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Structure of Primitive Pythagorean Triples in Generating Trees

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Abstract. A Pythagorean triple is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. If a, b are coprime, then it is called a primitive Pythagorean triple. It is known that every primitive Pythagorean triple can be generated from the triple $(3, 4, 5)$ using multiplication by unique number and order of three specific 3×3 matrices, which yields a ternary tree of triplets. Two such trees were described by Berggren and Price, respectively. A different approach is to view the primitive Pythagorean triples as points in the three-dimensional Euclidean space. In this paper, we prove that the triple of descendants of any primitive Pythagorean triple in Berggren's or Price's tree forms a triangle (and therefore defines a plane), and we present our results related to these triangles (and these planes).

2020 Mathematics Subject Classifications: 11D09, 11A41, 11C99

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1. Introduction

Primitive Pythagorean triples are an interesting object of the number theory, and they have potential to be also used while studying the properties in other areas of the mathematics (e.g. in geometry, or in graph theory). The description and knowledge of their properties can also be used practically, for example in cryptography, considering that there are infinitely many primitive Pythagorean triples, and the fact that such triples have the ability of creating probability events [15]. Moreover, sufficiently large primitive pythagorean triples can be used as keys by certification authority - sending two numbers to two parties as seeds to generate the key, and keeping the third number for audit purposes [5].

To use primitive pythagorean triples, we need to understand them and their generating better. Currently, there are various ways of generating integer solutions of Pythagorean

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equation $a^2 + b^2 = c^2$. Euclid's formula, Berggren's tree and Price's tree are among the best known ones, however there are many others. Some examples can be found e.g. in [13]. The matrices used to generate Berggren's tree produce solutions (a, b, c) with coprime elements, i.e. the primitive Pythagorean triples, which can be viewed as lengths of the sides of Pythagorean triangles. Another methods of gerating these triples use sequences, generating the following triples from a starting triple. One of the newest of these appraoches is described by [5]. Other approaches to primitive Pythagorean triples were introduced e.g. by [2].

Some properties of the Pythagorean triangles were already described. E.g., the number of primitive Pythagorean triples with a given inradius [11], triples with common lengths of leg [6] or height of primitive Pythagorean triples (the difference between length of the hypotenuse and length of even leg) [4]. Further, it was described how to find the pairs of Pythagorean triangles with equal areas, or primitive Pythagorean triangles having the same perimeter [1]. Also, some numerical properties of primitive Pythagorean triples were studied e.g. in [3], [8].

The primitive Pythagorean triples can be also viewed as coordinates of the points in the three-dimensional Euclidean space. In this paper, we consider the triples which are generated from a given triple in either Berggren's or Price's tree of primitive Pythagorean triples (such trees always produce three descendant triples from a given triple). Then we assign these descendant triples the vertices in three-dimensional space and prove that these points are not colinear, therefore they form a triangle (or a plane).

We focus on these triangles formed by the descendant triples in either Berggren's or Price's tree of primitive Pythagorean triples, and describe some of their properties. Namely we study the planes containing these triangles; and the types of these triangles.

2. Preliminary

First, we present some basic notations and definitions which we use through the paper.

Let us denote $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For a matrix M, denote its transpose as $M^\top$.

For a vector $\vec{u}$, we denote its norm as $|\vec{u}|$.

Definition 1. *A triple of positive integers (a, b, c) is called a Pythagorean triple if $a^2+b^2 = c^2$. Moreover, if $\gcd(a, b) = 1$, then we call (a, b, c) a primitive Pythagorean triple.*

Definition 2. *If (a, b, c) is a Pythagorean triple, we say that a triangle corresponds to this triple if its sides have lengths a, b, c. A triangle corresponding to a (primitive) Pythagorean triple is called a (primitive) Pythagorean triangle.*

For convenience, we use the abbreviation PPT for both primitive Pythagorean triple and primitive Pythagorean triangle.

It is easy to see the parity of the individual components of a PPT, see e.g. [8]:

Lemma 1. *If (a, b, c) is a primitive Pythagorean triple, then one of a, b is even and the other one is odd, and c is odd.*

In the following sections, we consider generating the primitive Pythagorean triples using the trees described by Berggren and Price, respectively. In these trees, the components of PPT (a, b, c) are ordered such that a is odd and b is even. We use this order of components in the rest of our article.

