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Euler Polynomials and Bi-univalent Functions

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Abstract. Our research introduces new subclasses of analytical functions that are defined by Euler polynomials. We then proceed to estimate the Fekete-Szegö functional problem and the Maclaurin coefficients for this specific subfamily, denoted as $|a_2|$ and $|a_3|$. Furthermore, we demonstrate several new results that emerge when we specialize the parameters used in our main findings.

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1. Preliminaries

Euler polynomials, which have their origins in Leonhard Euler's eighteenth-century research, are essential for understanding complex functions and their geometric properties. They play a key role in characterizing conformal mappings that preserve angles locally in geometric function theory. Additionally, they are widely utilized in various areas of geometric function theory, including the study of univalent functions, Schwarz-Christoffel mappings, and Riemann surface theory. These applications shed light on the intricate relationship between geometric transformations and analytic functions facilitated by Euler polynomials. This text explores the fundamental properties of Euler polynomials, providing an explanation of how they are employed to represent solutions to specific differential equations and to generate functions for different types of analytic functions.

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Due to the extensive usage of Euler polynomials in pure mathematics, numerous academics have started to explore various domains. The current research in geometric function theory primarily revolves around the geometric properties of special functions and their related counterparts. For further information on the geometric properties of these functions, please refer to [16, 28], and other relevant sources.

Let $\digamma$ be the class of analytic functions b in the unit disk $\Lambda = \{\kappa \in \mathbb{C} : |\kappa| < 1\}$ and normalized by $b(0) = b'(0) - 1 = 0$ of the form:

$$b(\kappa) = \kappa + \sum_{i=2}^{\infty} c_i \kappa^i, \quad (\kappa \in \Lambda). \tag{1}$$

We also let Ψ consisting of functions univalent in Λ.

Every mathematical function $b \in \Psi$ has an inverse b^{-1}, defined by

$$b^{-1}(b(\kappa)) = \kappa \text{ and } w = b(b^{-1}(w)) \qquad (\kappa \in \Lambda, \ |w| < r_0(b); \ r_0(b) \geq \frac{1}{4})$$

where

$$b^{-1}(w) = q(w) = w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (c_4 + 5, c_2^3 - 5c_3c_2)w^4 + \cdots . \tag{2}$$

A function b is said to be bi-univalent in Λ if both b and b^{-1} are univalent in Λ. Let Π denote the class of all bi-univalent functions in Λ given by (1).

Example in the class Π is $h(\kappa) = \frac{\kappa}{1-\kappa}$ but $h(\kappa) = \frac{\kappa}{1-\kappa^2}$ not members of Π. For interesting function classes in class Π, (see [1]).

Miller and Mocanu [21] introduced the first differential subordination problem, see [22] and [23]. We say that the function b is subordinate to q, written as $b \prec q$, if b and q are analytic in Λ and exists function $w \in \digamma$ in Λ with

$$w(0) = 0 \text{ and } |w(\kappa)| < 1, \qquad (\kappa \in \Omega)$$

such that

$$b(\kappa) = q(w(\kappa)).$$

Also, if q is univalent in Λ, then

$$b(\kappa) \prec q(\kappa) \quad \text{if and only if} \quad b(0) = q(0) \text{ and } b(\Lambda) \subset q(\Lambda).$$

Geometric function theory makes effective use of Euler polynomials, which is a fundamental tool in mathematical analysis. They are particularly important in the study of complex analysis and conformal mappings.

In this study, our focus is on the Euler polynomial, a specific special function. Our aim is to construct a new and comprehensive subclass of bi-univalent functions.

