EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 2562-2573
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Limited Role of Conservation Laws in the Double Reduction Routine for Partial Differential Equations

Winter Sinkala[1,*], Molahlehi Charles Kakuli[1]

[1] *Department of Mathematical Sciences and Computing, Faculty of Natural Sciences, Walter Sisulu University, South Africa*

Abstract. The double reduction method for finding invariant solutions of a given partial differential equation (PDE) provides for the reduction of a q-th order PDE that admits a Lie symmetry and an associated nontrivial conservation law to an ordinary differential equation (ODE) of order $q-1$. In all the articles we have seen where the method has been used, the algorithm has involved writing the conservation law in canonical variables determined by the associated symmetry. In this paper, we illustrate that it is not necessary to use or even have the associated conservation law. It is enough to know that there exists a conservation law associated with a given Lie symmetry. Canonical variables derived from the symmetry are sufficient to achieve double reduction. In the canonical variables, the PDE is transformed after routine calculations into an ODE of order one less than that of the PDE. We have outlined steps involved in this variation of the double reduction method and illustrated the routine using five PDEs.

2020 Mathematics Subject Classifications: 35C05, 58J70, 70H33, 35B06

Key Words and Phrases: Double reduction, Lie symmetry analysis, Conservation law, Invariant solution

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1. Introduction

The double reduction method, proposed by Sjöberg [19, 20], provides a powerful routine that exploits the relationship between Lie symmetries and conservation laws of a given PDE to find invariant solutions of the PDE. The theory of double reduction relies on the pioneering work by Kara and Mahomed [12, 13] on the relationship between Lie symmetries and conservation laws. For a scalar $(1+1)$-PDE of order q that admits a Lie symmetry and an associated nontrivial conservation law (in the sense defined in [12]), double reduction of the PDE amounts to a reduction of the equation to an ODE of order $q-1$.

All the articles we have examined that have used the double reduction method, as proposed by Sjöberg [19, 20], have exploited both the admitted Lie symmetries and the

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5318

Email addresses: `wsinkala@wsu.ac.za` (W. Sinkala), `ckakuli@wsu.ac.za` (M.C. Kakuli)

corresponding nontrivial conservation laws in the reduction algorithm (evident in references [1–4, 6–9, 11, 14, 15, 17–20]). In this article we demonstrate that it is not essential to utilise an associated conservation law in the reduction routine. Rather, it suffices to know that a given Lie symmetry has an associated nontrivial conservation law. For Euler-Lagrange PDEs, for example, every admitted Noether symmetry has an associated conservation law. Therefore, one could proceed to perform double reduction of an Euler-Lagrange PDE using only an admitted Noether symmetry.

The rest of the paper is organised as follows. In Section 2, the theory of double reduction is presented, based on $(1+1)$-scalar PDEs as proposed by Sjöberg [19, 20]. In Section 3, we present the steps involved in the proposed alternative double reduction algorithm and provide illustrative examples based on five PDEs. We give concluding remarks in Section 4.

2. Theory of double reduction of a scalar $(1+1)$-PDE

Preliminaries of the double reduction method are extensively covered in numerous articles. To avoid redundancy, these foundational details will not be repeated in this article. Interested readers can consult references [1–4, 6–8, 11, 15, 17–20]), and other related works for comprehensive information.

Consider a scalar qth-order ($q \geq 1$) PDE with two independent variables $(x^1, x^2) = (t, x)$ and one dependent variable $u = u(t, x)$,

$$F(t, x, u, \dots, u_{(q)}) = 0, \tag{1}$$

where $u_{(k)}$ denotes the collection $\{u_k\}$ of kth-order partial derivatives. Furthermore, suppose that Equation (1) admits a Lie point symmetry with an infinitesimal generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta(x, u)\frac{\partial}{\partial u}. \tag{2}$$