In the proofs concerning PPTs, it is often useful to express the primitive Pythagorean triples using Euclid's formula [10]:

Theorem 1 (Euclid's formula)**.** *Triple (a, b, c) is a primitive Pythagorean triple with odd a if and only if there exist $m, n \in \mathbb{N}$ of different parity, such that* $\gcd(m, n) = 1$, $m > n$ *and*

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = n^2 + m^2. \tag{1}$$

3. Descendants in Berggren's tree

According to Berggren, every primitive Pythagorean triple (a, b, c) with odd a can be generated from the triple $(3, 4, 5)$ by unique three-fold ascent using the three matrices U, A, D [7]:

$$U = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}.$$

Further, it is easy to show that for every $n \in \mathbb{N}$ the following holds:

$$U^n = \begin{pmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 2n^2+1 \end{pmatrix},$$

$$A^n = \begin{pmatrix} \frac{(-1)^n}{2} + a_1 & \frac{(-1)^{n+1}}{2} + a_1 & a_2 \\ \frac{(-1)^{n+1}}{2} + a_1 & \frac{(-1)^n}{2} + a_1 & a_2 \\ a_2 & a_2 & 2a_1 \end{pmatrix},$$

$$D^n = \begin{pmatrix} 1-2n^2 & 2n & 2n^2 \\ -2n & 1 & 2n \\ -2n^2 & 2n & 2n^2+1 \end{pmatrix},$$

where $a_1 = \frac{1}{4}[(3-2\sqrt{2})^n + (3+2\sqrt{2})^n]$, $\quad a_2 = -\frac{(3-2\sqrt{2})^n}{2\sqrt{2}} + \frac{(3+2\sqrt{2})^n}{2\sqrt{2}}$.

Theorem 2. *If (a, b, c) is a primitive Pythagorean triple with odd a, and M is a matrix such that $M \in \{U, A, D\}$, then $M \cdot (a, b, c)^\top$ is a primitive Pythagorean triple with an odd first component.*

Berggren's tree of PPT contains all PPTs, and each PPT is generated by a unique sequence of matrix multiplication [7]. The first few levels of Berggren's tree can be seen in the Figure 1.

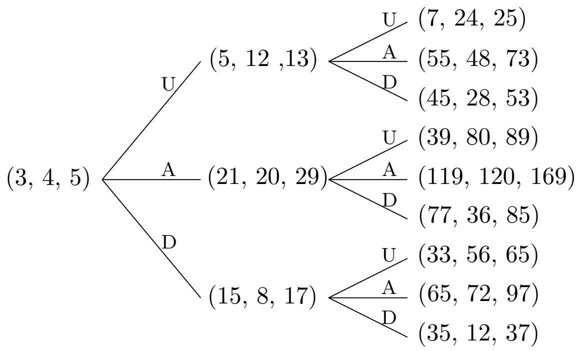

Figure 1: Berggren's tree of PPTs.

Definition 3. *Let P be a primitive Pythagorean triple. We say that P is the parent of the triples $UP^\top, AP^\top, DP^\top$. The triples $UP^\top, AP^\top, DP^\top$ are called the descendants of P in Berggren's tree.*

Proposition 1. *Let P be a primitive Pythagorean triple. Then the points with the coordinates $UP^\top, AP^\top, DP^\top$ are not collinear.*

Proof. Let $P = (a, b, c)$. We consider the vectors $\vec{u} = AP^\top - UP^\top$ and $\vec{v} = DP^\top - UP^\top$. It is easy to show that

$$\vec{u} = (A-U)P^\top = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4b \\ 2b \\ 4b \end{pmatrix},$$

$$\vec{v} = (D-U)P^\top = \begin{pmatrix} -2 & 4 & 0 \\ -4 & 2 & 0 \\ -4 & 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a+4b \\ -4a+2b \\ -4a+4b \end{pmatrix}.$$

By way of contradiction, assume that these vectors are linearly dependent, i.e., that there exists nonzero real number k such that $\vec{u} = k\vec{v}$. Then $(4b, 2b, 4b) = k(-2a+4b, -4a+2b, -4a+4b)$, hence $4b = k(-2a+4b)$ and $4b = k(-4a+4b)$. This implies that

$$\begin{aligned} k(-2a+4b) &= k(-4a+4b) \\ -2a+4b &= -4a+4b \\ 2a &= 0 \end{aligned}$$

which is a contradiction with $a \in \mathbb{N}$. Therefore, $\vec{u}, \vec{v}$ are linearly independent and the points $UP^\top, AP^\top, DP^\top$ are not collinear.

