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Double Inertial Krasnosel'skii-Mann-Type Method for Approximating Fixed Point of Nonexpansive Mappings

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Abstract. In this paper, we investigate a new method motivated by current advancements in general inertial algorithms. Specifically, we incorporate double inertial extrapolation terms into an iterative sequence, derived from Krasnosel'skii-Mann techniques. The weak convergence theorem for fixed points of nonexpansive mappings in real Hilbert spaces is established. The theoretical developments are rigorously proven, extending existing methods in literature. We also utilize our convergence analysis to solve real-world problems, such as convex minimization problems and zero finding for sums of monotone operators.

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1. Introduction

Consider a nonempty subset $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle$ and induced norm $||.||$. We refer to a selfmap $\mathcal{T}$ on $\mathcal{K}$ as:

(i) Nonexpansive (refer to, for example, [1, 11]), when $a, b \in \mathcal{K}$, then

$$||\mathcal{T}a - \mathcal{T}b|| \leq ||a - b||.$$

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(ii) If a real constant $L > 0$ exists, then for every $a, b \in \mathcal{K}$,

$$||\mathcal{T}a - \mathcal{T}b|| \leq L||a - b||,$$

is called $L-$Lipschitzian. Recall that all nonexpansive mappings are $L-$Lipschitzian mappings, where $L = 1$ (i.e., continuous).

In recent years, academics have become quite interested in the study of fixed points of nonexpansive mappings and their generalizations. The reason behind this is that fixed points of nonexpansive mappings have numerous practical uses in various fields, such as computer tomography and image recovery, mostly due to their close relationship with the accretive operator class (sometimes referred to as monotone operators in Hilbert spaces). Given a proper function $f : \mathcal{H} \to (-\infty, +\infty]$, let ∂f represent its subdifferential. Typically, one would demonstrate that a minimizer of f is any zero of ∂f. The fact that ∂f is a monotone operator is well known (see [3, Example 20.3]). In 1967, Browder [5] and Kato [11] introduced the accretive operators separately. According to Browder [5], if $\mathcal{A}$ is Lipschitzian and accretive, then $\frac{du}{dt} + \mathcal{A}u = 0,\ u(0) = u_0$ is solvable. This is a key result in the theory of accretive operators.
It is well known (see, for example, [20]) that if $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ is an accretive operator, then the resolvent of $\mathcal{A}$, given by $J^{\lambda}_{\mathcal{A}} := (I + \lambda\mathcal{A})^{-1}$, and denoted by $J^{\lambda}_{\mathcal{A}}$, is a nonexpansive operator for any real constant $\lambda > 0$. The set of zeros of $\mathcal{A}$ is denoted by $Zer(\mathcal{A})$ and defined by $Zer(\mathcal{A}) := \{a \in \mathcal{H} : 0 \in \mathcal{A}a\}$. It is simply provable that the fixed points of $J^{\lambda}_{\mathcal{A}}$ are the zeros of $\mathcal{A}$. As such, the study of fixed points of nonexpansive mappings unifies a number of application domains that are otherwise divided by the theory of accretive operators (see, for example, [27]).

Several iterative techniques for addressing fixed point problems of nonexpansive mappings have been presented and analyzed by numerous writers; see [2, 4, 6, 18, 32] and references therein. As we know, one of the most famous methods for approximating fixed points of nonexpansive mapping is the Mann [19] iterative process introduced in 1953 and establishing the weak convergence theorem for the sequence. The algorithm is of the form: for each random $a_0 \in \mathcal{K}$

$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n\mathcal{T}a_n, \tag{1}$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence. Under the condition $\sum \alpha_n(1 - \alpha_n) = +\infty$.

