EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2084-2091
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Study on Even Sum Domination Number of Some Graphs

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Abstract. Let G be a connected graph and $uv \in E(G)$. We say, the vertex v even sum dominates u (u even sum dominates v) if $deg(v)+deg(u)$ is an even number. A set S is an even sum dominating set (ESDS) if every vertex $v \in V$ is either in S or even sum dominated by a vertex in S. An even sum dominating set S is a mimimal even sum dominating set if no proper subset $S^{'} \subset S$ is an even sum dominating set. The even sum domination number $\gamma_{es}(G)$ of a graph G is the minimum cardinality of an even sum dominating set of G. In this paper, we discuss some properties and bounds for this concept. We also derive even sum domination number for some standard graphs.

2020 Mathematics Subject Classifications: 05C45, 05C69

Key Words and Phrases: Degree of a vertex, Domination number, Even sum domination number

1. Introduction

We consider simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. We follow West [9] for all standard terminology and notations while the terms related to the theory of domination in graphs are used in the sense of Haynes *et al.* [4]. We shall give brief summary of definitions which are useful for the present investigations. The domination number is a well studied parameter as observed by Hedetniemi and Laskar [5]. A set $S \subseteq V(G)$ of vertices in a graph $G = (V(G), E(G))$ is called a *dominating set* if every vertex $v \in V(G)$ is either an element of S or is adjacent to an

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5313

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element of S. A dominating set S is a *minimal dominating set* if no proper subset $S^{'} \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G. This concept is explored and reformed in various fields to solved many real life problems. Furthermore, a number of broad formulations of this idea have emerged recently; these definitions, however, rely on certain requirements that can be applied to the dominating set, outside of it, or both[1, 2, 8]. This paper is worked on one of the recently introduced parameter known as even sum domination [7]. This definition is dependent on the real-world situations when it is feasible to divide the set's elements into two partitions, with each partition being dominated by a same type of that partition.

The *degree of a vertex* v in graph G, denoted as $d(v)$ or $deg(v)$, is the number of edges incident to v, counting each loop twice. For any connected graph G and $uv \in E(G)$, the vertex v even sum dominates u (u even sum dominates v) if $deg(v) + deg(u)$ is an even number. Precisely, two adjacent vertices of odd degree as well as two adjacent vertices of even degree can even sum dominate each other. A set S is called even sum dominating set(ESDS) if every vertex $v \in V$ is either an element of S or it is even sum dominated by some vertex of S. An even sum dominating set S is a mimimal even sum dominating set if no proper subset $S^{'} \subset S$ is an even sum dominating set. The even sum domination number $\gamma_{es}(G)$ of a graph G is the minimum cardinality of an even sum dominating set of G.

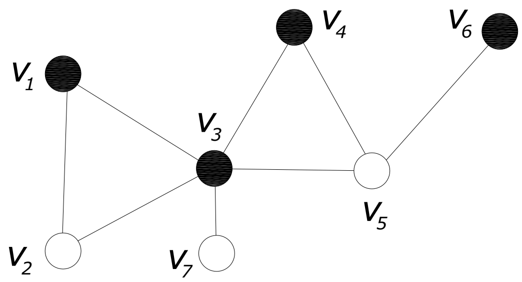

***Figure* 1** G

For the graph G given in Figure 1, the even sum dominating set $S = \{v_1, v_3, v_4, v_6\}$.

2. Main Results

Let $G = (V, E)$ be a connected graph and $u, v \in V$, then we say that u is even sum neighbor of v if $uv \in E$ and $deg(u) + deg(v)$ is an even number. The even sum open neighborhood $N_{es}(v)$ of the vertex v is the set of vertices which are even sum neighbors of v, that is $N_{es}(v)$={ $u \in V(G)$:$uv \in E$ and $(deg(u) + deg(v))$ is an even number}. We say a vertex $v \in V$ is an even sum isolate of S if $N_{es}(v) \subseteq V - S$. The vertex v is said to be an even sum isolate of V if v has no even sum neighbors in V, that is, there does not exist the vertex $u \in V$ which is adjacent to v in such a way that $deg(u) + deg(v)$ will be an even number. For the graph G given in Figure 1, the vertices v_1, v_3 and v_6 are even sum isolates of S while the vertex v_4 is an even sum isolate of V.