This implies that the points with the coordinates $UP^\top, AP^\top, DP^\top$ determine a plane, and a triangle. In the following, we prove some properties of this plane (and this triangle). For simplicity, we use the following notion:

Definition 4. *Let P be a primitive Pythagorean triple and let $UP^\top$, $AP^\top$, $DP^\top$ be its descendants in Berggren's tree. The triangle with vertices with the coordinates $UP^\top$, $AP^\top$, $DP^\top$ is called the descendant triangle of P in Berggren's tree, and we denote it by $\triangle_B(P)$.*

Proposition 2. *Let $P = (a, b, c)$ be a primitive Pythagorean triple. Then the triangle $\triangle_B(P)$ belongs to the plane*

$$2x + 2y - 3z + c = 0. \tag{2}$$

Proof. According to the Proposition 1, vertices $UP^\top$, $AP^\top$, $DP^\top$ form a triangle, therefore, they define a plane. Analogously to the previous proof, we consider vectors $\vec{u} = AP^\top - UP^\top = (4b, 2b, 4b)$ and $\vec{v} = DP^\top - UP^\top = (-2a + 4b, -4a + 2b, -4a + 4b)$.

Firstly, we compute the normal vector of the wanted plane as cross product of these vectors:

$$\vec{u} \times \vec{v} = (8ab, 8ab, -12ab) \approx (2, 2, -3).$$

Therefore, $2x + 2y - 3z + d = 0$ is the equation of wanted plane for some $d \in \mathbb{R}$. Since $UP^\top$ belongs to this plane, we get

$$2(a - 2b + 2c) + 2(2a - b + 2c) - 3(2a - 2b + 3c) + d = 0$$

which yields $d = c$. Hence, the wanted plane is $2x + 2y - 3z + c = 0$.

It directly follows that:

Corollary 1. *If P, Q are primitive Pythagorean triples, then the planes defined by triangles $\triangle_B(P)$ and $\triangle_B(Q)$ are parallel.*

Corollary 2. *Each primitive Pythagorean triple belongs to a plane $2x + 2y - 3z + d = 0$ for some odd $d \in \mathbb{N}$.*

Proof. According to [7], Berggren's tree contains all primitive Pythagorean triples. Therefore, if P is a PPT, then it is either $(3, 4, 5)$ or it is a descendant triple of some triple (e, f, g) in Berggren's tree such that $g \in \mathbb{N}$ is odd.

If $P = (3, 4, 5)$ then it clearly belongs to the plane $2x + 2y - 3z + 1 = 0$.

If P is a descendant triple of some triple (e, f, g) in Berggren's tree, then by Proposition 2, it belongs to the plane $2x + 2y - 3z + g = 0$ where $g \in \mathbb{N}$ is odd.

In [14], Tripathi proved the following Lemma:

Lemma 2. *For odd $n \in \mathbb{N}$, the number of primitive Pythagorean triples (a, b, n) is*

$$\mathscr{P}_2^*(n) = \begin{cases} 2^{\omega(n)-1}, & \text{if } n \geq 3 \text{ and no prime of the form } 4k+3 \text{ divides } n, \\ 0, & \text{if } n = 1 \text{ or } n \text{ has a prime divisor of the form } 4k+3, \end{cases}$$

where $\omega(n)$ is the number of prime divisors of n.

We use Lemmas 1 and 2 to determine the number of descendant triangles in the individual planes.

Proposition 3. *If $c \in \mathbb{N}$ is odd, then the plane $2x + 2y - 3z + c = 0$ contains $\mathscr{P}_2^*(c)$ descendant triangles $\triangle_B$.*

Proof. According to Lemma 2, for each odd $c \in \mathbb{N}$, there is $\mathscr{P}_2^*(c)$ primitive Pythagorean triples P with the third component c. Further, the Proposition 2 yields that for each such P, the descendant triangle $\triangle_B(P)$ belongs to the plane $2x + 2y - 3z + c = 0$.

It is easy to see that the plane $2x + 2y - 3z + c = 0$ cannot contain more descendant triangles. Assume that there is a descendant triangle $\triangle_B(Q)$ in $2x + 2y - 3z + c = 0$ such that $Q = (a, b, n), n \neq c$. However, by the Proposition 2, $\triangle_B(Q)$ also belongs to the plane $2x + 2y - 3z + n = 0$ which yields $n = c$, a contradiction.

Corollary 3. *For each odd integer $n \geq 3$, the plane $2x+2y-3z+n = 0$ contains $3\cdot\mathscr{P}_2^*(n)$ primitive Pythagorean triples. The plane $2x + 2y - 3z + 1 = 0$ contains a single primitive Pythagorean triple,* $(3, 4, 5)$.

Proof. Let $n = 1$. If (a, b, c) is a primitive Pythagorean triple belonging to $2x + 2y - 3z + 1 = 0$, then by Euclid's formula we get

$$\begin{aligned} 2(m^2 - n^2) + 2(2mn) - 3(m^2 + n^2) + 1 = 0 \\ m^2 - 4mn + 5n^2 - 1 = 0 \end{aligned}$$

Assuming that m is a variable and n is a parameter, we get

$$m = \frac{4n \pm \sqrt{4 - 4n^2}}{2}$$

According to Euclid's formula, $n, m \in \mathbb{N}$, which yields $4 - 4n^2 \geq 0 \iff n = 1$, therefore, $m = 2$. Hence, a primitive Pythagorean triple (a, b, c) belongs to the plane $2x+2y-3z+1 = 0 \iff [(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ where $n = 1$ and $m = 2] \iff (a, b, c) = (3, 4, 5)$.