However, the Mann sequence has a very slow rate of convergence. In [29], the authors noted that the rate at which the fixed points are approximated using a particular approach needs to be as fast as possible in order to make real systems stable and dependable (see, for instance, [14, 15]). This is the reason why a lot of writers have focused their attention on studying fast converging iteration algorithms. Many authors (e.g., [8, 9, 16, 17, 22, 30, 31]) have explored the iteration approach with inertial extrapolations in the recent past. The inertial term, which is added to these algorithms in an attempt to accelerate the convergence rate, is what distinguishes them. Typically, the inertial term has the form $\alpha_n(a_n - a_{n-1})$ and meets specific requirements. A portion of the inertial terms technique can be found in [2, 18], where the following theorems were presented:

Theorem 1. [2, Alvarez and Attouch] *Let $\mathcal{H}$ be a real Hilbert space. For any arbitrary points a_0, a_1 in $\mathcal{H}$, let $\{a_n\}$ be a sequence generated as*

$$a_{n+1} = J^{\mathcal{A}}_{\lambda_n}(a_n + \alpha_n(a_n - a_{n-1})),\ n \geq 1,$$

where $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator with $\mathcal{A}^{-1}(0) \neq \emptyset$, and the parameters α_n and λ_n satisfy
(i) there exists $\lambda > 0$ such that $\forall n \in N,\ \lambda_n \geq \lambda$.
(ii) there exists $\alpha \in [0, 1]$ such that $\forall n \in N,\ 0 \leq \alpha_n \leq \alpha$.
If the following condition holds

$$\sum_{n=1}^{+\infty} \alpha_n \|a_n - a_{n-1}\|^2 < +\infty,$$

then $\{a_n\}$ converges weakly to a point in $\mathcal{A}^{-1}(0)$ as $n \to +\infty$.

In 2008, Mainge [18] introduced the classical inertial Mann-type technique as follows:

$$\begin{cases} a_0, a_1 \in \mathcal{H}, \\ b_n = a_n + \alpha_n(a_n - a_{n-1}), \\ a_{n+1} = (1 - \lambda_n)y_n + \lambda_n \mathcal{T} b_n, \end{cases} \tag{2}$$

for each $n \in N$. He proved that under the conditions:

(i) $\alpha_n \in [0, \alpha]$ for each $n \geq 1$, where $\alpha \in [0, 1)$;

(ii) $\sum \alpha_n \|a_n - a_{n-1}\|^2 < +\infty$

(iii) $0 < \liminf_{n\to+\infty} \lambda_n \leq \limsup_{n\to+\infty} \lambda_n < 1$,

$\{a_n\}$ converges weakly to a fixed point of $\mathcal{T}$.

Later in 1015, Bot and Csetnek [4] replaced conditions (i) and (iii) above with:
(i') $\delta > \frac{\alpha^2(1+\alpha)+\alpha\sigma}{1-\alpha^2}$ (iii') $0 < \lambda \leq \lambda_n \leq \theta := \frac{\delta-\alpha[\alpha(1+\alpha)+\alpha\delta+\sigma]}{\delta[1+\alpha(1+\alpha)+\alpha\delta+\sigma]}$
We noted that condition (ii) was employed implicitly in the convergence results but was not explicitly listed as one of the hypotheses in either the Mainge [18] or the Bot and Csetnek [4]. Condition (ii) is simply implementable, according to the authors, because a_n and a_{n-1} are known at each stage. This allows α_n to be set so that it is dominated by a $\frac{1}{\|a_n - a_{n-1}\|^2}$ multiple of a summable sequence.

To further enhance convergence rates, many authors(see [12, 13, 21]) have studied iteration schemes with double inertial terms. In [7], the authors studied the following double inertial Mann-type method and established weak convergence of the process:

$$\begin{cases} a_0, a_1 \in \mathcal{H}, \\ b_n = a_n + \alpha_n(a_n - a_{n-1}), \\ c_n = a_n + \beta_n(a_n - a_{n-1}), \\ a_{n+1} = (1-\lambda_n)b_n + \lambda_n \mathcal{T} c_n, \end{cases} \tag{3}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ satisfy the following conditions:
(D_1) $\{\alpha_n\} \subset [0, \alpha]$ *and* $\beta_n \subset [0, \beta]$ are nondecreasing with $\alpha_1 = \beta_1 = 0$ and $\alpha, \beta \in [0, 1)$;
(D_2) for any $\lambda, \sigma, \delta > 0$, $\delta > \frac{\alpha\xi(1+\xi)+\alpha\sigma}{1-\alpha^2}$, $0 < \lambda \leq \lambda_n \leq \frac{\delta - \alpha[\xi(1+\xi)+\alpha\delta+\sigma]}{\delta[1+\xi(1+\xi)+\alpha\delta+\sigma]}$.

