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The Spectrum of a Certain Large Block Matrix

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Abstract. Large matrices appear in many applications in computer science, physics, chemistry and many other disciplines. This is because such matrices have the ability to hold huge amounts of memory. On of the main properties that researchers are interested is studying the spectral theory of these matrices. In this paper, we compute the spectrum of a certain large matrix that can serve as an adjacency matrix of a certain clean graph. In particular, we give a full characterization of the eigenvalues and eigenvectors of the intended matrix.

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1. Introduction

Large matrices are being used more and more in the big data era because of their potential to integrate and connect massive data sources across a wide range of industries, including social media, biology, communication networks, etc (see for example [2], [3], [4], [6], [12], [13], and [14]). For instance, large networks, such as the Internet, can be utilized to describe intriguing global patterns and occurrences. These networks attracted the mathematician who are interested in graph theory. This is because graphs are very useful ways of presenting information about these networks. In fact, the term graph, which represents an abstract mathematical concept, generally refers to an artificial formation of nodes and edges whereas the term network is then reserved for the graphs representing real-world objects in which the nodes represent units of the system and the edges represent the relationships between them, see [7].

The best way to dealing with graphs is the linear algebraic approach, which is to view graphs as matrices and use concepts in linear algebra to design and analyze algorithms for

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graph problems. Moreover, using matrices allows us to apply mathematical and computational tools to summarize and find patterns, especially when complex relationships exist between vertices and edges in a graph. In fact, many mathematical problems necessarily involve inputting certain coefficients into a matrix and studying its spectral properties. This includes studying the properties of a graph in terms of the characteristic polynomials, eigenvalues, and eigenvectors of a particular matrix associated with a graph in order to characterize the properties of a graph and extract information from its structure.

A variety of matrices associated with a graph are used, including adjacency matrices, Laplace matrices, and normalized Laplace matrices. The adjacency matrix of a simple undirected graph is a real symmetric matrix and is therefore orthogonally diagonalizable; its eigenvalues are real algebraic integers. Although the adjacency matrix depends on the vertex labels, its spectrum is a graph invariant, for more details, one can see [1] and [5].

The clean graph $Cl(R)$ is defined to be the graph in which every vertex has the form (a, v) where, a is an idempotent in the ring R and v is a unit. Nicholson [10] was the first to introduce the clean rings. The clean graph of a commutative ring was introduced by Petrovi´c and Pucanovic´ [11] in 2017 . Also, in 2021, Habibi et.al. [8], has determined the clique number, the chromatic number and the domination number of the clean graph $Cl(R)$ for some classes of rings. In this paper, we evaluate the spectrum of a certain large block matrix that forms an adjacency matrix of a clean graph.

2. The Main Result

2.1. General

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix of G, denoted by $A(G)$, is a square matrix of order $|V(G)|$ with ij-th entry equals 1 if v_iv_j in $E(G)$ and 0 otherwise, where v_i and v_j are vertices in $V(G)$. Let p be any prime number that is greater than or equal 5, $s = (p-1)^2$, I_s is the identity matrix of order s, and J_s denote the all-1 square matrix of order s.

Define A as a block matrix

$$A = \begin{pmatrix} K_s & J_s & J_s & J_s \\ J_s & Q_s & J_s & Q_s \\ J_s & J_s & Q_s & Q_s \\ J_s & Q_s & Q_s & Q_s \end{pmatrix} \tag{1}$$

of order $4s$, where $K_s = J_s - I_s$ and Q_s is a triadiognal matrix of oredr s defined by

$$(Q_s)_{i,j} = \begin{cases} 0, & \text{if} \quad i \ \text{ or } \ j \in \{1, 2, 3, 4\}, \\ 0, & \text{if} \quad i = j, \\ 1, & \text{if} \quad j = i + 1 \ \text{ and } \ j \geq 6 \ \text{ is even}, \\ 1, & \text{if} \quad j = i - 1 \ \text{ and } \ j \geq 7 \ \text{ is odd}, \\ 0, & \textit{Otherwise}. \end{cases}$$

In this section, we study the spectrum of the matrix A that is given by (1). This matrix appears in graph theory and it is an adjacency matrix of a clean graph [9].

