tamilnadu, India*

Abstract. The idea of Łukasiewicz t-norm is used to construct the concept of ε-Łukasiewicz fuzzy sets based on a given fuzzy set. The ε-Łukasiewicz fuzzy sets are applied to UP (BCC)-algebras. Moreover, the notion of ε-Łukasiewicz fuzzy UP (BCC)-subalgebras is introduced, and its various properties are investigated. Three subsets, so-called $\in$-set, q-set, and O-set, are constructed, and the conditions under which they can be UP (BCC)-subalgebras are explored.

2020 Mathematics Subject Classifications: 03G25; 08A72

Key Words and Phrases: UP (BCC)-algebra, ε-Łukasiewicz fuzzy set, ε-Łukasiewicz fuzzy UP (BCC)-subalgebra, $\in$-set, q-set, O-set.

---

1. Introduction

Zadeh [12] first proposed the idea of fuzzy sets. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [1–3]. The new technology allows very complex inferences about variations on a theme to be anticipated and fixed in a program. Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Łukasiewicz as a three-valued logic. Iampan [7] introduced a new algebraic structure called UP-algebra. Somjanta et al. [11] and Guntasow et al. [5] applied fuzzy set theory in UP-algebras. Dokkhamdang et al. [4] introduced the notion of fuzzy UP-subalgebras

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5311

Email addresses: aiyared.ia@up.ac.th (A. Iampan), subasinimaths@gmail.com (R. Subasini), nrajesh_topology@yahoo.co.in (N. Rajesh)

 © 2024 EJPAM All rights reserved.

with thresholds of UP-algebras. The concepts of UP-algebras (see [7]) and BCC-algebras (see [9]) are the same concept, as shown by Jun et al. [8] in 2022. In this publication and following investigations, our research team will refer to it as BCC rather than UP out of respect for Komori, who first characterized it in 1984.

In this paper, using the idea of Łukasiewicz t-norm, we construct the concept of ε-Łukasiewicz fuzzy sets based on a given fuzzy set and apply it to BCC-algebras. We define the concepts of ε-Łukasiewicz fuzzy BCC-subalgebras and investigate several properties. We provide conditions for an ε-Łukasiewicz fuzzy set to be an ε-Łukasiewicz fuzzy BCC-subalgebra. We discuss the characterizations of ε-Łukasiewicz fuzzy BCC-subalgebras. We construct three kinds of subsets, so-called $\in$-set, q-set, and O-set, and we find the conditions under which they can be BCC-subalgebras.

2. Preliminaries

The concept of BCC-algebras (see [9]) can be redefined without the condition (2.6) as follows:

An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see [6]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0) \tag{2.1}$$
$$(\forall x \in X)(0 * x = x) \tag{2.2}$$
$$(\forall x \in X)(x * 0 = 0) \tag{2.3}$$
$$(\forall x, y \in X)(x * y = 0 = y * x \Rightarrow x = y) \tag{2.4}$$

After this, we assign X instead of a BCC-algebra $(X, *, 0)$ until otherwise specified. We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [7]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z * x \leq z * y) \tag{2.8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y * z \leq x * z) \tag{2.9}$$
$$(\forall x, y, z \in X)(x \leq y * x, \text{ in particular, } y * z \leq x * (y * z)) \tag{2.10}$$
$$(\forall x, y \in X)(y * x \leq x \Leftrightarrow x = y * x) \tag{2.11}$$
$$(\forall x, y \in X)(x \leq y * y) \tag{2.12}$$
$$(\forall a, x, y, z \in X)(x * (y * z) \leq x * ((a * y) * (a * z))) \tag{2.13}$$
$$(\forall a, x, y, z \in X)(((a * x) * (a * y)) * z \leq (x * y) * z) \tag{2.14}$$
$$(\forall x, y, z \in X)((x * y) * z \leq y * z) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z * y) \tag{2.16}$$

$$(\forall x, y, z \in X)((x * y) * z \leq x * (y * z)) \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x * y) * z \leq y * (a * z)) \tag{2.18}$$