Let n be an odd integer, $n \geq 3$. By Proposition 3, the plane $2x + 2y - 3z + n = 0$ contains $\mathscr{P}_2^*(n)$ descendant triangles $\triangle_B$. Each vertex of $\triangle_B$ is a PPT, hence there are at least $3 \cdot \mathscr{P}_2^*(n)$ PPTs in the plane $2x + 2y - 3z + n = 0$.

By way of contradiction, we show that there are no more PPTs in $2x+2y-3z+n = 0$. Let us assume that there is a PPT (a, b, c) in $2x + 2y - 3z + n = 0$ such that (a, b, c) is not a descendant of any PPT with the third component n in Berggren's tree. Then $(a, b, c) = (3, 4, 5)$ or (a, b, c) is a descendant of some PPT (d, e, f) such that $f \neq n$.

If $(a, b, c) = (3, 4, 5)$, then from (a, b, c) belonging to $2x+2y-3z+n = 0$, we get $n = 1$, a contradiction with $n \geq 3$.

If (a, b, c) is a descendant of some PPT (d, e, f), such that $f \neq n$, then by Proposition 2, (a, b, c) belongs to the plane $2x + 2y - 3z + f = 0$. However, the triple (a, b, c) also belongs to the plane $2x + 2y - 3z + n = 0$, which yields $f = n$, a contradiction.

A primitive Pythagorean triple can be viewed as a right triangle, and the points corresponding to the descendants of a PPT in Berggren's tree also form a triangle. It is natural to ask whether the descendant triangle $\triangle_B(P)$ can also be a right triangle for some P. The following proposition offers the answer to this question.

Proposition 4. *Let P be a primitive Pythagorean triple. The triangle $\triangle_B(P)$ is a non-right triangle.*

Proof. Let $P = (a, b, c)$ and let the vectors $\vec{u}, \vec{v}, \vec{w}$ be as follows:

$$\begin{aligned}\vec{u} &= AP^\top - UP^\top,\\ \vec{v} &= DP^\top - UP^\top,\\ \vec{w} &= DP^\top - AP^\top.\end{aligned}$$

To show that the triangle with the vertices $UP^\top, AP^\top, DP^\top$ is a non-right triangle, it is sufficient to show that $\vec{u}\cdot\vec{v} \neq 0, \vec{u}\cdot\vec{w} \neq 0, \vec{v}\cdot\vec{w} \neq 0$. According to the proof of Proposition 1, $\vec{u} = (4b, 2b, 4b)$ and $\vec{v} = (-2a + 4b, -4a + 2b, -4a + 4b)$. Analogously,

$$\vec{w} = (D - A)P^\top = \begin{pmatrix} -2 & 0 & 0 \\ -4 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a \\ -4a \\ -4a \end{pmatrix}.$$

1.) By way of contradiction, assume that $\vec{u} \cdot \vec{v} = 0$. Clearly, $\vec{u} \cdot \vec{v} = (4b, 2b, 4b) \cdot (-2a + 4b, -4a + 2c, -4a + 4b) = -32ab + 36b^2$. Since $a, b > 0$, we get $b = \frac{8a}{9}$. However, (a, b, c) satisfies the Pythagorean equation which yields

$$\begin{aligned}a^2 + \frac{64a^2}{81} &= c^2\\ \sqrt{145}\frac{a}{9} &= c\end{aligned}$$

which is a contradiction with $a, c \in \mathbb{N}$.

2.) Similarly, $\vec{u} \cdot \vec{w} = (4b, 2b, 4b) \cdot (-2a, -4a, -4a) = -32ab$ and from $a, b > 0$ it follows that $\vec{u} \cdot \vec{w} \neq 0$.

3.) Finally, $\vec{v} \cdot \vec{w} = (-2a + 4b, -4a + 2c, -4a + 4b) \cdot (-2a, -4a, -4a) = 36a^2 - 32ab$ and we get a contradiction analogously to the case $\vec{u} \cdot \vec{v}$.

Also, it can be proven that the triangle $\triangle_B(P)$ is neither isosceles nor equilateral triangle.