Let X be an eigenvector of the matrix A corresponding to the eigenvalue λ, by looking deeply to the construction of the matrix A, we may consider the entries of the vector X to be

$$(X)_i = \begin{cases} a_i, & \text{if} \quad i = 1, 2, \ldots, s, \\ b_{i-s}, & \text{if} \quad i = s+1, s+2, \ldots, s+4, \\ b^*_{i-s-4}, & \text{if} \quad i = s+5, s+6, \ldots, 2s, \\ c_{i-2s}, & \text{if} \quad i = 2s+1, 2s+2, \ldots, 2s+4, \\ c^*_{i-2s-4}, & \text{if} \quad i = 2s+5, 2s+6, \ldots, 3s, \\ d_{i-3s}, & \text{if} \quad i = 3s+1, 3s+2, \ldots, 3s+4, \\ d^*_{i-3s-4}, & \text{if} \quad i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{2}$$

Since, we have to find λ so that $AX = \lambda X$, then X is an eigenvector of the matrix A corresponding to the eigenvalue λ if and only if all the following equations are satisfied:

$$\sum_{i=1}^{s} a_i + \sum_{j=1}^{4} b_j + \sum_{i=1}^{s-4} b^*_i + \sum_{j=1}^{4} c_j + \sum_{i=1}^{s-4} c^*_i + \sum_{j=1}^{4} d_j + \sum_{i=1}^{s-4} d^*_i = (\lambda+1)a_r, \tag{3}$$

$$\sum_{i=1}^{s} a_i + \sum_{j=1}^{4} c_j + \sum_{i=1}^{s-4} c^*_i = \lambda b_m, \tag{4}$$

$$\sum_{i=1}^{s} a_i + b^*_{k+1} + \sum_{j=1}^{4} c_j + \sum_{i=1}^{s-4} c^*_i + d^*_{k+1} = \lambda b^*_k, \tag{5}$$

$$\sum_{i=1}^{s} a_i + b^*_k + \sum_{j=1}^{4} c_j + \sum_{i=1}^{s-4} c^*_i + d^*_k = \lambda b^*_{k+1}, \tag{6}$$

$$\sum_{i=1}^{s} a_i + \sum_{j=1}^{4} b_j + \sum_{i=1}^{s-4} b^*_i = \lambda c_m, \tag{7}$$

$$\sum_{i=1}^{s} a_i + c^*_{k+1} + \sum_{j=1}^{4} b_j + \sum_{i=1}^{s-4} b^*_i + d^*_{k+1} = \lambda c^*_k, \tag{8}$$

$$\sum_{i=1}^{s} a_i + c^*_k + \sum_{j=1}^{4} b_j + \sum_{i=1}^{s-4} b^*_i + d^*_k = \lambda c^*_{k+1}, \tag{9}$$

$$\sum_{i=1}^{s} a_i = \lambda d_m, \tag{10}$$

$$\sum_{i=1}^{s} a_i + b^*_{k+1} + c^*_{k+1} + d^*_{k+1} = \lambda d^*_k, \tag{11}$$

and

$$\sum_{i=1}^{s} a_i + b_k^* + c_k^* + d_k^* = \lambda d_{k+1}^*, \tag{12}$$

where $r = 1, 2, \ldots, s, m = 1, 2, 3, 4,$ and $k = 1, 3, \ldots, s-5$.

Since equation (10) is true for all $m = 1, 2, 3, 4$, we get

$$4\sum_{i=1}^{s} a_i = \lambda \sum_{j=1}^{4} d_j. \tag{13}$$

In order to find the eigenvalues of the matrix A, we need first to prove the following lemma.

Lemma 1. *Suppose that X is given by equation (2). If X is an eigenvector of the matrix A corresponding to the eigenvalue λ such that $\lambda \neq 1$ and λ is not a root of the polynomial*

$$q(x) = x^5 - (2s+1)x^4 - (2s^2 - 2s - 1)x^3 + (s^3 - 2s^2 + 26s + 7)x^2 - (8s^2 - 8s - 4)x - 16s, \tag{14}$$

then $\sum_{j=1}^{4}(b_j + c_j) = 0$, $\sum_{j}^{4} d_j = 0$, and $\sum_{i=1}^{s-4} d_i^* = 0$.

