EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2142-2154
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Upper and Lower Slight $\alpha(\tau_1, \tau_2)$-Continuity

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Abstract. This paper deals with the notions of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions. Furthermore, several characterizations of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions are discussed.

2020 Mathematics Subject Classifications: 54C08, 54C60, 54E55

Key Words and Phrases: $\alpha(\tau_1, \tau_2)$-open set, upper slightly $\alpha(\tau_1, \tau_2)$-continuous multifunction, lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunction

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1. Introduction

In 1980, Jain [25] introduced the notion of slightly continuous functions. Nour [32] defined slightly semi-continuous functions as a weak form of slight continuity and investigated some characterizations of slightly semi-continuous functions. Noiri and Chae [30] have further investigated slightly semi-continuous functions. Pal and Bhattacharyya [33] introduced and studied the concept of faintly precontinuous functions. Slight continuity implies both slight semi-continuity and faint precontinuity. Noiri [29] introduced and studied the notion of slight β-continuity which is implied by both slight semi-continuity and faint precontinuity. Duangphui et al. [21] introduced and investigated the notion of almost $(\mu, \mu')^{(m,n)}$-continuous functions. Thongmoon and Boonpok [42] introduced and studied the notion of strongly $\theta(\Lambda, p)$-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, $\delta p(\Lambda, s)$-continuous functions, $(\Lambda, p(\star))$-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were presented in

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5321

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[40], [11], [37], [16], [10], [9], [5], [2], [44], [41], [8], [3], [17], [15] and [12], respectively. Sangviset et al. [39] introduced the notion of slightly (m, μ)-continuous functions as functions from an m-spaces into a generalized topological space and investigated several characterizations of slightly (m, μ)-continuous functions.

In 2005, Ekici [23] introduced and investigated the notion of upper (lower) slightly α-continuous multifunctions as a generalization of upper (lower) α-continuous multifunctions due to Neubrunn [28]. Popa and Noiri [36] introduced and studied the notion of upper (lower) β-continuous multifunctions. Furthermore, Ekici [22] introduced and studied upper (lower) slightly β-continuous multifunctions as a generalization of upper (lower) semicontinuous multifunctions, upper (lower) α-continuous multifunctions, upper (lower) precontinuous multifunctions [35], upper (lower) quasi-continuous multifunctions [34], upper (lower) γ-continuous multifunctions [24], upper (lower) β-continuous multifunctions and slightly β-continuous functions. Noiri and Popa [31] introduced the notion of slightly m-continuous multifunctions and established the relationships among m-continuity, almost m-continuity, weak m-continuity and slight m-continuity for multifunctions. Laprom et al. [27] introduced and investigated the notion of $\beta(\tau_1, \tau_2)$-continuous multifunctions. Furthermore, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly $\star$-continuous multifunctions, weakly $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $\imath^\star$-continuous multifunctions, almost $\beta(\star)$-continuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions and $(\tau_1, \tau_2)\alpha$-continuous multifunctions were investigated in [6], [18], [4], [14], [13], [7], [19], [26] and [43], respectively. Pue-on et al. [38] introduce and studied the notions of upper and lower (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the concepts of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions. We also investigate several characterizations of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [20] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [20] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [20] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [20] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [20] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [43] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [45] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\alpha(\tau_1, \tau_2)$-closed sets of X containing A is called the $\alpha(\tau_1, \tau_2)$-*closure* of A and is denoted by $\alpha(\tau_1, \tau_2)$-Cl(A). The union of all $\alpha(\tau_1, \tau_2)$-open sets of X contained in A is called the $\alpha(\tau_1, \tau_2)$-*interior* of A and is denoted by $\alpha(\tau_1, \tau_2)$-Int(A).

Lemma 2. *For subsets A and B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \alpha(\tau_1, \tau_2)$-Cl$(A)$ and $\alpha(\tau_1, \tau_2)$-Cl$(\alpha(\tau_1, \tau_2)$-Cl$(A)) = \alpha(\tau_1, \tau_2)$-Cl$(A)$.