Proposition 5. *Let P be a primitive Pythagorean triple. The triangle $\triangle_B(P)$ is not an isosceles triangle.*

Proof. Let $P = (a, b, c)$. Similarly like above, we consider the vectors $\vec{u}, \vec{v}, \vec{w}$:

$$\begin{aligned}\vec{u} &= AP^\top - UP^\top = (4b, 2b, 4b),\\ \vec{v} &= DP^\top - UP^\top = (-2a + 4b, -4a + 2b, -4a + 4b),\\ \vec{w} &= DP^\top - AP^\top = (-2a, -4a, -4a).\end{aligned}$$

It remains to prove that each of these vectors has a different norm. Clearly,

$$|\vec{u}| = \sqrt{36b^2},$$

$$
\begin{aligned}
|\vec{v}| &= \sqrt{36a^2 - 64ab + 36b^2}, \\
|\vec{w}| &= \sqrt{36a^2}.
\end{aligned}
$$

1.) By way of contradiction, let us assume that $|\vec{u}| = |\vec{w}|$. Then

$$\sqrt{36b^2} = \sqrt{36a^2} \implies 36b^2 = 36a^2 \implies a = b$$

which is a contradiction with (a, b, c) being a PPT.

2.) If $|\vec{u}| = |\vec{v}|$, then

$$
\begin{aligned}
\sqrt{36b^2} &= \sqrt{36a^2 - 64ab + 36b^2} \\
36b^2 &= 36a^2 - 64ab + 36b^2 \\
0 &= 36a^2 - 64ab \\
0 &= 9a - 16b.
\end{aligned}
$$

Applying Euclid's formula, we get

$$
\begin{aligned}
0 &= 9(m^2 - n^2) - 16(2mn) \\
16(2mn) &= 9(m^2 - n^2)
\end{aligned}
$$

where m, n have different parity. Then $16(2mn)$ is clearly even and $9(m^2 - n^2)$ is odd, which is a contradiction.

3.) Finally, let $|\vec{w}| = |\vec{v}|$. Then

$$
\begin{aligned}
\sqrt{36a^2} &= \sqrt{36a^2 - 64ab + 36b^2} \\
36a^2 &= 36a^2 - 64ab + 36b^2 \\
0 &= -16a + 9b
\end{aligned}
$$

Applying Euclid's formula, we get

$$
\begin{aligned}
0 &= -16(m^2 - n^2) + 9(2mn) \\
0 &= -8m^2 + 8n^2 + 9mn.
\end{aligned}
$$

Therefore

$$n = \frac{-9m \pm \sqrt{337}m}{16} \notin \mathbb{N},$$

which is a contradiction with $n \in \mathbb{N}$.

Corollary 4. *Let P be a primitive Pythagorean triple. The triangle $\triangle_B(P)$ is not an equilateral triangle.*

Further, we study the area, inradius and the radius of the circumcircle of the descendant triangles in Berggren's tree and present the related results.

Proposition 6. *Let $P = (a, b, c)$ be a primitive Pythagorean triple. The triangle $\triangle_B(P)$ has the area*

$$S = 2\sqrt{17}ab. \tag{3}$$

For proof, see [9].

Corollary 5. *Let P, Q be primitive Pythagorean triples. Then the triangles corresponding to P and Q have the same area if and only if the triangles $\triangle_B(P)$ and $\triangle_B(Q)$ have the same area.*

It would be interesting to know whether there even exist any PPTs with the same area. According to [13], the smallest PPTs with the same area are $(20, 21, 29)$ and $(12, 35, 37)$. Both of them have the area 210. Further, the following theorem was proved in [13]:

Theorem 3. *For every $n \in \mathbb{N}$, there exist n primitive Pythagorean triples with different third components and the same area.*

4. Descendants in Price's tree

Another well known tree of primitive Pythagorean triples was described by Price in [12]. Analogously like in Berggren's tree, Price proved that each PPT can be generated from $(3, 4, 5)$ by unique three-fold ascent using the three matrices M_1, M_2, M_3 :

$$M_1 = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

It is easy to show that

$$M_1^n = \begin{pmatrix} 2^n & -2^n + 1 & 2^n - 1 \\ 2^{2n} - 2^n & 2^n & 2^{2n} - 2^n \\ 2^{2n} - 2^n & 2^n - 1 & 2^{2n} - 2^n + 1 \end{pmatrix},$$

$$M_2^n = \begin{pmatrix} \frac{2^{2n+2}+(-1)^n\cdot 2^n+4}{9} & \frac{(-1)^{n+1}\cdot 2^n+1}{3} & \frac{2^{2n+2}+(-1)^n\cdot 2^n-5}{9} \\ \frac{2^{2n}+(-1)^n\cdot 2^n}{3} & (-1)^n \cdot 2^n & \frac{2^{2n}+(-1)^n\cdot 2^n}{3} \\ \frac{5\cdot 2^{2n}+(-1)^n\cdot 2^n-4}{9} & \frac{(-1)^n\cdot 2^n-1}{3} & \frac{5\cdot 2^{2n}+(-1)^n\cdot 2^n+5}{9} \end{pmatrix},$$

$$M_3^n = \begin{pmatrix} 2^n & 2^n - 1 & -2^n + 1 \\ -2^{2n} + 2^n & 2^n & 2^{2n} - 2^n \\ -2^{2n} + 2^n & 2^n - 1 & 2^{2n} - 2^n + 1 \end{pmatrix}.$$