Proof. Let X be given by (2). If X is an eigenvector of the matrix A corresponding to the eigenvalue λ, then all equations from (3) up to (12) have to be satisfied for all $r = 1, 2, \ldots, 5, m = 1, 2, 3, 4,$ and $k = 1, 3, \ldots, s-5$.

Substitute equations (4) and (7) in equation (3) to get

$$\lambda(b_m + c_m) - \sum_{i=1}^{s} a_i + \sum_{j=1}^{4} d_j + \sum_{i=1}^{s-4} d_i^* = (\lambda + 1)a_r. \tag{15}$$

Since this equation is true for all $r = 1, 2, \ldots, s$ and $m = 1, 2, 3, 4$, we get by the help of equation (13) that

$$s\lambda \sum_{j=1}^{4}(b_j + c_j) - (\lambda(\lambda + 1 + s) - 4s)\sum_{j=1}^{4} d_j + 4s\sum_{i=1}^{s-4} d_i^* = 0. \tag{16}$$

Subtract equation (4) from equation (5) and equation (4) from equation (6) and in the same way subtract equation (7) from equation (8) and equation (7) from equation (9) to get

$$(\lambda - 1)(b_k^* + b_{k+1}^* + c_k^* + c_{k+1}^*) = 2(d_k^* + d_{k+1}^*) + 2\lambda(b_m + c_m), \tag{17}$$

using equations (11) and (12), equation (17) becomes

$$(\lambda - 1)^2(d_k^* + d_{k+1}^*) - 2(\lambda - 1)\sum_{i=1}^{s} a_i = 2(d_k^* + d_{k+1}^*) + 2\lambda(b_m + c_m). \tag{18}$$

Since this equation is true for all $k = 1, 3, 5, \ldots, s-4$ and $m = 1, 2, 3, 4$, we get by the help of equation (13) that

$$4((\lambda - 1)^2 - 2)\sum_{i=1}^{s-4} d_i^* = \lambda(\lambda - 1)(s-4)\sum_{j=1}^{4} d_j + \lambda(s-4)\sum_{j=1}^{4}(b_j + c_j). \tag{19}$$

From equation (17), we get

$$(\lambda-1)\sum_{i=1}^{s-4}(b_i^*+c_i^*)=2\sum_{i=1}^{s-4}d_i^*+\lambda(s-4)(b_m+c_m). \tag{20}$$

Adding equation (4) to equation (7) and use (13), equation (28) becomes

$$8\sum_{i=1}^{s-4}d_i^*=2\lambda(1-\lambda)\sum_{j=1}^{4}d_j+(\lambda^2-(s+1)\lambda+4)\sum_{j=1}^{4}(b_j+c_j). \tag{21}$$

Equations (16), (19), and (21) form a homogeneous linear system with the variables $\sum_j^4 d_j, \sum_{i=1}^{s-4} d_i^*$, and $\sum_{j=1}^4(b_j+c_j)$ that has the coefficient matrix

$$B=\begin{pmatrix} s\lambda & -(\lambda(\lambda+1+s)-4s) & 4s \\ \lambda(s-4) & \lambda(\lambda-1)(s-4) & 4(2-(\lambda-1)^2) \\ (\lambda^2-(s+1)\lambda+4) & 2\lambda(1-\lambda) & -8 \end{pmatrix}. \tag{22}$$

Using Maple, we can compute the determinant of the matrix B to get

$$det(B)=4(\lambda-1)q(\lambda),$$

where

$$q(\lambda)=(\lambda^5-(2s+1)\lambda^4-(2s^2-2s-1)\lambda^3+(s^3-2s^2+26s+7)\lambda^2-(8s^2-8s-4)\lambda-16s).$$

Thus if $\lambda\neq 1$ and λ is not a root of the polynomial $q(x)$, then $\sum_{j=1}^4(b_j+c_j)=0$, $\sum_j^4 d_j=0$, and $\sum_{i=1}^{s-4} d_i^*=0$.