In [34], Yao et al improved the efficiency and accelerated the rate of convergence of methodologies for solving variational inequality problems by incoparating double inertial phases into their methods.

Furthermore, the authors in [23], extensively discussed the addition of inertial terms to speed up convergence rates of iteration schemes. They discussed the addition of one-step inertial term and the addition of two-step inertial terms(double inertial) to the proximal point algorithm (PPA). They exhibited an example from [26], which shows that the two step inertial Douglas-Rachford splitting method

$$a_{n+1} = F_{DR}(a_n + \theta(a_n - a_{n-1}) + \delta(a_{n-1} - a_{n-2})),$$

converges faster than the one-step inertial method

$$a_{n+1} = F_{DR}(a_n + \theta(a_n - a_{n-1})),$$

where F_{DR} is the Douglas-Rachford splitting operator. The authors posited, resulting from the example, that the one step inertial Douglas-Rachford method may fail to provide acceleration whereas the two-step (double step) method does.

Our aim in this paper is to further improve the control parameters on double inertial extrapolation Krasnosel'skii-Mann-type method. Within a convex subset of a real Hilbert space, we want to approximate fixed points of nonexpansive mappings. We firmly establish our suggested method's weak convergence theorem. We also use our findings to solve real-world applications, such as convex minimization and zero finding for sums of monotone operators. We conclude with numerical calculations of our suggested approach and a comparison with the techniques in [7, 18]. In comparison to the inertial Krasnosel'skii-Mann-type approaches in [7, 18], our suggested method converges more quickly in terms of CPU time and iterations, according to our preliminary computational results. This is the algorithm that we propose.

Given a real Hilbert space $\mathcal{H}$, let $\mathcal{K}$ be a convex subset of it. $\{a_n\}$ is generated by the rule from arbitrary $a_0, a_1 \in \mathcal{K}$.

$$\begin{cases} a_0, a_1 \in \mathcal{H}, \\ b_n = a_n + t_n(a_{n-1} - a_n), \\ c_n = a_n + r_n(a_{n-1} - a_n), \\ a_{n+1} = (1-\alpha_n)b_n + \alpha_n \mathcal{T} c_n, \ n \geq 1, \end{cases} \tag{4}$$

where $t_n, r_n, \alpha_n \in (0,1)$ satisfy certain conditions.
We will prove that our algorithm converges weakly to fixed points of nonexpansive mappings on $\mathcal{T}$ under mild conditions.

2. Preliminaries

Prior to stating and demonstrating our primary findings, we provide a definition and a few lemmas that will be helpful in the follow-up:

Definition 1. *Consider the Banach space $\mathcal{E}$. When $\{a_n\}$ is a sequence in $D(\mathcal{T})$ such that $\{a_n\}$ converges weakly to $a \in D(\mathcal{T})$ and $\{\mathcal{T}a_n\}$ converges strongly to u, then $\mathcal{T}a = u$. This mapping $\mathcal{T} : D(\mathcal{T}) \subseteq \mathcal{E} \to \mathcal{E}$ is said to be demiclosed at a point $u \in D(\mathcal{T})$. (see for example [25]).*

Lemma 1. *[10] Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{H}$. A nonexpansive mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$, is said to be demiclosed at zero if for any sequence $\{a_n\} \subset \mathcal{K}$ with $a_n \rightharpoonup a \in \mathcal{K}$ and*

$$||a_n - \mathcal{T}a_n|| \longrightarrow 0 \quad as \quad n \longrightarrow +\infty,$$

we have $\mathcal{T}a = a$.