Theorem 1. *An even sum dominating set S is a minimal even sum dominating set if and only if for every $u \in S$, one of the following condition holds.*
(a) *No vertex in S even sum dominates u.*
(b) *There exists a vertex $v \in V - S$ for which $N_{es}(v) \cap S = \{u\}$.*

Proof. Assume that S is a minimal even sum dominating set of G. Then for every vertex $u \in S$, $S - \{u\}$ is not an even sum dominating set. This means there is a vertex v in $V - S \cup \{u\}$ is not even sum dominated by any vertex of $S - \{u\}$. Now either $u = v$, in which case u is not even sum dominated by any vertex of S other than u, or $v \in V - S$. If v is not even sum dominated by any vertex of $S - \{u\}$ but even sum dominated by any vertex of S, then vertex v has only one even sum neighbor u in S, that is $N_{es}(v) \cap S = \{u\}$. Conversely, suppose that S is an even sum dominating set and for every $u \in S$, one of the two axioms holds. We will try to prove that S is an minimal even sum dominating set. Suppose that S is not a minimal even sum dominating set, that is, there is at least one vertex $u \in S$ such that $S - \{u\}$ is an even sum dominating set. Hence, u is even sum dominated by at least one vertex in $S - \{u\}$, that is, there exists a vertex $w \in S - \{u\}$ which even sum dominates u. Hence, axiom (a) does not hold. Also if $S - \{u\}$ is a dominating set then every vertex in $V - S$ is even sum dominated by some vertex of $S - \{u\}$, that is, there exists an another vertex x in S such that $x \in N_{es}(v)$ for some $v \in V - S$ and $x \neq u$. Hence, axiom (b) does not hold. So neither axiom (a) nor (b) holds that contradicts our assumption that one of these two axioms holds.

Theorem 2. *If there exists any isolate of V in G then it must be in every minimal even sum dominating set of G.*

Proof. As per the definition an even sum isolate of V, the vertex v is not even sum dominated by any vertex of V other than itself. Hence, it must be in every minimal even sum dominating set of G.

Theorem 3.([6]). *Every connected graph G of order $n \geq 2$ has a dominating set S whose complement $V - S$ is also a dominating set.*

For even sum dominating set there may not exist an even sum dominating set S whose complement $V - S$ is an even sum dominating set. For graph G, given in Figure 1, $S = \{v_1, v_3, v_4, v_6\}$ while $V - S = \{v_2, v_5, v_7\}$ is not an even sum dominating set.

Theorem 4. *Let G be a connected graph with even sum dominating set S. If G has no isolate of S and V then $V - S$ of every minimal even sum dominating set S is an even sum dominating set.*

Proof. As S being any minimal even sum dominating set of G, every vertex $v \in V - S$ is even sum dominated by atleast one vertex of S. So, by the definition of even sum domination, $\forall v \in V - S$ can also even sum dominate some $u \in S$. Assume that $V - S$ is not an even sum dominating set. Therefore, there exists at least one vertex $w \in S$ which is not even sum dominated by any vertex of $V - S$. So, w is even sum dominated by only some vertex of S that is, $S - \{w\}$ is also an even sum dominating set. In this case S is not minimal even sum dominating set of G which contradicts to our assumption that S is a minimal even sum dominating set of G. Hence, $V - S$ of every minimal even sum dominating set S is an even sum dominating set.

Theorem 5. $1 \leq \gamma_{es}(G) \leq n$

Proof. Let G be a simple graph and $|G|$ is an odd number. As per the definition of an even sum dominating set if G has atleast one vertex of $n-1$ degree then $\gamma_{es}(G) = 1$ and if the graph has all the vertices of odd degree than $\gamma_{es}(G) = n$.

Here, Star graph K_1, n achieves lower bound for any odd number n while it achieves the upper bounds for even number n. The graph containing the property given in below theorem also achieves the upper bound.