The generating function is commonly used to define the Eulers polynomials $\Theta_i(\ell)$ (see, [19, 27]):

$$\mathcal{B}(\ell, h) = \frac{2e^{h\ell}}{e^h + 1} = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{h^i}{i!}, \qquad \left(\frac{1}{2} < \ell \leq 1,\ |h| < \pi\right).$$

An explicit formula for $\Theta_i(\ell)$ is given by

$$\Theta_j(\ell) = \sum_{i=0}^{j} \frac{1}{2^i} \sum_{u=0}^{i} (-1)^u \binom{i}{u} (\ell + u)^j.$$

Now $\Theta_i(\ell)$ in terms of Θ_u can be obtained from the above equation as:

$$\Theta_i(\ell) = \sum_{u=0}^{i} \frac{\Theta_u}{2^u} \binom{i}{u} (\ell - \frac{1}{2})^{i-u}.$$

The initial values of Euler polynomials are:

$$\begin{aligned} \Theta_0(\ell) &= 1; \\ \Theta_1(\ell) &= \frac{2\ell - 1}{2}; \\ \Theta_2(\ell) &= \ell^2 - \ell; \\ \Theta_3(\ell) &= \frac{4\ell^3 - 6\ell^2 + 1}{4}; \\ \Theta_4(\ell) &= \ell^4 - 2\ell^3 + \ell. \end{aligned} \tag{3}$$

A lot of studies have looked at the geometric function theory in recent years, including coefficient estimates. Several subclasses of the class Π were introduced and non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were obtented in ([2–15, 18, 20, 24, 25, 29–31]).

In this study, we define new subclass of Π involving the Euler polynomials which are denote by $\mathcal{F}_\Pi(\zeta, \ell)$, and derive bounds for the $|a_2|$ and $|a_3|$ Taylor-Maclaurin coefficients and Fekete–Szegö functional problems. Furthermore, Several novel findings are shown to ensue.

2. Definition and Examples

At the beginning of this section, we present a definition of the new subclasses $\mathcal{F}_\Pi(\zeta, \ell)$ that is associated with Euler polynomials.

Definition 1. *If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}(\zeta, \ell)$:*

$$\frac{\kappa b'(\kappa) + \zeta\kappa^2 b''(\kappa)}{(1-\zeta)b(\kappa) + \zeta\kappa b'(\kappa)} \prec \mathcal{B}(\ell, \kappa) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{\kappa^i}{i!} \tag{4}$$

and

$$\frac{wg'(w) + \zeta w^2 g''(w)}{(1-\zeta)g(w) + \zeta wq'(w)} \prec \mathcal{B}(\ell, w) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{w^i}{i!}, \tag{5}$$

where $0 \le \zeta \le 1$, $\frac{1}{2} < \ell \le 1$ $\kappa, w \in \Lambda$ and $q = b^{-1}$.

Example 1. *If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}(0, \ell)$:*

$$\frac{\kappa b'(\kappa)}{b(\kappa)} \prec \mathcal{B}(\ell, \kappa) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{\kappa^i}{i!}$$

and

$$\frac{wg'(w)}{g(w)} \prec \mathcal{B}(\ell, w) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{w^i}{i!},$$

where $\frac{1}{2} < \ell \le 1$ $\kappa, w \in \Lambda$ and $q = b^{-1}$.

Example 2. *If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}(1, \ell)$:*

$$1 + \frac{\kappa b''(\kappa)}{b'(\kappa)} \prec \mathcal{B}(\ell, \kappa) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{\kappa^i}{i!}$$

and

$$1 + \frac{wg''(w)}{q'(w)} \prec \mathcal{B}(\ell, w) = \sum_{i=0}^{\infty} \Theta_i(\ell) \frac{w^i}{i!},$$

where $\frac{1}{2} < \ell \le 1$ $\kappa, w \in \Lambda$ and $q = b^{-1}$.

Lemma 1. *([26]) If $d \in \mathcal{D}$, then $|m_n| \le 2$ for each n, where $\mathcal{D}$ is the family of all analytic functions in Λ for which*

$$Re\,(d(\kappa)) > 0,\ d(\kappa) = 1 + m_1\kappa + m_2^2\kappa + \cdots \quad (\kappa \in \Lambda).$$

3. Bounds of the class $\mathcal{F}_{\Pi}(\zeta,\ell)$

For a function $b \in \Lambda$, we give the coefficient estimates and solve Fekete-Szegö problem(see [17]) for the class $\mathcal{F}_{\Pi}(\zeta,\ell)$, respectively.