A conservation law of (1)

$$D_t T^t + D_x T^x = 0 \tag{3}$$

is said to be associated with (2) if

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2. \tag{4}$$

In terms of canonical variables r, s and w, i.e., variables under which (2) is transformed into $X = \frac{\partial}{\partial s}$, the conservation law (3) can be written in canonical variables as [19, 20]

$$D_r T^r + D_s T^s = 0, \tag{5}$$

where

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)} \tag{6}$$

and

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{7}$$

The components T^t and T^x in (3) depend on $(t, x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q-1)})$, which means that T^r and T^s depend on $(r, s, w, w_r, w_{rr}, \ldots, w_{r^{q-1}})$ for solutions invariant with respect to X. Therefore, the conservation law in canonical variables (5) becomes

$$\frac{\partial T^s}{\partial s} + D_r T^r = 0. \tag{8}$$

From the association of X with $T = (T^r, T^s)$, it follows that

$$XT^r \equiv \frac{\partial T^r}{\partial s} = 0 \quad \text{and} \quad XT^s \equiv \frac{\partial T^s}{\partial s} = 0. \tag{9}$$

This leads to further reduction of the conservation law (8) to

$$D_r T^r = 0, \tag{10}$$

or, equivalently,

$$T^r = k, \tag{11}$$

where k is an arbitrary constant. Equation (11) is an ODE of order $q-1$, and its solution can easily be transformed into an invariant solution of (1).

3. An alternative double reduction algorithm

It follows from the double reduction routine that canonical variables, while reducing the conservation law (3) into the ODE (10), must necessarily transform the PDE (1) into an ODE equivalent to (10), i.e., of the same order as the original PDE. The reduction of order by one of this "equivalent" ODE is subsequently achieved through routine calculations. The steps for implementing the alternative double reduction routine are presented below:

(i) Identify a symmetry $X = \xi^i(x,u)\partial_{x^i} + \eta(x,u)\partial u$ of the PDE (1) for which there exists an associated nontrivial conservation law.

(ii) Find similarity variables r, s and w, i.e., variables under which the symmetry X is transformed to its canonical form $Y = \frac{\partial}{\partial s}$. The similarity variables can be determined from the conditions $X(r) = 0$, $X(s) = 1$, $X(w) = 0$. These variables establish an invertible mapping

$$r = R(t,x), \quad s = S(t,x), \quad w = W(t,x,u), \quad \frac{\partial w}{\partial u} \neq 0$$

from the space $(t, x, u, u_{(1)}, \ldots, u_{(q)})$ to the space $(r, s, w, w_r, \ldots, w_{r^q})$.

(iii) Express all partial derivatives of u in (1) in terms of r, s, w and derivatives of w.

(iv) Write the PDE (1) in canonical variables in the form

$$D_r\psi(r, w, w_r, w_{rr}, \ldots, w_{r^{q-1}}) = 0, \tag{12}$$

for some function ψ, where $w = w(r)$. It follows from (12) that

$$\psi(r, w, w_r, w_{rr}, \ldots, w_{r^{q-1}}) = k, \tag{13}$$

where k is an arbitrary constant. Equation (13) is the desired ODE of order $q-1$.

In the illustrative examples that follow, we perform double reduction of five PDEs following the algorithm outlined above. Although we have provided associated nontrivial conservation laws for each of the symmetries used in the examples, the conservation laws are not used in the double reduction procedure.

Example 1. The PDE [7]

$$u_{tt} - u_{txx} - 3u^2u_{xx} + u_{xxxx} - 6uu_x^2 - \frac{1}{x^5} = 0 \tag{14}$$

admits a Lie point symmetry with infinitesimal generator

$$X = 2t\partial_t + x\partial_x - u\partial_u \tag{15}$$

associated with the nontrivial conservation law $D_t(\phi^t) + D_x(\phi^x) = 0$, where

$$\phi^t = (u_t - u_{xx})t - \frac{t^2}{2x^2},\ \phi^x = tu_{xxx} - 3tu^2u_x + u_x. \tag{16}$$

To find canonical variables, we solve characteristic equations

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{-u} \tag{17}$$

corresponding to (15). We obtain canonical variables

$$r = \frac{x}{\sqrt{t}}, \quad s = \frac{\ln t}{2}, \quad w = u\sqrt{t} \ \text{ or } \ w = xu, \tag{18}$$

where $w = w(r)$. In (18) we essentially have two sets of canonical variables, one corresponding to $w = u\sqrt{t}$ and the other to $w = xu$.