Theorem 6. *Let G be the graph in which every vertex of even degree is adjacent to the vertices of odd degree only and every vertex of odd degree is adjacent to the vertices of even degree only then $\gamma_{es}(G) = |V(G)|$.*

Proof. Here sum of degree of any two adjacent vertices of G will be an odd number. So any vertex of G can not even sum dominate any other vertex of G. Therefore to construct an even sum dominating set of minimum cardinality we include all the vertices of G. Hence, $\gamma_{es}(G) = |V(G)|$.

Theorem 7 ([3])**.** *A connected graph G is Euler if and only if the degree of each vertex is even.*

Theorem 8. *If G is an Euler graph then $\gamma_{es}(G) = \gamma(G)$.*

Proof. Let G is an Euler graph then by Theorem 7, the degree of each vertex is even. So, any two vertices of G can even sum dominate each other if they are adjacent to each other. Hence, $\gamma_{es}(G) = \gamma(G)$.

Corollary 1. $\gamma_{es}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. As the cycle C_n is an Euler graph, according to Theorem 8, $\gamma_{es}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

3. Even sum domination number of some standard graphs

Theorem 1. $\gamma_{es}(P_n) = 2 + \left\lceil \frac{n-2}{3} \right\rceil$.

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the n vertices of P_n where v_1 and v_n are pendant vertices. So degree of v_1 and v_n is 1 while degree of v_2,...,v_{n-1} is 2. Therefore v_1 and v_n can even sum dominate themselves only and they are not even sum dominated by their neighbors. So, they must be in even sum dominating set S. Now as remaining vertices v_2,...,v_{n-1} having even degree they even sum dominate their selves and their neighbors. So to even sum dominate remaining $n-2$ vertices we need $\left\lceil \frac{n-2}{3} \right\rceil$ vertices. Therefore, we require $2 + \left\lceil \frac{n-2}{3} \right\rceil$ vertices to even sum dominate all the vertices of P_n.

Theorem 2. $$\gamma_{es}(K_{m,n}) = \begin{cases} 2, & \text{if } m+n \textit{ is an even number }, \\ & \textit{where } m, n > 1 . \\ |V(K_{m,n})|, & \text{if } m+n \textit{ is an odd number.} \end{cases}$$

Proof. Let V_1 and V_2 be two subsets of $K_{m,n}$ such that $V_1 \cup V_2 = V(K_{m,n})$ where $|V_1| = m$ and $|V_2| = n$ and v_1,v_2,...,v_m are vertices of V_1 and u_1,u_2,...,u_n are vertices of V_2. Now according to the values of m and n we consider two cases as below.

Case 1: $m + n$ is an even number.

That is either m and n both are even number or both are odd number. According to that we consider two subcases as given below.

Subcase 1.1:If m and n both are even number.

If m and n both are even number then v_1,v_2,...,v_m,u_1,u_2,...,u_n are vertices of even degree. According to the definition of even sum domination every v_i, $i = 1, 2, ..., m$ can even sum dominate itself and all the vertices of V_2. Similarly every u_j, $j = 1, 2, ..., n$ can even sum dominate itself and all the vertices of V_1. Therefore, it is enough to consider one vertex from V_1 and one vertex from V_2 to even sum dominate all the vertices of $K_{m,n}$.

Subcase 1.2:If m and n both are odd number.

If m and n both are odd number then $deg(v_i)+deg(u_j)$ is even number for $1 \leq i, j \leq m, n$. So from the definition of even sum domination every v_i, $i = 1, 2, ..., m$ can even sum dominate itself and all the vertices of V_2. Similarly every u_j, $j = 1, 2, ..., n$ can even sum dominate itself and all the vertices of V_1. Therefore, it is enough to consider one vertex from V_1 and one vertex from V_2 to even sum dominate all the vertices of $K_{m,n}$. Hence, from the above both subcases $\gamma_{es}(K_{m,n}) = 2$.

Case 2: $m + n$ is an odd number.