Lemma 2. *[33] Let $\mathcal{H}$ be a real Hilbert space. Then for all $a, b \in \mathcal{H}$, and for any real number, λ, the following well-known identity holds:*
$||(1-\lambda)a + \lambda b||^2 = (1-\lambda)||a||^2 + \lambda||b||^2 - \lambda(1-\lambda)||a-b||^2$

3. Main Results

Theorem 2. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Consider a nonexpansive mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ with a nonempty fixed points set, $F(\mathcal{T})$. The sequence generated by the (4) then converges weakly under the following conditions:*

(a) $\liminf \alpha_n(1-\alpha_n) > 0$

(b) $t_n \leq d$ *for some real constant* $d \in (0,1)$,

(c) $r_n \leq d_1$ *for some real constant* $d_1 \in (0,1)$.

Remark 1. *Our algorithm uses double inertial techniques because the combination of two inertial components reduces oscillations and produces smoother convergence behavior. Our method is now perfect for solving difficult monotone problems with inclusion because it can handle non-convex.*

Proof. **Convergence analysis of Theorem 2**
Let $p \in F(\mathcal{T})$. Using (4) and Lemma 2, we have

$$
\begin{aligned}
||a_{n+1}-p||^2 &= ||(1-\alpha_n)b_n+\alpha_n\mathcal{T}c_n-p||^2 \\
&= (1-\alpha_n)||b_n-p||^2+\alpha_n||\mathcal{T}c_n-p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2 \\
&\leq (1-\alpha_n)||b_n-p||^2+\alpha_n||c_n-p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2 \\
&= (1-\alpha_n)||(1-t_n)(a_n-p)+t_n(a_{n-1}-p)\|^2 \\
&\quad +\alpha_n\|(1-r_n)(a_n-p)+r_n(a_{n-1}-p)\|^2 \\
&\quad -\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2 \\
&= (1-\alpha_n)\Big[(1-t_n)\|a_n-p\|^2+t_n\|a_{n-1}-p\|^2 \\
&\quad -t_n(1-t_n)\|a_n-a_{n-1}\|^2\Big]+\alpha_n\Big[(1-r_n)\|a_n-p\|^2 \\
&\quad +r_n\|a_{n-1}-p\|^2-r_n(1-r_n)\|a_n-a_{n-1}\|^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2 \\
&= (1-\alpha_n)\Big[\|a_n-p\|^2+t_n(\|a_{n-1}-p\|^2-\|a_n-p\|^2) \\
&\quad -t_n(1-t_n)\|a_n-a_{n-1}\|^2\Big] \\
&\quad +\alpha_n\Big[\|a_n-p\|^2+r_n(\|a_{n-1}-p\|^2-\|a_n-p\|^2) \\
&\quad -r_n(1-r_n)\|a_n-a_{n-1}\|^2\|^2\Big]-\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2. \qquad (5)
\end{aligned}
$$

We can estimate $||a_{n-1}-p||^2-||a_n-p||^2$ in two ways, viz: (i) $||a_{n-1}-p||^2-||a_n-p||^2<0$ or (ii) $||a_{n-1}-p||^2-||a_n-p||^2 \geq 0$. If (i) holds (i.e $||a_{n-1}-p||^2<||a_n-p||^2$), then substituting this in (5), we get $||a_{n+1}-p||^2<||a_n-p||^2$. This is an absurdity. Hence (ii) holds. This implies $||a_{n+1}-p||^2 \leq ||a_n-p||^2$. Therefore $\{||a_n-p||^2\}$ is a monotone decreasing sequence so that $\lim ||a_n-p||^2$ exists. From this, we have that $||a_n-p||$ and hence $||a_n||$ are bounded. Therefore, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ which converges weakly to $z \in \mathcal{H}$. Since $\mathcal{H}$ is an Opial space, a standard argument (see eg [24]) yields that $\{a_n\}$ converges weakly to z.