Definition 1. *[7] A nonempty subset S of X is called a BCC-subalgebra of X if it satisfies the following properties:*

$$(\forall x, y \in S)(x * y \in S) \tag{2.19}$$

A *fuzzy set* [12] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2. *[11] A fuzzy set μ in X is called a fuzzy BCC-subalgebra of X if it satisfies the following property:*

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}). \tag{2.20}$$

A fuzzy set μ in a set X of the form

$$\mu(x) = \begin{cases} t \in (0, 1] & \text{if } x = a \\ 0 & \text{if } x \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set μ in a set X, we say that a fuzzy point $[a/t]$ is

(1) contained in μ, denoted by $[a/t] \in \mu$, (see [10]) if $\mu(a) \geq t$,

(2) quasi-coincident with μ, denoted by $[a/t]q\mu$, (see [10]) if $\mu(a) + t > 1$.

Proposition 1. *If μ is a fuzzy set in a set X and $\varepsilon \in (0, 1)$, then its ε-Łukasiewicz fuzzy set L^ε_μ satisfies the following property:*

(1) $(\forall x, y \in X)(\mu(x) \geq \mu(y) \Rightarrow L^\varepsilon_\mu(x) \geq L^\varepsilon_\mu(y))$

(2) $(\forall x \in X)([x/\varepsilon]q\mu \Rightarrow L^\varepsilon_\mu(x) = \mu(x) + \varepsilon - 1)$

(3) $(\forall x \in X, \forall \delta \in (0, 1))(\varepsilon \geq \delta \Rightarrow L^\varepsilon_\mu(x) \geq L^\delta_\mu(x))$

3. ε-Łukasiewicz fuzzy BCC-subalgebra of a BCC-algebra

In this section, we will recall the definition of ε-Łukasiewicz fuzzy sets and introduce a new concept called ε-Łukasiewicz fuzzy BCC-subalgebras.

Definition 3. *Let μ be a fuzzy set in a set X and let $\varepsilon \in [0, 1]$. A function $L^\varepsilon_\mu : X \to [0, 1]$; $x \mapsto \max\{0, \mu(x) + \varepsilon - 1\}$ is called an ε-Łukasiewicz fuzzy set of μ in X.*

Definition 4. *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is called an ε-Łukasiewicz fuzzy BCC-subalgebra of X if it satisfies the following property:*

$$(\forall x, y \in X, \forall t_a, t_b \in (0,1])([x/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}) \quad (3.1)$$

Theorem 1. *If μ is a fuzzy BCC-subalgebra of X, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.*

Proof. Assume that μ is a fuzzy BCC-subalgebra of X. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$. Thus

$$\begin{aligned} L^{\varepsilon}_{\mu}(x * y) &= \max\{0, \mu(x * y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{\mu(x), \mu(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, \mu(x) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} \\ &= \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \\ &\geq \min\{t_a, t_b\}. \end{aligned}$$

Hence, $[(x*y)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Therefore, L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.

The following example shows that the converse of Theorem 1 may not be true.

Example 1. *Let $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:*

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	4	2	3	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 1.0 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.2 & \text{if } x = 2 \\ 0.3 & \text{if } x = 3 \\ 0.6 & \text{if } x = 4. \end{cases}$$

Given $\varepsilon = 0.9$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu}(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1 \\ 0.1 & \text{if } x = 2 \\ 0.2 & \text{if } x = 3. \\ 0.5 & \text{if } x = 4. \end{cases}$$

Then L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.

Theorem 2. *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is an ε-Łukasiewicz fuzzy BCC-subalgebra of X if and only if it satisfies the following property:*

$$(\forall x, y \in X)(L^{\varepsilon}_{\mu}(x * y) \geq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\}) \tag{3.2}$$

Proof. Suppose L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then $[x/L^{\varepsilon}_{\mu}(x)] \in L^{\varepsilon}_{\mu}$ and $[y/L^{\varepsilon}_{\mu}(y)] \in L^{\varepsilon}_{\mu}$. Thus, $[(x * y)/\min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\}] \in L^{\varepsilon}_{\mu}$ by (3.1), which implies that $L^{\varepsilon}_{\mu}(x * y) \geq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\}$.