Case 1.1: Reduction using canonical variables with $w = u\sqrt{t}$

The inverse canonical variables of (18) in this case are given by

$$t = e^{2s}, \quad x = re^s, \quad u = e^{-s}w, \tag{19}$$

and the partial derivatives that appear in (14), expressed in terms of the canonical variables using the routine outlined in [10], are:

$$\begin{aligned} u_x &= e^{-2s}w_r, \quad u_{xx} = e^{-3s}w_{rr} \\ u_{tt} &= \tfrac{1}{4}e^{-5s}\left(r^2w_{rr} + 5rw_r + 3w\right) \\ u_{xxt} &= -\tfrac{1}{2}e^{-5s}(rw_{rrr} + 3w_{rr}), \quad u_{xxxx} = e^{-5s}w_{rrrr}. \end{aligned} \tag{20}$$

Substituting (19) and (20) into (14), we obtain an ODE of the same order as (14), namely

$$\begin{aligned} &r^7w_{rr} + 5r^6w_r + 2r^6w_{rrr} - 12r^5w^2w_{rr} - 24r^5ww_r^2 \\ &+ 3r^5w + 6r^5w_{rr} + 4r^5w_{rrrr} - 4 = 0. \end{aligned} \tag{21}$$

To obtain a lower order ODE, we find functions h and ψ so that (21) can be written in the form

$$h(r,w)D_r\psi(r,w,w_r,w_{rr},w_{rrr}) = 0. \tag{22}$$

Comparison of equations (21) and (22) leads to a system of determining equations which we solve to obtain $h = r^5$, and

$$\psi = \frac{1}{r^4} - 12w^2w_r + 3rw + r^2w_r + 4w_r + 2rw_{rr} + 4w_{rrr}. \tag{23}$$

Case 1.2: Reduction using canonical variables with $w = xu$

The inverse canonical variables of (18) in this case are given by

$$t = e^{2s}, \quad x = re^s, \quad u = \frac{e^{-s}w}{r}, \tag{24}$$

and partial derivatives that appear in (14), expressed in terms of the canonical variables are:

$$\begin{aligned} u_x &= e^{-2s}\left(\tfrac{w_r}{r} - \tfrac{w}{r^2}\right) \\ u_{xx} &= e^{-3s}\left(\tfrac{2w}{r^3} - \tfrac{2w_r}{r^2} + \tfrac{w_{rr}}{r}\right), \quad u_{tt} = \tfrac{1}{4}e^{-5s}(rw_{rr} + 3w_r) \\ u_{xxt} &= -\tfrac{1}{2}e^{-5s}w_{rrr} \\ u_{xxxx} &= e^{-5s}\left(\tfrac{24w}{r^5} - \tfrac{24w_r}{r^4} + \tfrac{12w_{rr}}{r^3} - \tfrac{4w_{rrr}}{r^2} + \tfrac{w_{rrrr}}{r}\right). \end{aligned} \tag{25}$$

Substituting (24) and (25) into (14), we obtain

$$\begin{aligned} &4r^4w_{rrrr} + 2r^3\left(r^2 - 8\right)w_{rrr} + \left(r^6 - 12r^2w^2 + 48r^2\right)w_{rr} - 24r^2ww_r^2 \\ &+ 3r\left(r^4 + 24w^2 - 32\right)w_r - 48w^3 + 96w - 4 = 2r^5D_r\psi = 0, \end{aligned} \tag{26}$$

where

$$\psi = \frac{1}{2r^4} + \frac{6w^3}{r^4} + \left(1 - \frac{12}{r^4}\right)w + \left(\frac{r}{2} + \frac{12}{r^3} - \frac{6w^2}{r^3}\right)w_r + \left(1 - \frac{6}{r^2}\right)w_{rr} + \frac{2w_{rrr}}{r}. \tag{27}$$

Example 2. The PDE [7]

$$u_{xxxx} - u_{txx} - ne^{nu}u_{xx} + u_{tt} - n^2e^{nu}u_x^2 - \frac{1}{x^4} = 0 \tag{28}$$

admits a Lie point symmetry with infinitesimal generator

$$X = t\partial_t + \frac{x}{2}\,\partial_x - \frac{1}{n}\,\partial_u. \tag{29}$$