Now, since (ii) holds, we have from (5) that

$$
\begin{aligned}
\sum_{n\geq 1}\alpha_n(1-\alpha_n)||\mathcal{T}c_n-b_n||^2 &\leq \sum_{n\geq 0}[||a_n-p||^2-||a_{n+1}-p||^2] \\
&\quad +\sum_{n\geq 0}[||a_{n-1}-p||^2-||a_n-p||^2] \qquad (6) \\
&\quad -\alpha_0(1-\alpha_0)||\mathcal{T}c_0-b_0||^2<+\infty. \qquad (7)
\end{aligned}
$$

This implies from condition (a) that

$$\lim ||\mathcal{T}c_n - b_n|| = 0 \tag{8}$$

Again, from (5) since (ii) holds and $\lim ||a_n - p||^2$ exists, we have

$$\begin{aligned} t_n(1-t_n)\|a_n - a_{n-1}\|^2 &\leq [||a_n - p||^2 - ||a_{n+1} - p||^2] \\ &\quad + [||a_{n-1} - p||^2 - ||a_n - p||^2] \to 0 \end{aligned}$$

This implies

$$\lim t_n(1-t_n)\|a_n - a_{n-1}\|^2 = 0 \tag{9}$$

Using (8), condition (b) and (9), we have

$$\begin{aligned} ||b_n - a_n|| &= t_n\|a_n - a_{n-1}\| \\ &= [[\frac{t_n(1-t_n)}{1-t_n}\|a_n - a_{n-1}\|]^2]^{\frac{1}{2}} \\ &\leq [[\frac{t_n(1-t_n)}{1-d}]^2\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \\ &= [\frac{[t_n(1-t_n)]^2}{[1-d]^2}\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \\ &\leq [\frac{t_n(1-t_n)}{[1-d]^2}\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \to 0. \end{aligned} \tag{10}$$

Again, from (5) since (ii) holds and $\lim ||a_n - p||^2$ exists, we have

$$\begin{aligned} r_n(1-r_n)\|a_n - a_{n-1}\|^2 &\leq [||a_n - p||^2 - ||a_{n+1} - p||^2] \\ &\quad + [||a_{n-1} - p||^2 - ||a_n - p||^2] \to 0 \end{aligned}$$

This implies

$$\lim r_n(1-r_n)\|a_n - a_{n-1}\|^2 = 0 \tag{11}$$

Using (8), condition (c) and (11), we have

$$\begin{aligned} ||c_n - a_n|| &= r_n\|a_n - a_{n-1}\| \\ &= [[\frac{r_n(1-r_n)}{1-r_n}\|a_n - a_{n-1}\|]^2]^{\frac{1}{2}} \\ &\leq [[\frac{r_n(1-r_n)}{1-d_1}]^2\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \\ &= [\frac{[r_n(1-r_n)]^2}{[1-d_1]^2}\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \\ &\leq [\frac{r_n(1-r_n)}{[1-d_1]^2}\|a_n - a_{n-1}\|^2]^{\frac{1}{2}} \to 0. \end{aligned} \tag{12}$$

Using the nonexpansiveness of $\mathcal{T}$, (9), (11) and (12), we have

$$||a_n - \mathcal{T}a_n|| = ||a_n - b_n + b_n - \mathcal{T}c_n + \mathcal{T}c_n - \mathcal{T}a_n||$$

$$\begin{aligned} &\leq ||a_n - b_n|| + \|b_n - \mathcal{T}c_n\| + \|\mathcal{T}c_n - \mathcal{T}a_n\| \\ &\leq ||a_n - b_n|| + \|b_n - \mathcal{T}c_n\| + \|c_n - a_n\| \to 0 \end{aligned}$$

Using this and the fact that $I - \mathcal{T}$ is demiclosed at 0 in Lemma 1, we have that $z \in F(\mathcal{T})$. Setting $z = p$ above, our proof is complete.

Theorem 3. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ be a nonexpansive mapping with a nonempty fixed points set, $F(\mathcal{T})$. Then, under following assumptions on the control parameter, the sequence $\{a_n\}\}$ generated by* (4) *converges weakly to an element of $F(\mathcal{T})$.*

(a) $0 < \epsilon_1 \leq \alpha_n \leq \epsilon_2 < 1$.

(b) $t_n \leq d$ *for some real constant* $d \in (0, 1)$,

(c) $r_n \leq d_1$ *for some real constant* $d_1 \in (0, 1)$.

