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Upper and Lower s-$(\tau_1, \tau_2)p$-Continuous Multifunctions

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Abstract. Our main purpose is to introduce the concepts of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, several characterizations of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08; 54C60; 54E55

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1. Introduction

In 1965, Lee [27] studied the notion of semiconnected functions. Kohli [24] introduced the notion of s-continuous functions and investigated several characterizations of semilocally connected spaces in terms of s-continuous functions. The class of s-continuity is a generalization of continuity and semiconnectedness. Furthermore, Kohli [25] introduced the concepts of s-regular spaces and completely s-regular spaces and proved that s-regularity and complete s-regularity are preserved under certain s-continuous functions. Duangphui et al. [21] introduced and investigated the notion of almost $(\mu, \mu')^{(m,n)}$-continuous functions. Thongmoon and Boonpok [35] introduced and studied the notion of strongly $\theta(\Lambda, p)$-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, $\delta p(\Lambda, s)$-continuous functions, $(\Lambda, p(\star))$-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were presented in [33], [11], [31], [16], [10], [9], [5], [2], [37], [34], [8], [3], [17], [15] and [12], respectively.

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In 1989, Lipski [28] extended the concept of s-continuous functions to the setting of multifunctions. Popa [29] introduced the concept of precontinuous multifunctions and showed that H-almost continuity and precontinuity are equivalent for multifunctions. Ewert and Lipski [22] introduced and investigated the concept of s-quasi-continuous multifunctions. Popa and Noiri [30] introduced and studied the notion of s-precontinuous multifunctions as a generalization of s-continuous multifunctions and precontinuous multifunctions. Laprom et al. [26] introduced and investigated the concept of $\beta(\tau_1, \tau_2)$-continuous multifunctions. In particular, some characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly $\star$-continuous multifunctions, weakly $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $\imath^\star$-continuous multifunctions, almost $\beta(\star)$-continuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions and $(\tau_1, \tau_2)\alpha$-continuous multifunctions were established in [6], [18], [4], [14], [13], [7], [19], [23] and [36], respectively. Pue-on et al. [32] introduce and studied the concepts of upper and lower (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the notions of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions. We also investigate several characterizations of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [20] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [20] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [20] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [20] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [20] if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [36] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6], $\alpha(\tau_1, \tau_2)$-*open*) [38]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq$

$\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)), $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A)), $A \subseteq \tau_1\tau_2$-Cl($\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl(A))), $A \subseteq \tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A)))). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*, $\alpha(\tau_1, \tau_2)$*-closed*). Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* of A and is denoted by (τ_1, τ_2)-pCl(A). The union of all $(\tau_1, \tau_2)p$-open sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* of A and is denoted by (τ_1, τ_2)-pInt(A).

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) A is $(\tau_1, \tau_2)p$-closed if and only if (τ_1, τ_2)-pCl$(A) = A$;

(2) (τ_1, τ_2)-pCl$(A) = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) \cup A$;

(3) (τ_1, τ_2)-pCl$((\tau_1, \tau_2)$-pCl$(A)) = (\tau_1, \tau_2)$-pCl(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions

In this section, we introduce the notions of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions. Moreover, some characterizations of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper s-$(\tau_1, \tau_2)p$-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(U) \subseteq V$.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper s-$(\tau_1, \tau_2)p$-continuous;

(2) $F^+(V)$ is $(\tau_1, \tau_2)p$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $F^-(K)$ *is* $(\tau_1, \tau_2)p$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2$*-Cl*$(\tau_1\tau_2$*-Int*$(F^-(B))) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(5) (τ_1, τ_2)*-pCl*$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(6) $F^+(\sigma_1\sigma_2$*-Int*$(B)) \subseteq (\tau_1, \tau_2)$*-pInt*$(F^+(B))$ *for every subset* B *of* Y *such that*

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$*-connected.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement and $x \in F^+(V)$. Then, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(U) \subseteq V$. Therefore, we have $x \in U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(V)))$. Thus,

$$F^+(V) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(V)))$$

and hence $F^+(V)$ is $(\tau_1, \tau_2)p$-open in X.

(2) ⇒ (3): The proof follows immediately from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. Then, $F^-(\sigma_1\sigma_2\text{-Cl}(B))$ is a $(\tau_1, \tau_2)p$-closed set of X. By Lemma 2, we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-(B))) &\subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(B))))\\ &\subseteq (\tau_1, \tau_2)\text{-pCl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))\\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)).\end{aligned}$$

(4) ⇒ (5): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. It follows from Lemma 2 that

$$\begin{aligned}(\tau_1, \tau_2)\text{-pCl}(F^-(B)) &= F^-(B) \cup \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(F^-(B)))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)).\end{aligned}$$

(5) ⇒ (6): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-connected. By (5),

$$\begin{aligned}X - (\tau_1, \tau_2)\text{-Int}(F^+(B)) &= (\tau_1, \tau_2)\text{-pCl}(X - F^+(B))\\ &= (\tau_1, \tau_2)\text{-pCl}(F^-(Y - B))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B))\\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(B))\\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)).\end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq (\tau_1, \tau_2)\text{-pInt}(F^+(B))$.