Theorem 1. *Let $b \in \Pi$ given by (1) belongs to the class $\mathcal{F}_{\Pi}(\zeta,\ell)$ where $0 \le \zeta \le 1$, $\frac{1}{2} < \ell \le 1$ $\kappa, w \in \Lambda$ and $q = b^{-1}$. Then*

$$|c_2| \le \sqrt{\Upsilon(\zeta,\ell)},$$

$$|c_3| \le \frac{(2\ell-1)^2}{(1+\zeta)^2} + \frac{2\ell-1}{4(1+2\zeta)}.$$

and

$$\left|c_3 - \varkappa c_2^2\right| \le \begin{cases} \frac{2\ell-1}{2(1+2\zeta)} & 0 \le |1-\varkappa|\,\Upsilon(\zeta,\ell) < \frac{2\ell-1}{4(1+2\zeta)}, \\ 2\,|1-\varkappa|\,\Upsilon(\zeta,\ell) & |1-\varkappa|\,\Upsilon(\zeta,\ell) \ge \frac{2\ell-1}{4(1+2\zeta)}. \end{cases}$$

where

$$\Upsilon(\zeta,\ell) = \frac{2\,(2\ell-1)^3}{\left|\left[(1+2\zeta-\zeta^2)\,(2\ell-1)^2 - 2(1+\zeta)^2\,(\ell^2-3\ell+1)\right]\right|}.$$

Proof. Since $b(\kappa) = \kappa + \sum\limits_{i=2}^{\infty} c_i \kappa^i \in \mathcal{F}_{\Pi}(\zeta,\ell)$, So from Definition 1, we can write

$$\frac{\kappa b'(\kappa) + \zeta\kappa^2 b''(\kappa)}{(1-\zeta)b(\kappa) + \zeta\kappa b'(\kappa)} \prec \mathcal{B}(\ell,\kappa) \tag{6}$$

and

$$\frac{wg'(w) + \zeta w^2 g''(w)}{(1-\zeta)g(w) + \zeta wq'(w)} \prec \mathcal{B}(\ell,w). \tag{7}$$

We can consider two functions $r,\, s : \Lambda \to \Lambda$, with $r(0) = s(0) = 0$ and $|r(\kappa)| < 1$, $|s(w)| < 1$ for all $\kappa, w \in \Lambda$. So we can define $\gamma,\, \lambda \in \mathcal{D}$ as following:

$$\gamma(\kappa) = \frac{r(\kappa)+1}{1-r(\kappa)} = 1 + \gamma_1\kappa + \gamma_2\kappa^2 + \gamma_3\kappa^3 + \cdots,\ |\gamma_i| \le 2 \text{ for all } i \in \mathbb{N}.$$

$$\Rightarrow r(\kappa) = \frac{\gamma(\kappa)-1}{\gamma(\kappa)+1} = \frac{\gamma_1}{2}\kappa + \left(\frac{\gamma_2}{2} - \frac{\gamma_1^2}{4}\right)\kappa^2 + \frac{1}{2}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{4}\right)\kappa^3 + \cdots \tag{8}$$

and

$$\lambda(w) = \frac{s(w)+1}{1-s(w)} = 1 + \lambda_1 w + \lambda_2 w^2 + \lambda_3 w^3 + \cdots,\ |\lambda_i| \le 2 \text{ for all } i \in \mathbb{N}.$$

$$\Rightarrow s(w) = \frac{\lambda(w)-1}{\lambda(w)+1} = \frac{\lambda_1}{2}w + \left(\frac{\lambda_2}{2} - \frac{\lambda_1^2}{4}\right)w^2 + \frac{1}{2}\left(\lambda_3 - \lambda_1\lambda_2 + \frac{\lambda_1^3}{4}\right)w^3 + \cdots. \tag{9}$$