Remark 1. *Let λ be an eigenvalue of the matrix A with corresponding eigenvector X given by* (2).

(I) It is clear from equation (3) that if $\lambda\neq -1$, then $a_1=a_2=\cdots=a_s=a$.

(II) Subtracting equation (5) from (6) and equation (8) from equation (9), we find that

$$(\lambda+1)(b_{k+1}^*-b_k^*)=(\lambda+1)(c_{k+1}^*-c_k^*)=d_k^*-d_{k+1}^*. \tag{23}$$

Also subtract equation (11) from equation (12) to get

$$(\lambda+1)(d_k^*-d_{k+1}^*)=b_{k+1}^*-b_k^*+c_{k+1}^*-c_k^*. \tag{24}$$

Thus from equations (23) and (24), we obtain

$$(\lambda^2+2\lambda-1)(d_k^*-d_{k+1}^*)=0. \tag{25}$$

This gives us that if $\lambda^2+2\lambda-1\neq 0$ and $\lambda\neq -1$, then from equations (23) and (25) we have $b_k^*=b_{k+1}^*, c_k^*=c_{k+1}^*$ and $d_k^*=d_{k+1}^*$ for all $k=1,3,\ldots,s-5$.

(III) If $\lambda^2 - 2\lambda - 1 \neq 0$, $\sum_{i=1}^{s} a_i = 0$ and $b_m + c_m = 0$ for all $m = 1, 2, 3, 4$, then from equation (18), $d_k^* = -d_{k+1}^*$ for all $k = 1, 3, \ldots, s-5$.

(IV) If $\lambda \neq 0$, we get from equations (4), (6), and (12) that $b_1 = b_2 = \cdots = b_4 = b$, $c_1 = c_2 = \cdots = c_4 = c$, and $d_1 = d_2 = \cdots = d_4 = d$.

(V) If $\lambda \neq 0$ and $\lambda^2 + 2\lambda - 1 \neq 0$ then by subtracting equation (4) from (5), we get $b_k^* + d_k^* = \lambda b_k^* - \lambda b$. Thus if $\sum_{i=1}^{s-4} d_i^* = 0$, then

$$(\lambda - 1)\sum_{i=1}^{s-4} b_i^* = \lambda(s-4)b. \tag{26}$$

(VI) If $\lambda \neq 0$, $\lambda \neq 1$, $\lambda^2 + 2\lambda - 1 \neq 0$, and $q(\lambda) \neq 0$, then from Lemma 1 and part (IV) of this remark, we have

$$\sum_{j}^{4} d_j = \sum_{i=1}^{s-4} d_i^* = 4(b+c) = 0,$$

so from equation (13), we obtain $\sum_{i=1}^{s} a_i = 0$. Thus, equation (7) implies that

$$4b + \sum_{i=1}^{s-4} b_i^* = -\lambda b. \tag{27}$$

From equations (26) and (27), we get

$$(\lambda^2 + (s-1)\lambda - 4)b = 0. \tag{28}$$

Based on this remark and Lemma 1, we have the following results.

Lemma 2. *$\lambda = -1$ is an eigenvalue of the matrix A with multiplicity is greater than or equal to $s - 1 + \frac{s-4}{2}$.*

Proof. We have to show that all equations from (3) to (12) are satisfied with $\lambda = -1$. If $\lambda = -1$, then from equation (13), Lemma 1, and Remark 1, we get $\sum_{i=1}^{s} a_i = 0$, $b_1 = b_2 = \cdots = b_4 = b = 0$, $c_1 = c_2 = \cdots = c_4 = c = 0$, and $d_1 = d_2 = d_3 = d_4 = d = 0$. Now, from equations (18) and (23), we get $d_k^* = 0(d_k^* = d_{k+1}^* = -d_{k+1}^*)$ for all $k = 1, 2, \ldots, s-4$. By subtracting equation (4) from equations (5) and equation (7) from equation (8), we get $b_k^* = -b_{k+1}^*$ and $c_k^* = -c_{k+1}^*$ for all $k = 1, 3, \ldots, s-5$. From equation (12), we get $b_k^* = -c_k^*$ for all $k = 1, 2, \ldots, s-4$. Based on these facts, the vector X with entries