(6) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By (6), we have

$$F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V)) \subseteq (\tau_1, \tau_2)\text{-pInt}(F^+(V)).$$

Put $U = (\tau_1, \tau_2)\text{-pInt}(F^+(V))$. Then, U is a $(\tau_1, \tau_2)p$-open set of X containing x such that $F(U) \subseteq V$. This shows that F is upper s-$(\tau_1, \tau_2)p$-continuous.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower s-$(\tau_1, \tau_2)p$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower s-$(\tau_1, \tau_2)p$-continuous;

(2) $F^-(V)$ is $(\tau_1, \tau_2)p$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;

(3) $F^+(K)$ is $(\tau_1, \tau_2)p$-closed in X for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(F^+(B))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $(\tau_1, \tau_2)\text{-}pCl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq (\tau_1, \tau_2)\text{-}pInt(F^-(B))$ for every subset B of Y such that

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$-connected.

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A function $: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be s-$(\tau_1, \tau_2)p$-continuous if for each point $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq V$.*

Corollary 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is s-$(\tau_1, \tau_2)p$-continuous;

(2) $f^{-1}(V)$ *is* $(\tau_1, \tau_2)p$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *having* $\sigma_1\sigma_2$*-connected complement;*

(3) $f^{-1}(K)$ *is* $(\tau_1, \tau_2)p$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(f^{-1}(B))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(5) $(\tau_1, \tau_2)\text{-}pCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y *having the* $\sigma_1\sigma_2$*-connected* $\sigma_1\sigma_2$*-closure;*

(6) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq (\tau_1, \tau_2)\text{-}pInt(f^{-1}(B))$ *for every subset* B *of* Y *such that*

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$*-connected.*

Corollary 2. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is upper* s*-*$(\tau_1, \tau_2)p$*-continuous if* $F^-(V)$ *is* $(\tau_1, \tau_2)p$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected set* V *of* Y*.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. Then, $Y - V$ is $\sigma_1\sigma_2$-connected and $F^-(Y - V)$ is $(\tau_1, \tau_2)p$-closed in X. Thus, $F^+(V)$ is $(\tau_1, \tau_2)p$-open in X and by Theorem 1, F is upper s-$(\tau_1, \tau_2)p$-continuous.

Corollary 3. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is lower* s*-*$(\tau_1, \tau_2)p$*-continuous if* $F^+(V)$ *is* $(\tau_1, \tau_2)p$*-closed in* X *for every* $\sigma_1\sigma_2$*-connected set* V *of* Y*.*

Proof. The proof is similar to that of Corollary 2.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\mathrm{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [20] we denote a multifunction defined as follows: $\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-}\mathrm{Cl}(F(x))$ for each $x \in X$.

Definition 4. [20] *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is said to be:*

(1) $\tau_1\tau_2$*-paracompact if every cover of* A *by* $\tau_1\tau_2$*-open sets of* X *is refined by a cover of* A *which consists of* $\tau_1\tau_2$*-open sets of* X *and is* $\tau_1\tau_2$*-locally finite in* X*;*

(2) $\tau_1\tau_2$*-regular if for each* $x \in A$ *and each* $\tau_1\tau_2$*-open set* U *of* X *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$*.*

Lemma 3. [20] *If* A *is a* $\tau_1\tau_2$*-regular* $\tau_1\tau_2$*-paracompact set of a bitopological space* (X, τ_1, τ_2) *and* U *is a* $\tau_1\tau_2$*-open neighbourhood of* A*, then there exists a* $\tau_1\tau_2$*-open set* V *of* X *such that* $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$*.*

Lemma 4. [20] *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is a multifunction such that* $F(x)$ *is* $\tau_1\tau_2$*-regular and* $\tau_1\tau_2$*-paracompact for each* $x \in X$*, then* $ClF^+_{\circledast}(V) = F^+(V)$ *for each* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Theorem 3. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, the following properties are equivalent:*

(1) F is upper s-$(\tau_1, \tau_2)p$-continuous;

(2) Cl$F_{\circledast}$ is upper s-$(\tau_1, \tau_2)p$-continuous.