Using (8) and (9), we get

$$\mathcal{B}(\ell, r(\kappa)) = \Theta_0(\ell) + \frac{\Theta_1(\ell)}{2}\gamma_1\kappa + \left(\frac{\Theta_1(\ell)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Theta_2(\ell)}{8}\gamma_1^2\right)\kappa^2 \tag{10}$$
$$+ \left(\frac{\Theta_1(\ell)}{2}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{4}\right) + \frac{\Theta_2(\ell)}{4}\left(\gamma_1\gamma_2 - \frac{\gamma_1^3}{2}\right) + \frac{\Theta_3(\ell)}{48}\gamma_1^3\right)\kappa^3 + \cdots$$

and

$$\mathcal{B}(\ell, s(w)) = \Theta_0(\ell) + \frac{\Theta_1(\ell)}{2}\lambda_1 w + \left(\frac{\Theta_1(\ell)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Theta_2(\ell)}{8}\lambda_1^2\right)w^2 \tag{11}$$
$$+ \left(\frac{\Theta_1(\ell)}{2}\left(\lambda_3 - \lambda_1\lambda_2 + \frac{\lambda_1^3}{4}\right) + \frac{\Theta_2(\ell)}{4}\left(\lambda_1\lambda_2 - \frac{\lambda_1^3}{2}\right) + \frac{\Theta_3(\ell)}{48}\lambda_1^3\right)w^3 + \cdots$$

From (6), (7) and the previous two equations, we have

$$(1+\zeta)c_2 = \frac{\Theta_1(\ell)}{2}\gamma_1, \tag{12}$$

$$2(1+2\zeta)c_3 - (1+\zeta)^2c_2^2 = \frac{\Theta_1(\ell)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Theta_2(\ell)}{8}\gamma_1^2, \tag{13}$$

$$-(1+\zeta)c_2 = \frac{\Theta_1(\ell)}{2}\lambda_1, \tag{14}$$

and

$$-2(1+2\zeta)c_3 - (\zeta^2 - 6\zeta - 3)c_2^2 = \frac{\Theta_1(\ell)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Theta_2(\ell)}{8}\lambda_1^2. \tag{15}$$

Adding equations (12) and (14) and some simplification, we get

$$\gamma_1 = -\lambda_1 \text{ and } \gamma_1^2 = \lambda_1^2 \tag{16}$$

and

$$2(1+\zeta)^2c_2^2 = \Theta_1^2(\ell)(\gamma_1^2 + \lambda_1^2). \tag{17}$$

$$\Rightarrow c_2^2 = \frac{\Theta_1^2(\ell)(\gamma_1^2 + \lambda_1^2)}{2(1+\zeta)^2} \tag{18}$$

Adding (13) to (15) gives

$$\left(2 + 4\zeta - 2\zeta^2\right)c_2^2 = 2\Theta_1(\ell)(\gamma_2 + \lambda_2) + (\gamma_1^2 + \lambda_1^2)\left(\frac{1}{2}\Theta_2(\ell) - \Theta_1(\ell)\right).$$

By (16), we have

$$\left(2 + 4\zeta - 2\zeta^2\right)c_2^2 = 2\Theta_1(\ell)(\gamma_2 + \lambda_2) + \gamma_1^2\left(\Theta_2(\ell) - 2\Theta_1(\ell)\right) \tag{19}$$

Also, applying (16) in (17)

$$\gamma_1^2 = \frac{(1+\zeta)^2 c_2^2}{\Theta_1^2(\ell)} \tag{20}$$

Replacing γ_1^2 in (19)

$$c_2^2 = \frac{2\Theta_1^3(\ell)(\gamma_2+\lambda_2)}{\left[(2+4\zeta-2\zeta^2)\,\Theta_1^2(\ell)-(1+\zeta)^2\left(\Theta_2(\ell)-2\Theta_1(\ell)\right)\right]} \tag{21}$$

$$\Rightarrow |c_2|^2 = \frac{2\Theta_1^3(\ell)\left(|\gamma_2|+|\lambda_2|\right)}{\left|\left[(2+4\zeta-2\zeta^2)\,\Theta_1^2(\ell)-(1+\zeta)^2\left(\Theta_2(\ell)-2\Theta_1(\ell)\right)\right]\right|}$$