$$(X)_i = \begin{cases} a_i, & \text{if } i = 1, 2, \ldots, s, \\ 0, & \text{if } i = s+1, s+2, \ldots, s+4, \\ b_{i-s-4}^*, & \text{if } i = s+5, s+6, \ldots, 2s, \\ 0, & \text{if } i = 2s+1, 2s+2, \ldots, 2s+4, \\ -b_{i-2s-4}^*, & \text{if } i = 2s+5, 2s+6, \ldots, 3s, \\ 0, & \text{if } i = 3s+1, 3s+2, \ldots, 3s+4, \\ 0, & \text{if } i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{29}$$

satisfy all equations from (3) to (12) with $\lambda = -1$ *provided that* $\sum_{i=1}^{s} a_i = 0$ *and* $b_k^* = -b_{k+1}^*$ *for all* $k = 1, 3, \dots s-5$. *Thus, X is the corresponding eigenvector for the eigenvalue* $\lambda = -1$ *of A. Therefore,* $\lambda = -1$ *is an eigenvalue of the matrix A with multiplicity is greater than or equal to* $s - 1 + \frac{s-4}{2}$.

Corollary 1. $\lambda = 0$ *is an eigenvalue of the matrix A with multiplicity is greater than or equal to 9.*

Proof. *If* $\lambda = 0$*, then subtraction equation (4) from (5) and (7) from (8), we get*

$$b_k^* + d_k^* = c_k^* + d_k^* = 0. \tag{30}$$

From Lemma 1, equation (13), equation (12), and equation (30), we get $a_1 = a_2 = \cdots = a_s = 0$, $b_1^* = b_2^* = \cdots = b_{s-4}^* = 0$, $c_1^* = c_2^* = \cdots = c_{s-4}^* = 0$, *and* $d_1^* = d_2^* = \cdots = d_{s-4}^* = 0$.

Thus, from equations (4) and (7), we get

$$\sum_{j=1}^{4} b_j = \sum_{j=1}^{4} c_j = \sum_{j=1}^{4} d_j = 0.$$

Under these conditions, the vector X that has the entries

$$(X)_i = \begin{cases} 0, & \textit{if} \quad i = 1, 2, \dots, s, \\ b_i, & \textit{if} \quad i = s+1, s+2, \dots, s+4, \\ 0, & \textit{if} \quad i = s+5, s+6, \dots, 2s, \\ c_i, & \textit{if} \quad i = 2s+1, 2s+2, \dots, 2s+4, \\ 0, & \textit{if} \quad i = 2s+5, 2s+6, \dots, 3s, \\ d_i, & \textit{if} \quad i = 3s+1, 3s+2, \dots, 3s+4, \\ 0, & \textit{if} \quad i = 3s+5, 3s+6, \dots, 4s. \end{cases} \tag{31}$$

such that $\sum_{j=1}^{4} b_j = \sum_{j=1}^{4} c_j = \sum_{j=1}^{4} d_j = 0$ *is the corresponding eigenvector for the eigenvalue* $\lambda = 0$ *of A. Thus,* $\lambda = 0$ *is an eigenvalue of the matrix A with multiplicity is greater than or equal to 9.*

Corollary 2. *The roots of the quadratic polynomial* $x^2 + 2x - 1 = 0$ *are eigenvalues of the matrix A with multiplicity is greater than or equal to* $\frac{s-4}{2}$.