Proof. We put $G = \mathrm{Cl}F_{\circledast}$. Suppose that F is upper s-$(\tau_1, \tau_2)p$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in G^+(V) = F^+(V)$ and hence there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3 there exists a $\tau_1\tau_2$-open set W of X such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$ for each $z \in U$. Thus, $G(U) \subseteq V$ and hence G is upper s-$(\tau_1, \tau_2)p$-continuous.

Conversely, suppose that G is upper s-$(\tau_1, \tau_2)p$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $G(U) \subseteq V$. Thus, $U \subseteq G^+(V) = F^+(V)$ and so $F(U) \subseteq V$. This shows that F is upper s-$(\tau_1, \tau_2)p$-continuous.

Lemma 5. [20] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, Cl$F^-_{\circledast}(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower s-$(\tau_1, \tau_2)p$-continuous;

(2) Cl$F_{\circledast}$ is lower s-$(\tau_1, \tau_2)p$-continuous.

Proof. By using Lemma 5 this can be shown similarly to that of Theorem 3.

The (τ_1, τ_2)*p-frontier* of a subset A of a bitopological space (X, τ_1, τ_2), denoted by (τ_1, τ_2)-pfr(A), is defined by

$$\begin{aligned}(\tau_1, \tau_2)\text{-pfr}(A) &= (\tau_1, \tau_2)\text{-pCl}(A) \cap (\tau_1, \tau_2)\text{-pCl}(X - A)\\ &= (\tau_1, \tau_2)\text{-pCl}(A) - (\tau_1, \tau_2)\text{-pInt}(A).\end{aligned}$$

Theorem 5. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper s-$(\tau_1, \tau_2)p$-continuous is identical with the union of the $(\tau_1, \tau_2)p$-frontier of the upper inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. Suppose that F is not upper s-$(\tau_1, \tau_2)p$-continuous at $x \in X$. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $(\tau_1, \tau_2)p$-open set U of X containing x. Therefore, we have $x \in (\tau_1, \tau_2)$-pCl$(X - F^+(V))$. On the other hand, we have

$$x \in F^+(V) \subseteq (\tau_1, \tau_2)\text{-pCl}(F^+(V))$$

and hence $x \in (\tau_1, \tau_2)$-pfr$(F^+(V))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $x \in (\tau_1, \tau_2)$-pfr$(F^+(V))$. If F is upper s-$(\tau_1, \tau_2)p$-continuous at $x \in X$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $x \in (\tau_1, \tau_2)$-pInt$(F^+(V))$. This is a contradiction and so F is not upper s-$(\tau_1, \tau_2)p$-continuous at x.

Theorem 6. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$$

is not lower s-$(\tau_1, \tau_2)p$-continuous is identical with the union of the $(\tau_1, \tau_2)p$-frontier of the lower inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets meeting $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. The proof is similar to that of Theorem 5.

4. Conclusion

This paper deals with the notions of upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, some characterizations and several properties concerning upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions are established. In the upcoming work, we plan to apply the concepts initiated in this paper to study a new generalization of upper (lower) s-$(\tau_1, \tau_2)p$-continuous multifunctions, namely upper (lower) almost s-$(\tau_1, \tau_2)p$-continuous multifunctions. A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called upper (lower) almost s-$(\tau_1, \tau_2)p$-continuous multifunctions if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $x \in F^+(V)$ $(x \in F^-(V))$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ $(U \subseteq F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$. The class of upper (lower) s-$(\tau_1, \tau_2)p$-continuous multifunctions included in the class of upper (lower) almost s-$(\tau_1, \tau_2)p$-continuous multifunctions.

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References

[1] C. Berge. *Espaces topologiques fonctions multivoques.* Dunod, Paris, 1959.

[2] C. Boonpok. Almost (g, m)-continuous functions. *International Journal of Mathematical Analysis*, 4(40):1957–1964, 2010.

[3] C. Boonpok. M-continuous functions in biminimal structure spaces. *Far East Journal of Mathematical Sciences*, 43(1):41–58, 2010.

[4] C. Boonpok. On continuous multifunctions in ideal topological spaces. *Lobachevskii Journal of Mathematics*, 40(1):24–35, 2019.

[5] C. Boonpok. On characterizations of $\star$-hyperconnected ideal topological spaces. *Journal of Mathematics*, 2020:9387601, 2020.

[6] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[7] C. Boonpok. Upper and lower $\beta(\star)$-continuity. *Heliyon*, 7:e05986, 2021.

[8] C. Boonpok. On some closed sets and low separation axioms via topological ideals. *European Journal of Pure and Applied Mathematics*, 15(3):300–309, 2022.

[9] C. Boonpok. On some spaces via topological ideals. *Open Mathematics*, 21:20230118, 2023.