Applying Lemma 1 and (3), we have:

$$|c_2| \le \sqrt{\frac{2\left(2\ell-1\right)^3}{\left|\left[(1+2\zeta-\zeta^2)\left(2\ell-1\right)^2-2(1+\zeta)^2\left(\ell^2-3\ell+1\right)\right]\right|}} = \sqrt{\Upsilon(\zeta,\ell)}.$$

Subtracting (15) from (13), then view (16) and with some computations, we obtain

$$c_3 = c_2^2 + \frac{\Theta_1(\ell)\left(\gamma_2-\lambda_2\right)}{8(1+2\zeta)} \tag{22}$$

By (18) and (16)

$$c_3 = \frac{\Theta_1^2(\ell)\gamma_1^2}{(1+\zeta)^2} + \frac{\Theta_1(\ell)\left(\gamma_2-\lambda_2\right)}{8(1+2\zeta)}. \tag{23}$$

Applying Lemma 1 and (3), we have:

$$|c_3| \le \frac{\left(2\ell-1\right)^2}{(1+\zeta)^2} + \frac{2\ell-1}{4(1+2\zeta)}.$$

From (22), we obtain

$$c_3 - \varkappa c_2^2 = \frac{\Theta_1(\ell)\left(\gamma_2-\lambda_2\right)}{8(1+2\zeta)} + (1-\varkappa)c_2^2$$

Applying the triangular inequality with assist (3), we obtain:

$$\left|c_3-\varkappa c_2^2\right| \le \frac{2\ell-1}{4(1+2\zeta)} + |1-\varkappa|\,\Upsilon(\zeta,\ell)$$

If

$$|1-\varkappa|\,\Upsilon(\zeta,\ell) \le \frac{2\ell-1}{4(1+2\zeta)}$$

we obtain

$$\left|c_3-\varkappa c_2^2\right| \le \frac{2\ell-1}{2(1+2\zeta)}$$

and if:

$$|1-\varkappa|\,\Upsilon(\zeta,\ell)\geq\frac{2\ell-1}{4(1+2\zeta)}$$

we obtain

$$\left|c_3-\varkappa c_2^2\right|\leq 2\,|1-\varkappa|\,\Upsilon(\zeta,\ell)$$

Which are asserted by the Theorem 1.

4. Some Corollaries

If we set $\zeta=1$ in Theorems 1, we get the next corollary.

Corollary 1. *Let $b\in\Pi$ given by (1) belongs to the class $\mathcal{F}_\Pi(1,\ell)$ where $\frac{1}{2}<\ell\leq 1$ $\kappa,w\ \in\ \Lambda$ and $q=b^{-1}$. Then*

$$|c_2|\leq\sqrt{\Upsilon(1,\ell)},$$

$$|c_3|\leq\frac{2\,(2\ell-1)\,(3\ell-1)}{12}.$$

and

$$\left|c_3-\varkappa c_2^2\right|\leq\begin{cases}\frac{2\ell-1}{6} & 0\leq|1-\varkappa|\,\Upsilon(1,\ell)<\frac{2\ell-1}{12},\\ 2\,|1-\varkappa|\,\Upsilon(1,\ell) & |1-\varkappa|\,\Upsilon(1,\ell)\geq\frac{2\ell-1}{12}.\end{cases}$$

where

$$\Upsilon(1,\ell)=\frac{(2\ell-1)^3}{\left|(2\ell-1)^2-4\left(\ell^2-3\ell+1\right)\right|}.$$

If we set $\zeta=0$ in Theorems 1, we get the next corollary.