Conversely, suppose that L^{ε}_{μ} satisfies the condition (3.2). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$, which implies from (3.2) that $L^{\varepsilon}_{\mu}(x*y) \geq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \geq \min\{t_a, t_b\}$. Thus, $[(x*y)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Hence, L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.

Proposition 2. *If μ is a fuzzy BCC-subalgebra of X, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} satisfies the following property:*

$$(\forall x \in X)(L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x)) \tag{3.3}$$

Proof. If μ is a fuzzy BCC-subalgebra of X, then $\mu(0) = \mu(x*x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$ for all $x \in X$. It follows from Proposition 1 (1) that $L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x)$ for all $x \in X$.

The following example shows that the converse of Proposition 2 is not true in general.

Example 2. *[5] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$*$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu : X \to [0,1]; x \mapsto \begin{cases} 1 & \textit{if } x = 0 \\ 0 & \textit{if } x = 1 \\ 1 & \textit{if } x = 2 \\ 1 & \textit{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.9$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu} : X \to [0,1]; x \mapsto \begin{cases} 0.9 & \textit{if } x = 0 \\ 0 & \textit{if } x = 1 \\ 0.9 & \textit{if } x = 2 \\ 0.9 & \textit{if } x = 3 \end{cases}$$

Then $L^{\varepsilon}_{\mu}(0) \geq= L^{\varepsilon}_{\mu}(x)$ for all $x \in X$ but μ is not a fuzzy BCC-subalgebra of X because $\mu(2 * 3) = \mu(1) = 0 \ngeq 1 = \min\{\mu(2), \mu(3)\}$.

Proposition 3. *If μ is a fuzzy BCC-subalgebra of X, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} satisfies the following property:*

$$(\forall x, y \in X)(L^{\varepsilon}_{\mu}(y) = L^{\varepsilon}_{\mu}(0) \Leftrightarrow L^{\varepsilon}_{\mu}(x * y) \geq L^{\varepsilon}_{\mu}(x)) \tag{3.4}$$

Proof. Assume that $L^{\varepsilon}_{\mu}(y) = L^{\varepsilon}_{\mu}(0)$ for all $y \in X$. Then $L^{\varepsilon}_{\mu}(x*y) \geq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} = \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(0)\} = L^{\varepsilon}_{\mu}(x)$ for all $x, y \in X$ by the combination of Theorem 1 and Proposition 2.

Conversely, suppose that $L^{\varepsilon}_{\mu}(x * y) \geq L^{\varepsilon}_{\mu}(x)$ for all $x, y \in X$. Using (2.2) induces $L^{\varepsilon}_{\mu}(y) = L^{\varepsilon}_{\mu}(0 * y) \geq L^{\varepsilon}_{\mu}(0)$. The combination of this and Proposition 2 leads to $L^{\varepsilon}_{\mu}(y) = L^{\varepsilon}_{\mu}(0)$ for all $y \in X$.

Proposition 4. *If μ is a fuzzy BCC-subalgebra of X, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} satisfies the following property:*

$$(\forall x, y \in X, \forall t_a, t_b \in (0, 1]) \left(\begin{array}{l} [x/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \\ \Rightarrow [(x * (0 * y))/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right) \tag{3.5}$$

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$. Thus

$$\begin{array}{rcl} L^{\varepsilon}_{\mu}(x * (0 * y)) & = & \max\{0, \mu(x * (0 * y)) + \varepsilon - 1\} \\ & \geq & \max\{0, \min\{\mu(x), \mu(0 * y)\} + \varepsilon - 1\} \\ & \geq & \max\{0, \min\{\mu(x), \min\{\mu(0), \mu(y)\}\} + \varepsilon - 1\} \\ & = & \max\{0, \min\{\mu(x), \mu(y)\} + \varepsilon - 1\} \\ & = & \max\{0, \min\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\ & = & \min\{\max\{0, \mu(x) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} \\ & = & \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \\ & \geq & \min\{t_a, t_b\}. \end{array}$$

Hence, $[(x * (0 * y))/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$.