[10] C. Boonpok. $\theta(\star)$-precontinuity. *Mathematica*, 65(1):31–42, 2023.

[11] C. Boonpok and J. Khampakdee. Almost strong $\theta(\Lambda, p)$-continuity for functions. *European Journal of Pure and Applied Mathematics*, 17(1):300–309, 2024.

[12] C. Boonpok and C. Klanarong. On weakly (τ_1, τ_2)-continuous functions. *European Journal of Pure and Applied Mathematics*, 17(1):416–425, 2024.

[13] C. Boonpok and P. Pue-on. Continuity for multifunctions in ideal topological spaces. *WSEAS Transactions on Mathematics*, 19:624–631, 2020.

[14] C. Boonpok and P. Pue-on. Upper and lower weakly α-$\star$-continuous multifunctions. *International Journal of Analysis and Applications*, 21:90, 2023.

[15] C. Boonpok and P. Pue-on. Characterizations of almost (τ_1, τ_2)-continuous functions. *International Journal of Analysis and Applications*, 22:33, 2024.

[16] C. Boonpok and N. Srisarakham. Weak forms of (Λ, b)-open sets and weak (Λ, b)-continuity. *European Journal of Pure and Applied Mathematics*, 16(1):29–43, 2023.

[17] C. Boonpok and N. Srisarakham. (τ_1, τ_2)-continuity for functions. *Asia Pacific Journal of Mathematics*, 11:21, 2024.

[18] C. Boonpok and C. Viriyapong. Almost weak continuity for multifunctions in ideal topological spaces. *WSEAS Transactions on Mathematics*, 19:367–372, 2020.

[19] C. Boonpok and C. Viriyapong. Upper and lower almost weak (τ_1, τ_2)-continuity. *European Journal of Pure and Applied Mathematics*, 14(1):1212–1225, 2021.

[20] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[21] T. Duangphui, C. Boonpok, and C. Viriyapong. Continuous functions on bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 5(24):1165–1174, 2011.

[22] J. Ewert and T. Lipski. On s-quasi-continuous multivalued maps. *Review of Research, Faculty of Science, Mathematics Series*, 20(1):167–183, 1990.

[23] C. Klanarong, S. Sompong, and C. Boonpok. Upper and lower almost (τ_1, τ_2)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(2):1244–1253, 2024.

[24] J. K. Kohli. A class of mappings containing all continuous and all semi-connected mappings. *Proceedings of the American Mathematical Society*, 72:175–181, 1978.

[25] J. K. Kohli. S-continuous functions and certain weak forms of regularity and complete regularity. *Mathematics Nachrichten*, 97:189–196, 1980.

[26] K. Laprom, C. Boonpok, and C. Viriyapong. $\beta(\tau_1, \tau_2)$-continuous multifunctions on bitopological spaces. *Journal of Mathematics*, 2020:4020971, 2020.

[27] Y. L. Lee. Some characterizations of semilocally connected spaces. *Proceedings of the American Mathematical Society*, 16:1318–1320, 1965.

[28] T. Lipski. S-continuous multivalued maps. *Mathematical Chronicle*, 18:57–61, 1989.

[29] V. Popa. Some properties of H-almost continuous multifunctions. *Problemy Matematyczne*, 10:9–26, 1988.

[30] V. Popa and T. Noiri. On s-precontinuous multifunctions. *Demonstratio Mathematica*, 33(3):679–687, 2000.

[31] P. Pue-on and C. Boonpok. $\theta(\Lambda, p)$-continuity for functions. *International Journal of Mathematics and Computer Science*, 19(2):491–495, 2024.

[32] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower (τ_1, τ_2)-continuous multifunctions. *International Journal of Mathematics and Computer Science*, 19(4):1305–1310, 2024.

[33] N. Srisarakham and C. Boonpok. Almost (Λ, p)-continuous functions. *International Journal of Mathematics and Computer Science*, 18(2):255–259, 2023.

[34] N. Srisarakham and C. Boonpok. On characterizations of $\delta p(\Lambda, s)$-$\mathscr{D}_1$ spaces. *International Journal of Mathematics and Computer Science*, 18(4):743–747, 2023.

[35] M. Thongmoon and C. Boonpok. Strongly $\theta(\Lambda, p)$-continuous functions. *International Journal of Mathematics and Computer Science*, 19(2):475–479, 2024.

[36] C. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[37] C. Viriyapong and C. Boonpok. (Λ, sp)-continuous functions. *WSEAS Transactions on Mathematics*, 21:380–385, 2022.

[38] N. Viriyapong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-extremal disconnectedness in bitopological spaces. *International Journal of Mathematics and Computer Science*, 19(3):855–860, 2024.