The symmetry (29) has an associated nontrivial conservation law $D_t(\phi^t) + D_x(\phi^x) = 0$, where

$$\phi^t = x(u_t - u_{xx}),\ \phi^x = -nxe^{nu}u_x + e^{nu} - u_{xx} + xu_{xxx} + \frac{1}{2x^2}. \tag{30}$$

Canonical variables arising from (29) are

$$r = \frac{x}{\sqrt{t}}, \quad s = \ln t, \quad w = \frac{\ln t}{n} + u \ \text{ or } \ w = \frac{2\ln x}{n} + u, \tag{31}$$

where $w = w(r)$.

Case 2.1: Reduction using canonical variables with $w = n^{-1}\ln t + u$

The inverse canonical variables of (31) in this case are given by

$$t = e^s, \quad x = re^{s/2}, \quad u = \frac{nw - s}{n}, \tag{32}$$

and the following partial derivatives of u in terms of the canonical variables are obtained:

$$\begin{aligned} u_x &= e^{-\frac{s}{2}}w_r, \quad u_{xx} = e^{-s}w_{rr}, \quad u_{tt} = \tfrac{e^{-2s}}{4n}\left(nr^2w_{rr} + 3nrw_r + 4\right) \\ u_{xxt} &= -\tfrac{1}{2}e^{-2s}(rw_{rrr} + 2w_{rr}), \quad u_{xxxx} = e^{-2s}w_{rrrr}. \end{aligned} \tag{33}$$

Substituting (32) and (33) into (28), we obtain

$$\begin{aligned} & w_{rrrr} + \frac{rw_{rrr}}{2} + \frac{r^2w_{rr}}{4} + w_{rr} - ne^{nw}w_{rr} \\ & + \frac{3rw_r}{4} - n^2e^{nw}w_r^2 - \frac{1}{r^4} + \frac{1}{n} = r^{-1}D_r\psi = 0, \end{aligned} \tag{34}$$

where

$$\psi = rw_{rrr} + \frac{r^2w_{rr}}{2} - w_{rr} + \frac{r^3w_r}{4} - nre^{nw}w_r + e^{nw} + \frac{r^2}{2n} + \frac{1}{2r^2}. \tag{35}$$

Case 2.2: Reduction using canonical variables with $w = 2n^{-1}\ln x + u$

The inverse canonical variables of (31) in this case are given by

$$t = e^s, \quad x = re^{s/2}, \quad u = w - \frac{2\ln r + s}{n}. \tag{36}$$

Therefore, the partial derivatives of u expressed in terms of the canonical variables are:

$$\begin{aligned} u_x &= e^{-\frac{s}{2}}\left(w_r - \tfrac{2}{nr}\right) \\ u_{xx} &= e^{-s}\left(\tfrac{2}{nr^2} + w_{rr}\right), \quad u_{tt} = \tfrac{1}{4}re^{-2s}(rw_{rr} + 3w_r) \\ u_{xxt} &= -\tfrac{1}{2}e^{-2s}(rw_{rrr} + 2w_{rr}), \quad u_{xxxx} = e^{-2s}\left(\tfrac{12}{nr^4} + w_{rrrr}\right). \end{aligned} \tag{37}$$

Upon substituting (36) and (37) into (28), we obtain

$$
\begin{aligned}
&4nr^4 w_{rrrr} + 2nr^5 w_{rrr} - nr^2\left(4ne^{nw} - r^4 - 4r^2\right) w_{rr} - 4n^3 r^2 w_r^2 e^{nw} \\
&+ nr\left(16ne^{nw} + 3r^4\right) w_r - 24ne^{nw} - 4n + 48 = 4nr^3 D_r \psi = 0,
\end{aligned} \tag{38}
$$

where

$$
\psi = r w_{rrr} + \left(\frac{r^2}{2} - 1\right) w_{rr} + \left(\frac{r^3}{4} - \frac{ne^{nw}}{r}\right) w_r + \frac{3e^{nw}}{r^2} + -\frac{6}{nr^2} + \frac{1}{2r^2} + \frac{1}{n}. \tag{39}
$$