(2) If $A \subseteq B$, then $\alpha(\tau_1, \tau_2)$-Cl$(A) \subseteq \alpha(\tau_1, \tau_2)$-Cl$(B)$.

(3) $\alpha(\tau_1, \tau_2)$-Cl(A) is $\alpha(\tau_1, \tau_2)$-closed.

(4) A is $\alpha(\tau_1, \tau_2)$-closed if and only if $A = \alpha(\tau_1, \tau_2)$-Cl$(A)$.

(5) $\alpha(\tau_1, \tau_2)$-Cl$(X - A) = X - \alpha(\tau_1, \tau_2)$-Int$(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions

In this section, we introduce the notions of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions. Moreover, some characterizations of upper and lower slightly $\alpha(\tau_1, \tau_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper slightly $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-clopen set V of Y containing $F(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper slightly $\alpha(\tau_1, \tau_2)$-continuous if F has this property at every point of X.*

Recall that a net (x_γ) in a topological space (X, τ) is said to be *eventually* in the set $U \subseteq X$ if there exists an index $\gamma_0 \in \nabla$ such that $x_\gamma \in U$ for all $\gamma \geq \gamma_0$.

Definition 2. *A sequence (x_n) is called $\alpha(\tau_1, \tau_2)$-converge to a point x if for every $\alpha(\tau_1, \tau_2)$-open set V containing x, there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^+(V)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq F^+(V)$;

(3) for each $x \in X$ and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^+(Y - V)$, there exists an $\alpha(\tau_1, \tau_2)$-closed set H of X such that $x \in X - H$ and $F^-(V) \subseteq H$;

(4) $F^+(V)$ is $\alpha(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(5) $F^-(V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(6) $F^-(Y - V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(7) $F^+(Y - V)$ is $\alpha(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(8) for each $x \in X$ and for each net (x_γ) which $\alpha(\tau_1, \tau_2)$-converges to x in X and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^+(V)$, the net (x_γ) is eventually in $F^+(V)$.

Proof. (1) $\Leftrightarrow$ (2): Obvious.

(2) $\Leftrightarrow$ (3): Let $x \in X$ and V be any $\sigma_1\sigma_2$-clopen set of Y such that $x \in F^+(Y - V)$. By (2), there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq F^+(Y - V)$. Then, $F^-(V) \subseteq X - U$. Put $H = X - U$. Then, H is $\alpha(\tau_1, \tau_2)$-closed in X and $x \in X - H$. The converse is similar.

(1) $\Leftrightarrow$ (4): Let V be any $\sigma_1\sigma_2$-clopen set of Y and $x \in F^+(V)$. By (1), there exists an $\alpha(\tau_1, \tau_2)$-open set U_x of X containing x such that $U_x \subseteq F^+(V)$. It follows that $F^+(V) = \cup_{x \in F^+(V)} U_x$ and hence $F^+(V)$ is $\alpha(\tau_1, \tau_2)$-open in X. The converse can be shown easily.

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-clopen set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-clopen in Y and by (4), $F^+(Y - V) = X - F^-(V)$ is $\alpha(\tau_1, \tau_2)$-open in X. Thus, $F^-(V)$ is $\alpha(\tau_1, \tau_2)$-closed in X.

(5) ⇒ (4): It is similar to that of (4) ⇒ (5).

(4) ⇔ (6) and (5) ⇔ (7): It follows from the fact that $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(1) ⇒ (8): Let (x_γ) be a net which $\alpha(\tau_1, \tau_2)$-converges to x in X and let V be any $\sigma_1\sigma_2$-clopen set of Y such that $x \in F^+(V)$. Since F is an upper slightly $\alpha(\tau_1, \tau_2)$-continuous multifunction, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq F^+(V)$. Since (x_γ) $\alpha(\tau_1, \tau_2)$-converges to x, it follows that there exists an index $\gamma_0 \in \nabla$ such that $x_\gamma \in U$ for all $\gamma \geq \gamma_0$. Therefore, $x_\gamma \in U \subseteq F^+(V)$ for all $\gamma \geq \gamma_0$. Thus, the net (x_γ) is eventually in $F^+(V)$.