We provide conditions for an ε-Łukasiewicz fuzzy set to be an ε-Łukasiewicz fuzzy BCC-subalgebra.

Theorem 3. *Let μ be a fuzzy set in X. If its ε-Łukasiewicz fuzzy set L^{ε}_{μ} satisfies the following property:*

$$[y/t_b] \in L^{\varepsilon}_{\mu}, [z/t_c] \in L^{\varepsilon}_{\mu} \Rightarrow [(x * y)/\min\{t_b, t_c\}] \in L^{\varepsilon}_{\mu} \tag{3.6}$$

for all $t_b, t_c \in (0, 1]$ and $x, y, z \in X$ with $z \leq x$, then L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Since $x \leq x$ for all $x \in X$, it follows from (3.6) that $[(x * y)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Hence, L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.

Proposition 5. *Let μ be a fuzzy set in X. Then every ε-Łukasiewicz fuzzy BCC-subalgebra L^{ε}_{μ} of X satisfies the following property:*

$$(\forall x, y \in X, \forall t_a, t_b \in (0,1])([x/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \Rightarrow [(x*(0*y))/\min\{t_b, t_c\}] \in L^{\varepsilon}_{\mu}) \tag{3.7}$$

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$. It follows from Theorem 2 and Proposition 2 that

$$\begin{aligned} L^{\varepsilon}_{\mu}(x*(0*y)) &\geq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(0*y)\} \\ &\geq \min\{L^{\varepsilon}_{\mu}(x), \min\{L^{\varepsilon}_{\mu}(0), L^{\varepsilon}_{\mu}(y)\}\} \\ &= \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \\ &\geq \min\{t_a, t_b\}. \end{aligned}$$

Hence, $[(x*(0*y))/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$.

Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X and $t \in (0,1]$, consider the sets

$$(L^{\varepsilon}_{\mu}, t)_{\in} = \{x \in X : [x/t] \in L^{\varepsilon}_{\mu}\},$$
$$(L^{\varepsilon}_{\mu}, t)_{q} = \{x \in X : [x/t] q L^{\varepsilon}_{\mu}\},$$

which are called the $\in$-set and q-set, respectively, of L^{ε}_{μ} (with value t).

We explore the conditions under which the $\in$-set and q-set of ε-Łukasiewicz fuzzy sets can be BCC-subalgebras.

Theorem 4. *Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. Then the $\in$-set $(L^{\varepsilon}_{\mu}, t)_{\in}$ of L^{ε}_{μ} with value $t \in (0.5, 1]$ is a BCC-subalgebra of X if and only if the following assertion is valid:*

$$(\forall x, y \in X)(\min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \leq \max\{L^{\varepsilon}_{\mu}(x*y), 0.5\}) \tag{3.8}$$

Proof. Assume that the $\in$-set $(L^{\varepsilon}_{\mu}, t)_{\in}$ of L^{ε}_{μ} with value $t \in (0.5, 1]$ is a BCC-subalgebra of X. If the condition (3.8) is not valid, then there exist $a, b \in X$ such that $\min\{L^{\varepsilon}_{\mu}(a), L^{\varepsilon}_{\mu}(b)\} > \max\{L^{\varepsilon}_{\mu}(a*b), 0.5\}$. If we take $s = \min\{L^{\varepsilon}_{\mu}(a), L^{\varepsilon}_{\mu}(b)\}$, then $s \in (0.5, 1]$ and $[a/s], [b/s] \in L^{\varepsilon}_{\mu}$, that is, $a, b \in (L^{\varepsilon}_{\mu}, s)_{\in}$. Since $(L^{\varepsilon}_{\mu}, s)_{\in}$ is a BCC-subalgebra of X, we have $a * b \in (L^{\varepsilon}_{\mu}, s)_{\in}$. But $[(a*b)/s] \notin L^{\varepsilon}_{\mu}$ implies $a * b \notin (L^{\varepsilon}_{\mu}, s)_{\in}$, a contradiction. Thus, $\min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \leq \max\{L^{\varepsilon}_{\mu}(x*y), 0.5\}$ for all $x, y \in X$.