Corollary 2. *Let $b\in\Pi$ given by (1) belongs to the class $\mathcal{F}_\Pi(0,\ell)$ where $\frac{1}{2}<\ell\leq 1$ $\kappa,w\ \in\ \Lambda$ and $q=b^{-1}$. Then*

$$|c_2|\leq\sqrt{\Upsilon(0,\ell)},$$

$$|c_3|\leq(2\ell-1)^2+\frac{2\ell-1}{4}.$$

and

$$\left|c_3-\varkappa c_2^2\right|\leq\begin{cases}\frac{2\ell-1}{2} & 0\leq|1-\varkappa|\,\Upsilon(0,\ell)<\frac{2\ell-1}{4},\\ 2\,|1-\varkappa|\,\Upsilon(0,\ell) & |1-\varkappa|\,\Upsilon(0,\ell)\geq\frac{2\ell-1}{4}.\end{cases}$$

where

$$\Upsilon(0,\ell)=\frac{2\,(2\ell-1)^3}{\left|(2\ell-1)^2-2\left(\ell^2-3\ell+1\right)\right|}.$$

5. Conclusions

Because polynomials and special functions are used in various mathematical and scientific fields, many prominent mathematicians have recently focused on studying them. This paper aims to define new subclasses of analytical and univalent functions using Euler polynomials. For functions belonging to these classes $\mathcal{F}_{\Pi}(\zeta,\ell)$, $\mathcal{F}_{\Pi}(0,\ell)$ and $\mathcal{F}_{\Pi}(1,\ell)$, we have established an upper bound estimate for the coefficients and successfully solved the Fekete-Szegö problem. The sharp upper bounds for $|c_2|$, $|c_3|$ and $\left|c_3 - \varkappa c_2^2\right|$ are still an interesting challenge to discover, as well as the open problem regarding $|c_i|$, $i \geq 3$.

References

[1] T. Al-Hawary. Coefficient bounds and fekete-szegö problem for qualitative subclass of bi-univalent functions. *Afr. Mat.*, 33(1):1–9.

[2] T. Al-Hawary, A. Amourah, A. Alsoboh, and O. Alsalhi. A new comprehensive subclass of analytic bi-univalent functions related to gegenbauer polynomials. *Symmetry*, 15(3):1–11, 2023.

[3] T. Al-Hawary, A. Amourah, and B. A. Frasin. Fekete-szegö inequality for bi-univalent functions by means of horadam polynomials. *Boletin de la Sociedad Matematica Mexicana*, 27:1–12, 2021.

[4] T. Al-Hawary and B. A. Frasin. Coefficient estimates and subordination properties for certain classes of analytic functions of reciprocal order. *Studia Universitatis Babes-Bolyai Mathematica*, 63(2), 2018.

[5] E. Almuhur, M. Khandaqji, M. Al-Labadi, and A. Alboustanji. Predicting pandemic curve distribution using statistical models. *Indian Journal of Forensic Medicine and Toxicology*, 16(1):427–432, 2022.

[6] A. Amourah, T. Al-Hawary, and B. A. Frasin. Application of chebyshev polynomials to certain class of bi-bazilevič functions of order $\alpha + i\beta$. *Afr. Mat.*, pages 1–8, 2021.

[7] A. Amourah, M. Alomari, F. Yousef, and A. Alsoboh. Consolidation of a certain discrete probability distribution with a subclass of bi-univalent functions involving gegenbauer polynomials. *Mathematical Problems in Engineering*, 2022:2022, 2022.

[8] A. Amourah, A. Alsoboh, D. Breaz, and S. M. El-Deeb. A bi-starlike class in a leaf-like domain defined through subordination via q-calculus. *Mathematics*, 12(11):1735, 2024.

[9] A. Amourah, N. Anakira, M. J. Mohammed, and Malath Jasim. Jacobi polynomials and bi-univalent functions. *Int. J. Math. Comput. Sci.*, 19(4):957–968, 2024.

[10] A. Amourah, B. A. Frasin, M. Ahmad, and F. Yousef. Exploiting the pascal distribution series and gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions. *Symmetry*, 14(1):147, 2022.