Theorem 4. *If (a, b, c) is a primitive Pythagorean triple with odd a, and M is a matrix such that $M \in \{M_1, M_2, M_3\}$, then $M \cdot (a, b, c)^\top$ is a primitive Pythagorean triple with an odd first component.*

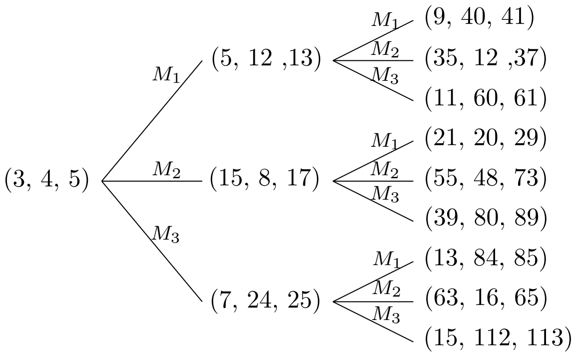

Figure 2: Price's tree of PPTs.

Price's tree of PPT contains all PPTs, and each PPT is generated by a unique sequence of matrix multiplication [12]. The first levels of Price's tree are in the Figure 2.

Similar relations that hold in Berggren's tree of PPTs can be also proved in Price's tree. We will use the analogous terms and notations.

Definition 5. *Let R be a primitive Pythagorean triple. We say that R is the parent of the triples $M_1R^\top, M_2R^\top, M_3R^\top$. The triples $M_1R^\top, M_2R^\top, M_3R^\top$ are called the descendants of R in Price's tree.*

Proposition 7. *Let R be a primitive Pythagorean triple. Then the points with the coordinates $M_1R^\top, M_2R^\top, M_3R^\top$ are not collinear.*

Proof. Let $R = (a, b, c)$. We consider the vectors $\vec{p} = M_2R^\top - M_1R^\top$ and $\vec{q} = M_3R^\top - M_1R^\top$. It is easy to show that

$$\vec{p} = (M_2 - M_1)R^\top = \begin{pmatrix} 0 & 0 & 2 \\ 4 & -4 & 0 \\ 4 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c \\ 4a - 4b \\ 4a - 2b \end{pmatrix},$$

$$\vec{q} = (M_3 - M_1)R^\top = \begin{pmatrix} 0 & -2 & 2 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2b + 2c \\ 4a \\ 4a \end{pmatrix}.$$

By way of contradiction, assume that these vectors are linearly dependent, i.e., that there exists nonzero real number k such that $(2c, 4a - 4b, 4a - 2b) = k(-2b + 2c, 4a, 4a)$, hence $4a - 4b = k(4a)$ and $4a - 2b = k(4a)$. This implies

$$\begin{aligned} 4a - 4b &= 4a - 2b \\ -4b &= -2b \\ b &= 0 \end{aligned}$$

which is a contradiction with $b \in \mathbb{N}$. Therefore, $\vec{p}, \vec{q}$ are linearly independent and the points $M_1R^\top, M_2R^\top, M_3R^\top$ are not collinear.

Therefore, the points $M_1R^\top, M_2R^\top, M_3R^\top$ determine a plane, and a triangle. In the following, we prove some properties of this plane (and this triangle).

Definition 6. *Let R be a primitive Pythagorean triple and let $M_1R^\top, M_2R^\top$, $M_3R^\top$ be its descendants in Price's tree. The triangle with vertices with the coordinates $M_1R^\top, M_2R^\top, M_3R^\top$ is called the descendant triangle of R in Price's tree, and we denote it by $\triangle_P(R)$.*

Proposition 8. *Let $R = (a, b, c)$ be a primitive Pythagorean triple. Then the triangle $\triangle_P(R)$ belongs to the plane*

$$(2a)x + (2a - b + c)y + (-2a + 2b - 2c)z + 2ab - 2ac + 4bc - 4b^2 = 0. \tag{4}$$

Proof. According to the Proposition 7, the vertices $M_1R^\top, M_2R^\top, M_3R^\top$ form a triangle, therefore, they define a plane. We consider the vectors $\vec{p} = (M_2 - M_1)R^\top = (2c, 4a - 4b, 4a - 2b)$ and $\vec{q} = (M_3 - M_1)R^\top = (-2b + 2c, 4a, 4a)$ and we compute the normal vector of the wanted plane as cross product of these vectors: $\vec{p} \times \vec{q} = (-8ab, -8ab + 4b^2 - 4bc, 8ab + 8bc - 8b^2) \approx (2a, 2a - b + c, -2a + 2b - 2c)$.