Example 3. The BBM equation [19]

$$
u_{txx} - u_t + u u_x = 0 \tag{40}
$$

admits Lie point symmetries with infinitesimal generators

$$
X_1 = \partial_x \quad \text{and} \quad X_2 = \partial_t. \tag{41}
$$

These symmetries have an associated nontrivial conservation law $D_t(\phi^t) + D_x(\phi^x) = 0$, where

$$
\phi^t = \frac{u^3}{3}, \ \phi^x = u_t^2 - u_{tx}^2 - u^2 u_{tx} - \frac{u^4}{4}. \tag{42}
$$

Let $X = \alpha X_1 + X_2$, where α is an arbitrary constant. Canonical variables derived from X are

$$
r = x - \alpha t, \quad s = t, \quad w = u, \tag{43}
$$

where $w = w(r)$. From (43), we obtain the inverse canonical variables

$$
t = s, \quad x = \alpha s + r, \quad u = w, \tag{44}
$$

and the following partial derivatives of u expressed in terms of the canonical variables:

$$
u_x = w_r, \quad u_t = -\alpha w_r, \quad u_{xxt} = -\alpha w_{rrr}. \tag{45}
$$

Substituting (44) and (45) into (40), we obtain

$$
\alpha w_r - \alpha w_{rrr} + w w_r = D_r \psi = 0, \tag{46}
$$

where

$$
\psi = \alpha w - \alpha w_{rr} + \frac{w^2}{2}. \tag{47}
$$

Example 4. The PDE [5]

$$
u_{xxx} + u_t + u^2 u_x + u u_x = 0 \tag{48}
$$

admits Lie point symmetries with infinitesimal generators

$$
X_1 = \partial_x \quad \text{and} \quad X_2 = \partial_t. \tag{49}
$$

The nontrivial conservation law of (48) $D_t(\phi^t) + D_x(\phi^x) = 0$, where

$$\phi^t = \frac{u^2}{2} + \frac{u}{2} + \frac{1}{2},\ \phi^x = \frac{u^4}{4} + \frac{u^3}{2} + \frac{u^2}{4} - \frac{u_x^2}{2} + \left(u - \frac{1}{2}\right) u_{xx} \tag{50}$$

is associated with the symmetries in (49).

Let $X = \alpha X_1 + X_2$, where α is an arbitrary constant. Canonical variables derived from X are

$$r = x - \alpha t, \quad s = t, \quad w = u, \tag{51}$$

where $w = w(r)$. From (51), we obtain inverse canonical variables

$$t = s, \quad x = \alpha s + r, \quad u = w, \tag{52}$$

and the partial derivatives:

$$u_x = w_r, \quad u_t = -\alpha w_r, \quad u_{xxx} = w_{rrr}. \tag{53}$$

Substituting (52) and (53) into (48), we obtain

$$w_{rrr} + w^2 w_r + w w_r - \alpha w_r = D_r \psi = 0, \tag{54}$$

where

$$\psi = w_{rr} + \frac{w^3}{3} + \frac{w^2}{2} - \alpha w. \tag{55}$$

Example 5. The PDE [14] (see also [16])

$$u_t - u u_x - u_{xxx} = 0 \tag{56}$$

admits a Lie point symmetry with infinitesimal generator

$$X = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{57}$$

Furthermore, the nontrivial conservation law of (56) $D_t(\phi^t) + D_x(\phi^x) = 0$, where

$$\phi^t = \frac{tu^2}{2} + ux, \quad \phi^x = -\frac{tu^3}{3} - tuu_{xx} + \frac{tu_x^2}{2} - \frac{xu^2}{2} + u_x - xu_{xx}. \tag{58}$$

is associated with (57).

Canonical variables derived from (57) are

$$r = \frac{x^3}{t}, \quad s = \frac{\ln t}{3}, \quad w = ut^{2/3} \ \text{ or } \ w = ux^2, \tag{59}$$

where $w = w(r)$.