If $m+n$ is an odd number then either m is an odd number and n is an even number or n is an odd number and m is an even number. Therefore, in both the cases $deg(v_i)+deg(u_j)$ will be an odd number for $1 \leq i, j \leq m, n$. So from the definition of even sum domination every v_i, $i = 1, 2, ..., m$ can even sum dominate itself only and similarly for every u_j, $j = 1, 2, ..., n$ can even sum dominate itself only. Therefore to even sum dominate all the vertices of $K_{m,n}$ we must include all the vertices of $K_{m,n}$. Hence, $\gamma_{es}(K_{m,n}) = m + n = |V(K_{m,n})|$.

Definition 1. *The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.*

Theorem 3. $\gamma_{es}(M(P_n)) = 4 + \left\lceil \frac{n-6}{2} \right\rceil$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. Then $V(M(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\}$ and let $V(M(P_n)) = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{e_1, e_2, \ldots, e_{n-1}\}$. Here, $d(v_1) = d(v_n) = 1$, $d(v_i) = 2$ for $i = 2, 3, \ldots, n-1$, $d(e_1) = 2$ for $n = 2$, $d(e_1) = d(e_2) = 3$ for $n = 3$, $d(e_1) = d(e_{n-1}) = 3$ for $i = 3, 4, \ldots, n$ and $d(e_i) = 4$ for $i = 2, 3, \ldots, n-2$. As per the definition of even sum domination the vertex v_1 and the vertex e_1 can even sum dominate each other and the vertex v_n and the vertex e_{n-1} can even sum dominate each other. So, let's consider e_1 and e_{n-1} in even sum dominating set S. Now v_2 and v_{n-1} can even sum dominate e_2 and e_{n-2} respectively other than themselves while e_2 and e_{n-2} can even sum dominate their three neighbor vertices other than themselves. Therefore $e_2, e_{n-2} \in S$. So after considering e_1, e_{n-1}, e_2 and e_{n-2} in S, in total six vertices $v_1, v_2, v_3, v_n, v_{n-1}, v_{n-2}$ from V_1 and six vertices $e_1, e_2, e_3, e_{n-1}, e_{n-2}, e_{n-3}$ from V_2 will be even sum dominated. Now to even sum dominate remaining $n-6$ vertices from V_1 and $n-7$ vertices from V_2 it is enough to consider $\left\lceil \frac{n-6}{2} \right\rceil$ from V_2. Thus, $e_1, e_{n-1}, e_2,\ e_{n-2}$ and $\left\lceil \frac{n-6}{2} \right\rceil$ vertices from V_2 even sum dominate all the vertices of $M(P_n)$.

Theorem 4. $\gamma_{es}(M(C_n)) = \begin{cases} \frac{n}{2}, & \text{if } n \textit{ is an even number}, \\ \frac{n+1}{2}, & \text{if } n \textit{ is an odd number}. \end{cases}$

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_1, e_2, e_3, \ldots, e_n$ be the edges of cycle C_n. Then $V(M(C_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_n\}$. Here, $d(v_i) = 4$ for $1 \leq i \leq n$, $d(e_i) = 2$ for $1 \leq i \leq n$. Therefore, every v_i even sum dominates four vertices other than itself while every e_i even sum dominates two vertices other than itself. If n is an even number then in order to form an even sum dominating set of minimum cardinality it is enough to consider either $\frac{n}{2}$ vertices, either $v_1, v_3, v_5, ..., v_{n-1}$ or $v_2, v_4, v_6, ..., v_n$ from $M(C_n)$. Now if n is an odd number then to form an even sum dominating set of minimum cardinality it is enough to consider $\frac{n+1}{2}$ vertices, which may be $v_1, v_3, ..., v_n$ or $v_2, v_4, ..., v_{n-1}, v_n$ from $M(C_n)$.

Definition 2. *Let G be a graph with $V(G) = S_1 \cup S_2 \cup \ldots S_t \cup T$ where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$ where $i = 1, 2, \ldots, t$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.*

Theorem 5. $\gamma_{es}(DS(P_n)) = \begin{cases} 2, & \text{if } n \textit{ is an odd number}, \\ 2 + \left\lceil \frac{n-2}{3} \right\rceil, & \text{if } n \textit{ is an even number}. \end{cases}$

Proof. The path P_n has two pendant vertices and the remaining $n-2$ vertices are of degree 2. Thus, $V(P_n) = \{v_i; 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i; 2 \leq i \leq n-1\}$. To obtain $DS(P_n)$ from P_n, add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus, $V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $|V(DS(P_n))| = n+2$. We shall consider two cases according to values of n.