Hence, $(2a)x + (2a - b + c)y + (-2a + 2b - 2c)z + d = 0$ is the equation of wanted plane for some $d \in \mathbb{R}$. Since $M_1R^\top$ belongs to this plane, we get

$$(2a)(2a + b - c) + (2a - b + c)(-2a + 2b + 2c) + (-2a + 2b - 2c)(-2a + b + 3c) + d = 0.$$

Moreover, $c^2 = a^2 + b^2$, which yields $d = 2ab - 2ac + 4bc - 4b^2$. Therefore, $\triangle_P(R)$ belongs to the plane

$$(2a)x + (2a - b + c)y + (-2a + 2b - 2c)z + 2ab - 2ac + 4bc - 4b^2 = 0.$$

Notice the difference between Price's tree and Berggren's tree. In Berggren's tree, all descendant triangles $\triangle_B(P)$ belong to planes parallel with the plane $2x + 2y - 3z = 0$ (see the Proposition 2). In general, the results in Berggren's tree are usually nicer than analogous results in Price's tree.

Also, similarly to Berggren's tree, no plane defined by a descendant triangle $\triangle_P(R)$ in Price's tree passes through the origin $(0, 0, 0)$.

Proposition 9. *Let (a, b, c) be a primitive Pythagorean triple. The plane $(2a)x + (2a - b + c)y + (-2a + 2b - 2c)z + 2ab - 2ac + 4bc - 4b^2 = 0$ does not pass through the origin $(0, 0, 0)$.*

Proof. If $(0, 0, 0)$ belongs to this plane, then $2ab - 2ac + 4bc - 4b^2 = 0$. After some modifications, we get

$$\begin{aligned} ab - ac + 2bc - 2b^2 &= 0 \\ b(a - 2b) - c(a - 2b) &= 0 \\ (b - c)(a - 2b) &= 0 \end{aligned}$$

which holds iff $b - c = 0$ or $a - 2b = 0$. However, $b - c = 0 \implies b = c$, a contradiction with (a, b, c) being a PPT, and $a - 2b = 0 \implies a = 2b$ which is a contradiction with a being odd and $2b$ being even.

From the Propositions 4 and 5, we already know that in Berggren's tree, the descendant triangle $\triangle_B(P)$ fails to be either a right triangle or an isosceles triangle. The following propositions offer the analogous results in Price's tree.

Proposition 10. *Let R be a primitive Pythagorean triple. The triangle $\triangle_P(R)$ is a non-right triangle.*

Proof. Let us consider the vectors

$$\begin{aligned}\vec{p} &= M_2R^\top - M_1R^\top = (2c, 4a-4b, 4a-2b),\\ \vec{q} &= M_3R^\top - M_1R^\top = (-2b+2c, 4a, 4a),\\ \vec{r} &= M_3R^\top - M_2R^\top = (-2b, 4b, 2b).\end{aligned}$$

To prove that $\triangle_P(R)$ is a non-right triangle, it is sufficient to show that $\vec{p}\cdot\vec{q} \neq 0$, $\vec{p}\cdot\vec{r} \neq 0$ and $\vec{q}\cdot\vec{r} \neq 0$.

1.) By way of contradiction, let us assume that $\vec{p}\cdot\vec{q} = 0$. This and $a^2+b^2=c^2$ yield

$$\begin{aligned}(2c, 4a-4b, 4a-2b)\cdot(-2b+2c, 4a, 4a) = 16a^2 - 8b^2 + 4c^2 - 4cb &= 0\\ 20c^2 - 24b^2 - 4bc &= 0\\ 5c^2 - 6b^2 - bc &= 0\\ 6b^2 + bc &= 5c^2.\end{aligned}$$

In the beginning, we set the order of the components of (a, b, c) to be odd, even, odd. Hence, there is an even natural number on the left side, and an odd natural number on the right side, which is a contradiction.

2.) Analogously, let us assume that $\vec{p}\cdot\vec{r} = 0$. Then

$$\begin{aligned}\vec{p}\cdot\vec{r} = (2c, 4a-4b, 4a-2b)\cdot(-2b, 4b, 2b) &= 0\\ -20b^2 + 24ab - 4bc &= 0\\ 5b - 6a + c &= 0.\end{aligned}$$

Using Euclid's formula, we get

$$\begin{aligned}5(2mn) - 6(m^2-n^2) + (m^2+n^2) &= 0\\ 10mn - 5m^2 &= -7n^2\end{aligned}$$

where $m, n \in \mathbb{N}$ have different parity. If m is even and n is odd, then the left side of this equation is even and the right side is odd, which is a contradiction. If m is odd and n is even, then the left side of this equation is odd and the right side is even, also a contradiction. Hence, $\vec{p}\cdot\vec{r} \neq 0$.