Proof. If $\lambda = -1 \pm \sqrt{2}$ are the roots of the quadratic equation $x^2 + 2x - 1 = 0$, then from equation (18), Remark 1 and Lemma 1, we get $d_k^* = -d_{k+1}^*$ for all $k = 1, 3, \dots s-5$. Then, if we add equation (5) to (6) and equation (7) to (8), we get $b_k^* = -b_{k+1}^*$ and $c_k^* = -c_{k+1}^*$, respectively, for all $k = 1, 3, \dots s-5$ and so $\sum_{i=1}^{s-4} b_i^* = \sum_{i=1}^{s-4} c_i^* = 0$. Thus, from equations (4) and (7), we obtain $b = c = 0$. From equation (23), we get $b_k^* = c_k^*$ and $d_i^* = (\lambda + 1) b_i^*$

for all $k = 1, 2, \ldots, s-5$. Hence, one can show that the vector X with entries

$$(X)_i = \begin{cases} 0, & \text{if } \; i = 1, 2, \ldots, s, \\ 0, & \text{if } \; i = s+1, s+2, \ldots, s+4, \\ b^*_{i-s-4}, & \text{if } \; i = s+5, s+6, \ldots, 2s, \\ 0, & \text{if } \; i = 2s+1, 2s+2, \ldots, 2s+4, \\ b^*_{i-2s-4}, & \text{if } \; i = 2s+5, 2s+6, \ldots, 3s, \\ 0, & \text{if } \; i = 3s+1, 3s+2, \ldots, 3s+4, \\ d^*_{i-2s-4}, & \text{if } \; i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{32}$$

such that $b^*_k = -b^*_{k+1}$ for all $k = 1, 3, \ldots s-5$ and $d^*_i = (\lambda+1)b^*_i$ for all $i = 1, 2, \ldots, s-4$ is the corresponding eigenvector for the eigenvalues $\lambda = -1 \pm \sqrt{2}$. Thus, If $\lambda = -1 \pm \sqrt{2}$, then they are eigenvalues of the matrix A with multiplicity is greater than or equal to $\frac{s-4}{2}$.

Similar to corollary 2.5, one can prove that the vector X with entries

$$(X)_i = \begin{cases} 0, & \text{if } \; i = 1, 2, \ldots, s, \\ 0, & \text{if } \; i = s+1, s+2, \ldots, s+4, \\ b^*_{i-s-4}, & \text{if } \; i = s+5, s+6, \ldots, 2s, \\ 0, & \text{if } \; i = 2s+1, 2s+2, \ldots, 2s+4, \\ b^*_{i-2s-4}, & \text{if } \; i = 2s+5, 2s+6, \ldots, 3s, \\ 0, & \text{if } \; i = 3s+1, 3s+2, \ldots, 3s+4, \\ d^*_{i-2s-4}, & \text{if } \; i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{33}$$

with $\sum_{i=1}^{s-4} b^*_i = 0$ and $b^*_k = b^*_{k+1}$ for all $k = 1, 3, \ldots s-5$ and $d^*_i = (\lambda-1)b^*_i$ for all $i = 1, 2, \ldots, s-4$ is an eigenvector of A associated with the eigenvalues $\lambda = 1 \pm \sqrt{2}$ which are the roots of the equation $x^2 - 2x - 1 = 0$. Therefor, we have the following result.

Corollary 3. *$\lambda = 1 \pm \sqrt{2}$ are eigenvalues of the matrix A with multiplicity is greater than or equal to $\frac{s-4}{2} - 1$.*

Corollary 4. *The roots of the equation $x^2 + (s-1)x - 4 = 0$ are eigenvalues of the matrix A with multiplicity is greater than or equal to one.*

Proof. *If λ is a root of $x^2 + (s-1)x - 4 = 0$, then from Lemma 1 and equation (10), we get $\sum_{j=1}^{4}(b_j + c_j) = \sum_j^4 d_j = \sum_{i=1}^{s-4} d^*_i = 0$ and $d_1 = d_2 = d_3 = d_4 = 0$, respectively. From equations (18) and (23), we get $d^*_k = 0 (d^*_k = d^*_{k+1} = -d^*_{k+1})$ for all $k = 1, 2, \ldots, s-4$. Then, from equations (5) and (8), we get $b^*_1 = b^*_2 = \cdots = b^*_{s-4} = b^*$ and $c^*_1 = c^*_2 = \cdots = c^*_{s-4} = c^*$. Thus from equation (17), we have $c^* = -b^*$. Finally, from* *equations (26), we get* $b^* = \frac{\lambda}{\lambda-1}b$*. Hence, the vector* X *with entries*