(8) ⇒ (1): Suppose that F is not upper slightly $\alpha(\tau_1, \tau_2)$-continuous. There exists a point x and a $\sigma_1\sigma_2$-clopen set V of Y with $x \in F^+(V)$ such that $U \not\subseteq F^+(V)$ for each $\alpha(\tau_1, \tau_2)$-open set U of X containing x. Let $x_U \in U$ and $x_U \notin F^+(V)$ for each $\alpha(\tau_1, \tau_2)$-open set U of X containing x. Then, for the $\alpha(\tau_1, \tau_2)$-neighbourhood net (x_U), (x_U) $\alpha(\tau_1, \tau_2)$-converges to x, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. Thus, F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous.

Definition 3. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower slightly $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-clopen set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower slightly $\alpha(\tau_1, \tau_2)$-continuous if F has this property at every point of X.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower slightly $\alpha(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^-(V)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq F^-(V)$;

(3) for each $x \in X$ and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^-(Y - V)$, there exists an $\alpha(\tau_1, \tau_2)$-closed set H of X such that $x \in X - H$ and $F^+(V) \subseteq H$;

(4) $F^-(V)$ is $\alpha(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(5) $F^+(V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(6) $F^+(Y - V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(7) $F^-(Y - V)$ is $\alpha(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-clopen set V of Y;

(8) for each $x \in X$ and for each net (x_γ) which $\alpha(\tau_1, \tau_2)$-converges to x in X and for each $\sigma_1\sigma_2$-clopen set V of Y such that $x \in F^-(V)$, the net (x_γ) is eventually in $F^-(V)$.

Proof. The proof is similar to that of Theorem 1.

Definition 4. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be slightly $\alpha(\tau_1, \tau_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-clopen set V of Y containing $f(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$.*

Corollary 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is slightly $\alpha(\tau_1, \tau_2)$-continuous;

(2) $f^{-1}(V)$ is $\alpha(\tau_1, \tau_2)$-open in X for each $\sigma_1\sigma_2$-clopen set V of Y;

(3) $f^{-1}(V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for each $\sigma_1\sigma_2$-clopen set V of Y;

(4) for each $x \in X$ and for each $\sigma_1\sigma_2$-clopen set V of Y containing $f(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$.

Definition 5. *A bitopological space (X, τ_1, τ_2) is said to be mildly $\tau_1\tau_2$-compact if every cover of X by $\tau_1\tau_2$-clopen sets of X has a finite subcover.*

Definition 6. *A bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-compact if if every cover of X by $\alpha(\tau_1, \tau_2)$-open sets of X has a finite subcover.*

Theorem 3. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an upper slightly $\alpha(\tau_1, \tau_2)$-continuous surjective multifunction such that $F(x)$ is mildly $\sigma_1\sigma_2$-compact for each $x \in X$. If (X, τ_1, τ_2) is $\alpha(\tau_1, \tau_2)$-compact, then (Y, σ_1, σ_2) is mildly $\sigma_1\sigma_2$-compact.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any $\sigma_1\sigma_2$-clopen cover of Y. Since $F(x)$ is mildly $\sigma_1\sigma_2$-compact for each $x \in X$, there exists a finite subset $\Gamma(x)$ of Γ such that

$$F(x) \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}.$$

Put $V(x) = \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Since F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous, there exists an $\alpha(\tau_1, \tau_2)$-open set $U(x)$ of X containing x such that $F(U(x)) \subseteq V(x)$. Then, the family $\{U(x) \mid x \in X\}$ is an $\alpha(\tau_1, \tau_2)$-open cover of X. Since (X, τ_1, τ_2) is $\alpha(\tau_1, \tau_2)$-compact, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{U(x_i) \mid 1 \leq i \leq n\}$. Thus,

$$\begin{aligned} Y = F(X) &= \cup\,\{F(U(x_i)) \mid 1 \leq i \leq n\} \\ &\subseteq \cup\{V(x_i) \mid 1 \leq i \leq n\} \\ &\subseteq \cup\{V_\gamma \mid \gamma \in \Gamma(x_i), 1 \leq i \leq n\}. \end{aligned}$$

This shows that (Y, σ_1, σ_2) is mildly $\sigma_1\sigma_2$-compact.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [20] if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets.