Case 5.1: Reduction using canonical variables with $w = ut^{2/3}$

From (59), the inverse canonical variables in this case are given by

$$t = e^{3s}, \quad x = r^{1/3}e^{s}, \quad u = e^{-2s}w. \tag{60}$$

Therefore, the following partial derivatives of u expressed in terms of the canonical variables are obtained from (60):

$$\begin{array}{rcl} u_x &=& 3r^{2/3}e^{-3s}w_r, \quad u_r = -\frac{1}{3}e^{-5s}(3rw_r + 2w) \\ u_{xxx} &=& e^{-5s}\left(27r^2w_{rrr} + 54rw_{rr} + 6w_r\right). \end{array} \tag{61}$$

Substituting (60) and (61) into (56), we obtain

$$\begin{aligned} 81r^2w_{rrr} + 162rw_{rr} + 3\left(3r^{2/3}w + r + 6\right)w_r + 2w& \\ = \frac{3r^{2/3}}{r^{1/3} + w}D_r\psi = 0,& \end{aligned} \tag{62}$$

where

$$\begin{aligned} \psi \;=\; & r^{2/3}w + 2r^{1/3}w^2 + w^3 + 9\left(r^{2/3} + 2r^{1/3}w\right)w_r \\ & + 27\left[\left(r^{4/3}w + r^{5/3}\right)w_{rr} - \frac{1}{2}r^{4/3}w_r^2\right]. \end{aligned} \tag{63}$$

Case 5.2: Reduction using canonical variables with $w = x^2u$

We obtain from (59), in this case, inverse canonical variables

$$t = e^{3s}, \quad x = r^{1/3}e^{s}, \quad u = \frac{e^{-2s}w}{r^{2/3}}, \tag{64}$$

and the following partial derivatives of u expressed in terms of the canonical variables:

$$\left.\begin{array}{c} u_x = \frac{e^{-3s}(3rw_r - 2w)}{r}, \quad u_t = -r^{1/3}e^{-5s}w_r \\ u_{xxx} = e^{-5s}\left(27r^{4/3}w_{rrr} + \frac{24w_r}{r^{2/3}} - \frac{24w}{r^{5/3}}\right). \end{array}\right\} \tag{65}$$

Substituting (64) and (65) into (56), we obtain

$$6w^2 + w(72 - 9rw_r) - 3(r + 24)rw_r - 81r^3w_{rrr} = \frac{3r^3}{r + w}D_r\psi = 0, \tag{66}$$

where

$$\psi = -\frac{w^3}{r^2} - \frac{2(r + 6)w^2}{r^2} - \frac{(r + 24)w}{r} + \frac{27w_r^2}{2} + 27w_r - 27(r + w)w_{rr}. \tag{67}$$

4. Concluding remarks

In the standard double reduction method as proposed by Sjöberg [19, 20], both a Lie symmetry of a PDE and an associated nontrivial conservation law are used to find invariant solutions of the PDE. The algorithm involves writing the conservation law in terms of canonical variables derived from the associated Lie symmetry. The association of the conservation law with the symmetry ensures that, in canonical variables, the conservation law is reduced to an ODE of order one less than that of the PDE. In this article, we have demonstrated that double reduction can be realised by simply transforming the PDE, in stead of the conservation law, using the constructed canonical variables. This means that any symmetry of the PDE known to have an associated nontrivial conservation law may be used to perform double reduction of the PDE, without explicitly using the conservation law in the algorithm. We have provided examples involving five $(1+1)$-PDEs.

For each of the PDEs in the illustrative examples, we started by identifying an admitted Lie point symmetry that has an associated nontrivial conservation law. We then found canonical variables r, s and w, i.e., variables under which the Lie symmetry is transformed into $X = \partial/\partial s$. The canonical variables constitute an invertible mapping from the $(t, x, u, u_{(1)}, \ldots, u_{(q)})$-space to the $(r, s, w, w_r, \ldots, w_{r^q})$-space. In the latter space, the PDE is reduced to an ODE of the same order, but one which could be written in the form $D_r(\cdot) = 0$ to complete the reduction.

Conflict of interest

The authors declare that they have no competing interests.

Author contributions

WS conceived the presented idea. WS and MCK performed the computations, discussed the results, and contributed to the final manuscript. Both authors approved the submitted version.

Acknowledgements

We would like to thank the Directorate of Research Development and Innovation of Walter Sisulu University for continued support. We also thank the anonymous reviewers for their careful reading of our manuscript and their useful comments.

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