$$(X)_i = \begin{cases} 0, & \text{if} \quad i = 1, 2, \ldots, s, \\ b, & \text{if} \quad i = s+1, s+2, \ldots, s+4, \\ b^*, & \text{if} \quad i = s+5, s+6, \ldots, 2s, \\ -b, & \text{if} \quad i = 2s+1, 2s+2, \ldots, 2s+4, \\ -b^*, & \text{if} \quad i = 2s+5, 2s+6, \ldots, 3s, \\ 0, & \text{if} \quad i = 3s+1, 3s+2, \ldots, 3s+4 \\ 0, & \text{if} \quad i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{34}$$

such that $b^* = \frac{\lambda}{\lambda-1}b$ *is an eigenvectorA associated with the eigenvalue* λ*, where* $\lambda^2 + (s-1)\lambda - 4 = 0$*. Hence, If* λ *is a root of the equation* $x^2 + (s-1)x - 4 = 0$*, then it is an eigenvalue of the matrix* A *with multiplicity is greater than or equal to one.*

Remark 2. *(1) It is not difficult to show that the eigenvalues that we so far discovered none of them is a root of the polynomial* $q(x)$ *that is given by (14).*

(2) Suppose $\lambda = 1$ *or* λ *is a root of the polynomial* $q(x)$*. Then, from equation (3), we have* $a_1 = a_2 = \cdots = a_s = a$*. Since* $\lambda^2 + 2\lambda - 1 \neq 0$*, then we get from Remark (1) that* $d^*_k = d^*_{k+1}$*,* $b^*_k = b^*_{k+1}$ *and* $c^*_k = c^*_{k+1}$ *for all* $k = 1, 3, \ldots s-5$*. Since* $\lambda \neq 0$*, then from equations (4), (7), and (10), we obtain* $b_1 = b_2 = \cdots = b$*,* $c_1 = c_2 = \cdots = c$*, and* $d_1 = d_2 = d_3 = d_4 = 0$*. Moreover, from equation (18), we get* $d^*_1 = d^*_2 = \cdots = d^*_{s-4} = d^*$ *and if* $\lambda \neq 1$*, we get from equation (5) and (8) that* $b^*_1 = b^*_2 = \cdots = b^*_{s-4} = b^*$ *and* $c^*_1 = c^*_2 = \cdots = c^*_{s-4} = c^*$*. Therefore, we have the following lemmas.*

Lemma 3. $\lambda = 1$ *is an eigenvalue of the matrix* A *with multiplicity is greater than or equal to* $\frac{s-4}{2} - 1$*.*

Proof. *If* $\lambda = 1$*, then using Remark (2) and by subtracting equation (4) from equation (5) and equation (7) from equation (8), we get* $d^* = -c = -b$*. Equation (12) gives* $sa = -(b^*_k + c^*_k)$ *for all* $k = 1, 2, \ldots s-4$*. Add equation (4) to equation (7) to get*

$$6b + (6s - s^2)a = 0$$

and from equation (10) and (15), we get

$$(6-s)b + (3s-2)a = 0.$$

Since $s > 16$*, then we can show that* $b = a = 0$ *and* $d^* = -c = d = 0$ *and so* $c^*_k = -b^*_k$ *for all* $k = 1, 2, \ldots s-4$*. Moreover, from equation (7), we get* $\sum_{j=1}^{4} b^*_j = 0$*. Therefore, the* *vector X with entries*

$$(X)_i = \begin{cases} 0, & if \quad i = 1, 2, \ldots, s, \\ 0, & if \quad i = s+1, s+2, \ldots, s+4, \\ b^*_{i-s-4}, & if \quad i = s+5, s+6, \ldots, 2s, \\ 0, & if \quad i = 2s+1, 2s+2, \ldots, 2s+4, \\ -b^*_{i-2s-4}, & if \quad i = 2s+5, 2s+6, \ldots, 3s, \\ 0, & if \quad i = 3s+1, 3s+2, \ldots, 3s+4, \\ 0, & if \quad i = 3s+5, 3s+6, \ldots, 4s. \end{cases} \tag{35}$$

such that $\sum_{i=1}^{s-4} b^*_i = 0$ and $b^*_k = b^*_{k+1}$ for all $k = 1, 3, \ldots, s-5$ is an eigenvector for $\lambda = 1$ of the matrix A, which means that $\lambda = 1$ is an eigenvalue of the matrix A with multiplicity is greater than or equal to $\frac{s-4}{2} - 1$.