Definition 7. *A bitopological space* (X, τ_1, τ_2) *is said to be* $\alpha(\tau_1, \tau_2)$*-connected provided that* X *is not the union of two disjoint nonempty* $\alpha(\tau_1, \tau_2)$*-open sets.*

Definition 8. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called punctually* $\tau_1\tau_2$*-connected if, for each* $x \in X$*,* $F(x)$ *is* $\sigma_1\sigma_2$*-connected.*

Theorem 4. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an upper slightly* $\alpha(\tau_1, \tau_2)$*-continuous multifunction such that* F *is punctually* $\tau_1\tau_2$*-connected. If* (X, τ_1, τ_2) *is* $\alpha(\tau_1, \tau_2)$*-connected, then* (Y, σ_1, σ_2) *is* $\sigma_1\sigma_2$*-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. Then, there exist nonempty $\sigma_1\sigma_2$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous, $F^+(U)$ and $F^+(V)$ are $\alpha(\tau_1, \tau_2)$-open sets of X. In view of the fact that $F^+(U)$, $F^+(V)$ are disjoint and F is punctually $\tau_1\tau_2$-connected, $X = F^+(U) \cup F^+(V)$ is a partition of X. This is contrary to the $\alpha(\tau_1, \tau_2)$-connectedness of (X, τ_1, τ_2). This shows that (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.

Theorem 5. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a lower slightly* $\alpha(\tau_1, \tau_2)$*-continuous multifunction such that* F *is punctually* $\tau_1\tau_2$*-connected. If* (X, τ_1, τ_2) *is* $\alpha(\tau_1, \tau_2)$*-connected, then* (Y, σ_1, σ_2) *is* $\sigma_1\sigma_2$*-connected.*

Proof. The proof is similar to that of Theorem 4.

Definition 9. *A bitopological space* (X, τ_1, τ_2) *is called strongly* (τ_1, τ_2)*-normal if, for any disjoint* $\tau_1\tau_2$*-closed sets* F *and* K *of* X*, there exist* $\tau_1\tau_2$*-clopen sets* U *and* V *of* X *such that* $F \subseteq U$*,* $K \subseteq V$ *and* $U \cap V = \emptyset$*.*

Definition 10. *A bitopological space* (X, τ_1, τ_2) *is called* $\alpha(\tau_1, \tau_2)$*-Hausdorff if, for each pair of distinct points* x *and* y *in* X*, there exist disjoint* $\alpha(\tau_1, \tau_2)$*-open sets* U *and* V *of* X *such that* $x \in U$ *and* $y \in V$*.*

Definition 11. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called punctually* (τ_1, τ_2)*-closed if, for each* $x \in X$*,* $F(x)$ *is* $\sigma_1\sigma_2$*-closed.*

Theorem 6. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an upper slightly* $\alpha(\tau_1, \tau_2)$*-continuous multifunction and punctually* (τ_1, τ_2)*-closed from a bitopological space* (X, τ_1, τ_2) *to a strongly* (σ_1, σ_2)*-normal bitopological space* (Y, σ_1, σ_2) *and let* $F(x) \cap F(y) = \emptyset$ *for each pair of distinct points* $x, y \in X$*. Then,* (X, τ_1, τ_2) *is an* $\alpha(\tau_1, \tau_2)$*-Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then, we have $F(x) \cap F(y) = \emptyset$. Since (Y, σ_1, σ_2) is strongly (σ_1, σ_2)-normal, it follows that there exist disjoint $\sigma_1\sigma_2$-clopen sets U and V of Y containing $F(x)$ and $F(y)$, respectively. Thus, $F^+(U)$ and $F^+(V)$ are disjoint $\alpha(\tau_1, \tau_2)$-open sets of X containing x and y, respectively. This shows that (X, τ_1, τ_2) is an $\alpha(\tau_1, \tau_2)$-Hausdorff space.