[11] A. Amourah, B. A. Frasin, G. Murugusundaramoorthy, and T. Al-Hawary. Bi-bazilevič functions of order $\vartheta + i\delta$ associated with $(p,q)-$lucas polynomials. *AIMS Math.*, 6(5):4296–4305, 2021.

[12] A. Amourah and M. Illafe. A comprehensive subclass of analytic and bi-univalent functions associated with subordination. *Palest. J. Math.*, 9(1):187–193, 2020.

[13] N. Anakira, M. J. Mohammed, I. Irianto, and A. Amourah. Exact solution of system of multi-photograph type delay differential equations via new algorithm based on homotopy perturbation method. *Results in Nonlinear Analysis*, 7(2):187–197, 2024.

[14] S. Bulut, N. Magesh, and C. Abirami. A comprehensive class of analytic bi-univalent functions by means of chebyshev polynomials. *J. Fract. Calc. Appl.*, 8(2):32–39, 2017.

[15] A. Burqan, H. Dbabesh, A. Qazza, and M. Khandaqji. New bounds for the eigenvalues of matrix polynomials. *European Journal of Pure and Applied Mathematics*, 16(2):806–818, 2023.

[16] J. Dziok and H. M. Srivastava. Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Integral Transforms Spec. Funct.*, 14(1):7–18, 2003.

[17] M. Fekete and G. Szegö. Eine bemerkung über ungerade schlichte funktionen. *J. Lond. Math. Soc.*, 1(2):85–89, 1933.

[18] B. A. Frasin. Subordination results for a class of analytic functions defined by a linear operator. *J. Inequal. Pure Appl. Math.*, 7(4):1–7, 2006.

[19] V. Kac and P. Cheung. *Quantum Calculus.* Universitext. Springer, New York, NY, USA, 2002.

[20] M. Khandaqji, E. AlMuhur, M. Al-Labadi, and A. Alboustanji. Further results of best simultaneous approximation on function spaces. *Journal of Analysis and Applications*, 20(2):91–104, 2022.

[21] S. S. Miller and P. T. Mocanu. Second order differential inequalities in the complex plane. *J. Math. Anal. Appl.*, 65:289–305, 1978.

[22] S. S. Miller and P. T. Mocanu. Differential subordinations and univalent functions. *Mich. Math. J.*, 28:157–172, 1981.

[23] S. S. Miller and P. T. Mocanu. *Differential Subordinations. Theory and Applications.* Marcel Dekker, Inc., New York, NY, USA, 2000.

[24] G. Murugusundaramoorthy, N. Magesh, and V. Prameela. Coefficient bounds for certain subclasses of bi-univalent function. *Abst. Appl. Anal.*, 2013:573017, 3 pages.

[25] Z. Peng, G. Murugusundaramoorthy, and T. Janani. Coefficient estimate of bi-univalent functions of complex order associated with the hohlov operator. *J. Complex Analysis*, 2014:693908, 6 pages.

[26] Ch. Pommerenke. *Univalent Functions*. Vandenhoeck and Rupercht, Göttingen, 1975.

[27] H. M. Srivastava. Some formulas for the bernoulli and euler polynomials at rational arguments. *Math. Proc. Camb. Philos. Soc.*, 129:77–84, 2000.

[28] H. M. Srivastava. Some families of mittag-leffler type functions and associated operators of fractional calculus. *TWMSJ. Pure Appl. Math.*, 7:123–145, 2016.

[29] H. M. Srivastava, A. K. Mishra, and P. Gochhayat. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.*, 23(10):1188–1192, 2010.

[30] F. Yousef, T. Al-Hawary, and G. Murugusundaramoorthy. Fekete-szegö functional problems for some subclasses of bi-univalent functions defined by frasin differential operator. *Afr. Mat.*, 30(3–4):495–503, 2019.

[31] F. Yousef, S. Alroud, and M. Illafe. New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems. *Anal. Math. Phys.*, 11(2):1–12, 2021.