Conversely, suppose that L^{ε}_{μ} satisfies the condition (3.8). Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $x \in (L^{\varepsilon}_{\mu}, t)_{\in}$ and $y \in (L^{\varepsilon}_{\mu}, t)_{\in}$. Then $L^{\varepsilon}_{\mu}(x) \geq t$ and $L^{\varepsilon}_{\mu}(y) \geq t$, which imply from (3.8) that $0.5 < t \leq \min\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\} \leq \max\{L^{\varepsilon}_{\mu}(x*y), 0.5\}$. Thus, $[(x*y)/t] \in L^{\varepsilon}_{\mu}$, that is, $x * y \in (L^{\varepsilon}_{\mu}, t)_{\in}$. So, $(L^{\varepsilon}_{\mu}, t)_{\in}$ is a BCC-subalgebra of X for $t \in (0.5, 1]$.

Theorem 5. *Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-subalgebra of X, then the nonempty q-set $(L^{\varepsilon}_{\mu}, t)_{q}$ of L^{ε}_{μ} with value $t \in (0,1]$ is a BCC-subalgebra of X.*

Proof. Let $t \in (0,1]$ and $x, y \in (L^\varepsilon_\mu, t)_q$. Then $[x/t]qL^\varepsilon_\mu$ and $[y/t]qL^\varepsilon_\mu$, that is, $L^\varepsilon_\mu(x) + t > 1$ and $L^\varepsilon_\mu(y) + t > 1$. It follows from Theorems 1 and 2 that $L^\varepsilon_\mu(x * y) + t \geq \min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\} + t = \min\{L^\varepsilon_\mu(x) + t, L^\varepsilon_\mu(y) + t\} > 1$. Thus, $[(x * y)/t]qL^\varepsilon_\mu$. So, $x * y \in (L^\varepsilon_\mu, t)_q$. Hence, $(L^\varepsilon_\mu, t)_q$ is a BCC-subalgebra of X.

The following example shows that the converse of Theorem 5 is not true in general.

Example 3. *From the BCC-algebra X in Example 2, define a fuzzy set μ as follows:*

$$\mu : X \to [0,1]; x \mapsto \begin{cases} 0.1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ 0.1 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.2$, the ε-Łukasiewicz fuzzy set L^ε_μ of μ in X is given as follows:

$$L^\varepsilon_\mu : X \to [0,1]; x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ 0 & \text{if } x = 2 \\ 0 & \text{if } x = 3 \end{cases}$$

Then the q-set $(L^\varepsilon_\mu, t)_q$ of L^ε_μ with value $t \in (0,1]$ is empty but μ is not a fuzzy BCC-subalgebra of X because $\mu(2 * 3) = \mu(1) = 0 \not\geq 0.1 = \min\{\mu(2), \mu(3)\}$.

Theorem 6. *Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L^ε_μ of μ in X, if the q-set $(L^\varepsilon_\mu, t)_q$ is a BCC-subalgebra of X, then L^ε_μ satisfies the following property:*

$$(\forall x, y \in X, \forall t_a, t_b \in (0, 0.5]) \left(\begin{array}{l} [x/t_a]qL^\varepsilon_\mu, [y/t_b]qL^\varepsilon_\mu \\ \Rightarrow [(x * y)/\max\{t_a, t_b\}] \in L^\varepsilon_\mu \end{array} \right) \tag{3.9}$$

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $[x/t_a]qL^\varepsilon_\mu$ and $[y/t_b]qL^\varepsilon_\mu$. Then $x \in (L^\varepsilon_\mu, t_a)q \subseteq (L^\varepsilon_\mu, \max\{t_a, t_b\})_q$ and $y \in (L^\varepsilon_\mu, t_b)_q \subseteq (L^\varepsilon_\mu, \max\{t_a, t_b\})_q$. Thus, $x * y \in (L^\varepsilon_\mu, \max\{t_a, t_b\})_q$. Since $\max\{t_a, t_b\} \leq 0.5$, we have $L^\varepsilon_\mu(x * y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}$. Hence, $[(x * y)/\max\{t_a, t_b\}] \in L^\varepsilon_\mu$.

Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L^ε_μ of μ in X, consider the set: $O(L^\varepsilon_\mu) = \{x \in X : L^\varepsilon_\mu(x) > 0\}$, which is called the O-set of L^ε_μ. It is observed that $O(L^\varepsilon_\mu) = \{x \in X : \mu(x) + \varepsilon - 1 > 0\}$.

Theorem 7. *Let L^ε_μ be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-subalgebra of X, then the nonempty O-set $O(L^\varepsilon_\mu)$ of L^ε_μ is a BCC-subalgebra of X.*

Proof. Let $x, y \in O(L^\varepsilon_\mu)$. Then $\mu(x) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. If μ is a fuzzy BCC-subalgebra of X, then L^ε_μ is an ε-Łukasiewicz fuzzy BCC-subalgebra of X by Theorem 1. It follows from Theorem 2 that $L^\varepsilon_\mu(x * y) \geq \min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\} = \min\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\} > 0$. Thus, $x * y \in O(L^\varepsilon_\mu)$. Hence, $O(L^\varepsilon_\mu)$ is a BCC-subalgebra of X.

Theorem 8. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X satisfies the following property:*

$$[x/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \Rightarrow [(x * y)/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu} \tag{3.10}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the nonempty O-set $O(L^{\varepsilon}_{\mu})$ of L^{ε}_{μ} is a BCC-subalgebra of X.

Proof. Assume that L^{ε}_{μ} satisfies the condition (3.10) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in O(L^{\varepsilon}_{\mu})$. Then $\mu(x) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Since $[x/L^{\varepsilon}_{\mu}(x)] \in L^{\varepsilon}_{\mu}$ and $[y/L^{\varepsilon}_{\mu}(y)] \in L^{\varepsilon}_{\mu}$, it follows from (3.10) that

$$[(x * y)/\max\{L^{\varepsilon}_{\mu}(x * (y * z)), L^{\varepsilon}_{\mu}(y)\}]qL^{\varepsilon}_{\mu}. \tag{3.11}$$

If $x * y \notin O(L^{\varepsilon}_{\mu})$, then $L^{\varepsilon}_{\mu}(x * y) = 0$. Thus,

$$\begin{aligned}
&L^{\varepsilon}_{\mu}(x * y) + \max\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\}\\
&= \max\{L^{\varepsilon}_{\mu}(x), L^{\varepsilon}_{\mu}(y)\}\\
&= \max\{\max\{0, \mu(x) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\}\\
&= \max\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\\
&= \max\{\mu(x), \mu(y)\} + \varepsilon - 1\\
&\leq 1 + \varepsilon - 1\\
&= \varepsilon\\
&\leq 1,
\end{aligned}$$

which shows that (3.11) is not valid. This is a contradiction. Hence, $x * y \in O(L^{\varepsilon}_{\mu})$. Therefore, $O(L^{\varepsilon}_{\mu})$ is a BCC-subalgebra of X.