3.) Finally, we assume that $\vec{q}\cdot\vec{r} = 0$. Then

$$\begin{aligned}\vec{q}\cdot\vec{r} = (-2b+2c, 4a, 4a)\cdot(-2b, 4b, 2b) &= 0\\ 4b^2 - 4bc + 24ab &= 0\\ b - c + 6a &= 0.\end{aligned}$$

Using Euclid's formula, we get

$$2mn - (m^2+n^2) + 6(m^2-n^2) = 0$$

$$2mn + 5m^2 = 7n^2$$

where $m, n \in \mathbb{N}$ have different parity. We get the contradiction analogously to the previous case.

Proposition 11. *Let R be a primitive Pythagorean triple. The triangle $\triangle_P(R)$ is not an isosceles triangle.*

Proof. Let $R = (a, b, c)$. Similarly like above, we consider the vectors $\vec{p}, \vec{q}, \vec{r}$:

$$\begin{aligned}
\vec{p} &= M_2 R^\top - M_1 R^\top = (2c, 4a - 4b, 4a - 2b),\\
\vec{q} &= M_3 R^\top - M_1 R^\top = (-2b + 2c, 4a, 4a),\\
\vec{r} &= M_3 R^\top - M_2 R^\top = (-2b, 4b, 2b).
\end{aligned}$$

Then

$$\begin{aligned}
|\vec{p}| &= 2\sqrt{9a^2 - 12ab + 6b^2}\\
|\vec{q}| &= 2\sqrt{(c - b)^2 + 8a^2}\\
|\vec{r}| &= 2\sqrt{6b^2}.
\end{aligned}$$

1.) By way of contradiction, let us assume that $|\vec{p}| = |\vec{q}|$. This yields

$$\begin{aligned}
9a^2 - 12ab + 6b^2 &= c^2 - 2bc + b^2 + 8a^2\\
9a^2 - 12ab + 6b^2 &= a^2 + b^2 - 2bc + b^2 + 8a^2\\
bc - 6ab + 2b^2 &= 0.
\end{aligned}$$

Using the Euclid's formula, we get

$$\begin{aligned}
(2mn)(m^2 + n^2) - 6(m^2 - n^2)(2mn) + 2(2mn) &= 0\\
m^2 + n^2 - 6m^2 + 6n^2 + 1 &= 0\\
7n^2 + 1 &= 5m^5
\end{aligned}$$

which implies that $7n^2 + 1 \equiv 0 \pmod 5 \iff n^2 \equiv 2 \pmod 5$. However, this congruence has no solutions in $\mathbb{N}$, hence $|\vec{p}| \neq |\vec{q}|$.

2.) Further, we assume that $|\vec{p}| = |\vec{r}|$. Hence,

$$\begin{aligned}
9a^2 - 12ab + 6b^2 &= 6b^2\\
3a - 4b &= 0\\
3a &= 4b.
\end{aligned}$$

However, $3a$ is odd and $4b$ is even, a contradiction.

3.) Finally, we assume that $|\vec{q}| = |\vec{r}|$. Then

$$c^2 - 2bc + b^2 + 8a^2 = 6b^2$$

$$\begin{aligned} c^2 - 2bc - 5b^2 + 8(c^2 - b^2) &= 0 \\ 9c^2 - 2bc - 12b^2 &= 0 \\ 9c^2 &= b(2c + 13b) \end{aligned}$$

where c is odd and b is even, which implies that the left side is odd, and the right side is even, a contradiction.

Therefore, $\triangle_P(R)$ is not an isosceles triangle.

Corollary 6. *Let R be a primitive Pythagorean triple. The triangle $\triangle_P(R)$ is not an equilateral triangle.*

5. Concluding remarks

To consider the three descendants of a primitive Pythagorean triple in a tree of primitive Pythagorean triples as points in the three-dimensional space can provide further insight into the structure of primitive Pythagorean triples. As these three descendants form a triangle in either Berggren's or Price's tree, it is interesting to study the properties of these triangles. This new approach offers numerous open problems, which we would like to explore in our further research.

For example, we proved that the descendant triangle fails to be right triangle in both Berggren's and Price's tree. If it always fails to be a right triangle, could it always be acute? Is it obtuse in some cases? Can the set of all inner angles be somehow described? These and further problems remain open and we intend to study them further.

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