4. Slight $\alpha(\tau_1, \tau_2)$-continuity and other forms of $\alpha(\tau_1, \tau_2)$-continuity

We begin this section by introducing the concept of upper $\alpha(\tau_1, \tau_2)$-continuous multifunctions.

Definition 12. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.*

Theorem 7. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $\alpha(\tau_1, \tau_2)$-continuous, then F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-clopen set of Y containing $F(x)$. Since F is upper $\alpha(\tau_1, \tau_2)$-continuous, there exists an $\alpha(\tau_1, \tau_2)$-open set of X containing x such that $F(U) \subseteq V$. This shows that F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous.

Definition 13. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.*

Theorem 8. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $\alpha(\tau_1, \tau_2)$-continuous, then F is lower slightly $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. The proof is similar to that of Theorem 7.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-*extremally disconnected* [45] if the $\tau_1\tau_2$-closure of every $\tau_1\tau_2$-open set U of X is $\tau_1\tau_2$-open.

Lemma 3. [45] *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected.

(2) Every $(\tau_1, \tau_2)r$-open set of X is $\tau_1\tau_2$-closed.

(3) Every $(\tau_1, \tau_2)r$-closed set of X is $\tau_1\tau_2$-open.

Definition 14. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that*

$$F(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)).$$

A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.

Lemma 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\alpha(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set of X containing x such that $F(U) \subseteq V$.

Theorem 9. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper slightly $\alpha(\tau_1, \tau_2)$-continuous and (Y, σ_1, σ_2) is (σ_1, σ_2)-extremally disconnected, then F is upper almost $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Then, by Lemma 3 we have V is $\sigma_1\sigma_2$-clopen in Y. Since F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous, there exists an $\alpha(\tau_1, \tau_2)$-open set of X containing x such that $F(U) \subseteq V$. By Lemma 4, F is upper almost $\alpha(\tau_1, \tau_2)$-continuous.

Definition 15. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that*

$$\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)) \cap F(z) \neq \emptyset$$

for every $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.

Lemma 5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\alpha(\tau_1, \tau_2)$-continuous;

(2) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\tau_1, \tau_2)$-open set of X containing x such that $U \subseteq F^-(V)$.

Theorem 10. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower slightly $\alpha(\tau_1, \tau_2)$-continuous and (Y, σ_1, σ_2) is (σ_1, σ_2)-extremally disconnected, then F is lower almost $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. By utilizing Lemma 5, this can be proved similarly to that of Theorem 9.

Definition 16. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that*

$$F(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V).$$

A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.

Theorem 11. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper weakly $\alpha(\tau_1, \tau_2)$-continuous, then F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-clopen set of Y containing $F(x)$. Since F is upper weakly $\alpha(\tau_1, \tau_2)$-continuous, there exists an $\alpha(\tau_1, \tau_2)$-open set of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V) = V$. This shows that F is upper slightly $\alpha(\tau_1, \tau_2)$-continuous.

Definition 17. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\alpha(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\tau_1, \tau_2)$-open set U of X containing x such that*

$$\sigma_1\sigma_2\text{-}Cl(V) \cap F(z) \neq \emptyset$$

for every $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called lower weakly $\alpha(\tau_1, \tau_2)$-continuous if F has this property at each point of X.

Theorem 12. *If a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly $\alpha(\tau_1, \tau_2)$-continuous, then F is lower slightly $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. The proof is similar to that of Theorem 11.

Acknowledgements

This research project was financially supported by Mahasarakham University.

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