Theorem 9. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X satisfies the condition (3.9) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the nonempty O-set $O(L^{\varepsilon}_{\mu})$ of L^{ε}_{μ} is a BCC-subalgebra of X.*

Proof. Let $x, y \in O(L^{\varepsilon}_{\mu})$. Then $\mu(x) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Thus, $L^{\varepsilon}_{\mu}(x) + 1 = \max\{0, \mu(x) + \varepsilon - 1\} + 1 = \mu(x) + \varepsilon - 1 + 1 = \mu(x) + \varepsilon > 1$ and $L^{\varepsilon}_{\mu}(y) + 1 = \max\{0, \mu(y) + \varepsilon - 1\} + 1 = \mu(y) + \varepsilon - 1 + 1 = \mu(y) + \varepsilon > 1$, that is, $[x/1]qL^{\varepsilon}_{\mu}$ and $[y/1]qL^{\varepsilon}_{\mu}$. It follows from (3.9) that

$$[(x * y)/1] = [(x * y)/\max\{1, 1\}] \in L^{\varepsilon}_{\mu}. \tag{3.12}$$

If $x * y \notin O(L^{\varepsilon}_{\mu})$, then $L^{\varepsilon}_{\mu}(x * y) = 0 < 1$ and so (3.12) is not valid. This is a contradiction. Thus, $x * y \in O(L^{\varepsilon}_{\mu})$. Hence, $O(L^{\varepsilon}_{\mu})$ is a BCC-subalgebra of X.

4. Conclusions

The concept of ε-Łukasiewicz fuzzy sets using Łukasiewicz t-norm was introduced by Jun [8]. In this paper, the ε-Łukasiewicz fuzzy set has been applied to BCC-subalgebras in BCC-algebras, introducing the concept of ε-Łukasiewicz fuzzy BCC-subalgebras, and examining several properties. We discussed the characterization of ε-Łukasiewicz fuzzy BCC-subalgebras and considered the relationship between fuzzy BCC-subalgebras and ε-Łukasiewicz fuzzy BCC-subalgebras. We provided conditions under which ε-Łukasiewicz fuzzy sets can be ε-Łukasiewicz fuzzy BCC-subalgebras and further explored conditions under which three subsets: $\in$-set, q-set, and O-set, will be BCC-subalgebras. The ideas and results obtained in this paper will be applied to the relevant algebraic systems in the future, further examining their usability as a mathematical tool applicable to decision theory, medical diagnosis systems, automation systems, etc.

Acknowledgements

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2024).

References

[1] B. Ahmad and A. Kharal. On fuzzy soft sets. *Adv. Fuzzy Syst.*, 2009:Article ID 586507, 6 pages, 2009.

[2] M. Atef, M. I. Ali, and T. Al-Shami. Fuzzy soft covering based multi-granulation fuzzy rough sets and their applications. *Comput. Appl. Math.*, 40(4):115, 2021.

[3] N. Cağman, S. Enginoğlu, and F. Citak. Fuzzy soft set theory and its application. *Iran. J. Fuzzy Syst.*, 8(3):137–147, 2011.

[4] N. Dokkhamdang, A. Kesorn, and A. Iampan. Generalized fuzzy sets in UP-algebras. *Ann. Fuzzy Math. Inform.*, 16(2):171–190, 2018.

[5] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan. Fuzzy translations of a fuzzy set in UP-algebras. *J. Indones. Math. Soc.*, 23(2):1–19, 2017.

[6] Y. Huang. *BCI-algebra*. Science Press, Beijing, China, 2006.

[7] A. Iampan. A new branch of the logical algebra: UP-algebras. *J. Algebra Relat. Top.*, 5(1):35–54, 2017.

[8] Y. B. Jun, B. Brundha, N. Rajesh, and R. K. Bandaru. $(3,2)$-Fuzzy UP (BCC)-subalgebras and $(3,2)$-fuzzy UP (BCC)-filters. *J. Mahani Math. Res. Cent.*, 11(3):1–14, 2022.

[9] Y. Komori. The class of BCC-algebras is not a variety. *Math. Japon.*, 29(3):391–394, 1984.

[10] P. M. Pu and Y. M. Liu. Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. *J. Math. Anal. Appl.*, 76(2):571–599, 1980.

[11] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan. Fuzzy sets in UP-algebras. *Ann. Fuzzy Math. Inform.*, 12(6):739–756, 2016.

[12] L. A. Zadeh. Fuzzy sets. *Inf. Control*, 8(3):338–353, 1965.