Lemma 4. *Let λ be a root of the polynomial $q(x)$ that is given by (14), then λ is an eigenvalue of the matrix A with multiplicity is greater than or equal to one.*

Proof. *If $q(\lambda) = 0$, then from Remark 2 and since $\lambda \neq 1$, we have $a_1 = a_2 = \cdots = a_s = a$, $b_1 = b_2 = \cdots = b$, $c_1 = c_2 = \cdots = c$, $b^*_1 = b^*_2 = \cdots = b^*_{s-4} = b^*$ and $c^*_1 = c^*_2 = \cdots = c^*_{s-4} = c^*$. Now subtraction equation (7) from equation (4) gives*

$$4(c-b) + (s-4)(c^* - b^*) = \lambda(b-c).$$

Also, subtraction equation (8) from equation (6) gives

$$(b^* - c^*) + 4(c-b) + (s-4)(c^* - b^*) = \lambda(b^* - c^*).$$

Solve these equations to get $b = c$ and $b^* = c^*$. Equations (3) to (12) will be reduced to the following linear system:

$$\begin{array}{lllllll} (s-(\lambda+1))a + & 8b & + \ 2(s-4)b^* & + \ 4d & + \ (s-4)d^* & = 0 \\ sa & + \ (4-\lambda)b + & (s-4)b^* & & & = 0 \\ sa & + \ 4b & + \ (s-3-\lambda)b^* & & + \ d^* & = 0 \\ sa & & & - \ \lambda d & & = 0 \\ sa & & + \ 2b^* & & + \ (1-\lambda)d^* & = 0. \end{array} \tag{36}$$

Thus, λ is an eigenvalue of A if the system given by (36) has a nontrivial solution. Using Maple, we can see that this system has only one free variable if $q(\lambda) = 0$ and has only the trivial solution if $q(\lambda) \neq 0$. Thus the roots of the polynomial $q(x)$ form eigenvalues of the matrix A with multiplicity is at least one.

Considering the above Lemmas and corollaries, we reach to the main theorem of the paper.

Theorem 1. *The spectral radius of the matrix A is given by*

$$\sigma(A) = \begin{pmatrix} -1 & 0 & -1 \pm \sqrt{2} & 1 \pm \sqrt{2} & 1 & \lambda_1 & \lambda_2 & \dots & \lambda_7 \\ \frac{3s-6}{2} & 9 & \frac{s-4}{2} & \frac{s-6}{2} & \frac{s-6}{2} & 1 & 1 & \dots & 1 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_7$ *are the distinct real roots of the polynomial*

$$(x^2 + (s-1)x - 4)q(x),$$

where $q(x)$ *is given by (14).*

Proof. It is not difficult to show that the sign of $q(0), q(s)$, and $q(-s)$ is negative and the sign of $q(1), q(-1)$, and $q(3s)$ is positive, thus by applying the intermediate value theorem, the polynomial $q(x)$ has five different real roots, and since the roots of the quadratic polynomial $(x^2 - (s-1)x + 4)$ are not roots of $q(x)$, we get that the polynomial $(x^2 - (s-1)x + 4)q(x)$ has seven distinct real roots. Moreover, the summation of the lower bound of the multiplicity of each eigenvalues of the matrix A that we found through this section is $4s$ which is the size of the matrix A. Therefore, the multiplicity of each eigenvalue is exactly the lower bound. This completes the proof.

3. Conclusions

The spectrum of a certain large block matrix has been determined. This matrix can be considered as an adjacency matrix of a certain graph. More precisely, it has been proved that the proposed matrix has fourteen distinct eigenvectors. In addition the eigenspace of each eigenvalue has been determined.

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