Case 1: n is an odd number.

According to the definition of even sum domination the vertex w_1 even sum dominates both the vertices v_1, v_n and itself also. Therefore, by considering w_1 in S, all the vertices of S_1 will be even sum dominated. So, $w_1 \in S$. Now w_2 can even sum dominate every v_i from S_2 while every v_i from S_2 can even sum dominate only its three neighbors other than itself. Therefore, we also consider w_2 in S. Hence $S = \{w_1, w_2\}$ will be an even sum dominating set of minimum cardinality. Therefore, $\gamma_{es}(DS(P_n)) = 2$.

Case 2: n is an even number.

According to the definition of even sum domination the vertex w_1 even sum dominates both the vertices v_1, v_n and itself also. Therefore, if we consider w_1 in S then all the vertices of S_1 are even sum dominated. In S_2, $deg(v_i) = 3$ which is odd number while $deg(w_2) = n-2$ which is even number w_2 even sum dominates itself only and every v_i can even sum dominates only its neighbor from S_2 and itself. So, from $n-2$ vertices of

S_2 it is enough to consider $\left\lceil \dfrac{n-2}{3} \right\rceil$ vertices from S_2. Therefore, S forms an even sum dominating set of minimum cardinality. Hence, $\gamma_{es}(DS(P_n)) = 2 + \left\lceil \dfrac{n-2}{3} \right\rceil$.

Definition 3. *The* wheel W_n *is defined to be the join* $K_1 + C_n$*. The vertex corresponding to* K_1 *is known as the apex and the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.*

Theorem 6. $\gamma_{es}(W_n) = \begin{cases} 1, & \text{if } n \text{ is an odd number}, \\ \gamma_{es}(C_n) + 1, & \text{if } n \text{ is an even number.} \end{cases}$

Proof. Consider the wheel W_n with the vertices $v_1, v_2, \ldots, v_n$ as its rim vertices and v_0 as its apex vertex. According to value of n we consider following two cases.

Case 1: n is an odd number.

In this case, $deg(v_i) = 3$, $1 \leq i \leq n$ and $deg(v_0) = n$, which is an odd number. So, $deg(v_i) + deg(v_0)$ will be an even number. Therefore, the vertex v_0 even sum dominates every rim vertex. Thus it is enough to consider an apex vertex v_0 in even sum dominating set S. Hence, $\gamma_{es}(W_n) = 1$.

Case 2: n is an even number.

In this case, $deg(v_i) = 3$, $1 \leq i \leq n$ and $deg(v_0) = n$, which is an even number. The apex vertex v_0 does not even sum dominate any rim vertex and it is not even sum dominated by any rim vertex also. So the vertex v_0 must be in even sum dominating set S while each rim vertex can even sum dominate its both neighbors likewise cycle. Thus we need $\left\lceil \dfrac{n}{3} \right\rceil$ vertices to even sum dominate all the rim vertices. Therefore, it is enough to consider $\left\lceil \dfrac{n}{3} \right\rceil + 1$ vertices to construct the even sum dominating set of minimum cardinality. Hence, $\gamma_{es}(W_n) = \gamma_{es}(C_n) + 1$.

4. Concluding Remarks

We have derived some basic results on even sum domination number. In different sport centers of any sport academy even sum domination can be used to schedule match without wasting time of players and using minimum resources for some particular game like Carrom, Tennis, Badminton in which we need even number of players. We have discussed some properties and boundaries for this concept. We have also discussed even sum domination number for some standard graphs. In future research, It may be fascinating to find even sum domination number of some graphs obtain by binary operations and compare various domination models with even sum domination.

Acknowledgements

The authors thank the reviewers and editors of European Journal of Pure and Applied Mathematics, for reviewing the paper and for the comments and suggestions they provided.

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