Proof. Computing as in the proof of Theorem 2 above, we arrive at (5) Since (ii) holds and $\|x_n - p\|$ exists, we have from condition (a) and (5) that

$$\begin{aligned} \epsilon_1(1-\epsilon_2)||\mathcal{T}c_n - b_n||^2 &\leq \alpha_n(1-\alpha_n)||\mathcal{T}c_n - b_n||^2 \\ &\leq [||a_n - p||^2 - ||a_{n+1} - p||^2] \\ &\quad +[||a_{n-1} - p||^2 - ||a_n - p||^2]. \end{aligned} \tag{13}$$

This implies $\lim ||\mathcal{T}c_n - b_n|| = 0$.
The rest of the proof now follows as in that of Theorem 2 above.

Theorem 4. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{H}$. Let $\mathcal{A} : \mathcal{K} \subseteq \mathcal{H} \to \mathcal{K} \subseteq \mathcal{H}$ be a maximally monotone operator such that $Zer(\mathcal{A}) \neq \emptyset$. Let $J_{\mathcal{A}}^{\lambda} := (I + \lambda\mathcal{A})^{-1}$ be the resolvent of $\mathcal{A}$, for some real constant $\lambda > 0$. Then the sequence $\{x_n\}$ generated from $a_0, a_1 \in \mathcal{K}$ by*

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1-\alpha_n)b_n + \alpha_n J_{\mathcal{A}}^{\lambda} c_n, n \geq 1 \end{cases}$$

where $\{\alpha_n\}, \{r_n\}$ and $\{t_n\}$ are real sequences in $(0, 1)$ satisfying:

(a) $\liminf \alpha_n(1 - \alpha_n) > 0$

(b) $t_n \leq d$ *for some real constant* $d \in (0, 1)$,

(c) $r_n \leq d_1$ *for some real constant* $d_1 \in (0, 1)$, *converges weakly to an element of* $F(J_{\mathcal{A}}^{\lambda})$, *which is also an element of* $Zer(\mathcal{A})$.

Proof. Since $J_{\mathcal{A}}^{\lambda}$ is nonexpansive, the proof follows like that of Theorem 2 above.

Theorem 5. *Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed and convex subset of $\mathcal{H}$. Let $\mathcal{A} : \mathcal{K} \subseteq \mathcal{H} \to \mathcal{C} \subseteq \mathcal{H}$ be a maximally monotone operator such that $Zer(\mathcal{A}) \neq \emptyset$. Let $J_{\mathcal{A}}^{\lambda} := (I + \lambda\mathcal{A})^{-1}$ be the resolvent of $\mathcal{A}$, for some real constant $\lambda > 0$. Then the sequence $\{a_n\}$ generated from $a_0, a_1 \in \mathcal{K}$ by*

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1 - \alpha_n)b_n + \alpha_n J_{\mathcal{A}}^{\lambda} c_n, n \geq 1 \end{cases}$$

where $\{\alpha_n\}, \{r_n\}$ and $\{t_n\}$ are real sequences in $(0, 1)$ satisfying:

(a) $0 < \sigma_1 \leq \alpha_n \leq \sigma_2 < 1$ *for some real constants $a, b \in (0, 1)$,*

(b) $t_n \leq d$ *for some real constant $d \in (0, 1)$,*

(c) $r_n \leq d_1$ *for some real constant $d_1 \in (0, 1)$,*

converges weakly to an element of $F(J_A^{\lambda})$, which is also an element of $Zer(\mathcal{A})$.

Proof. Since J_A^{λ} is nonexpansive, the proof follows like that of Theorem 3 above.

4. Further Results and Applications

In the subsequent steps, we utilize the convergence outcomes we previously examined to determine the zeros of monotone operator sums. We next utilize these findings to solve convex minimization issues. We first remember the following:

Suppose $\alpha \in (0, 1)$, an $\alpha-$averaged operator is defined as follows: $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ if $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator such that $\mathcal{T} = (1 - \alpha)I + \alpha\mathcal{G}$, where I is the identity operator. It is simple to demonstrate that each $\alpha-$averaged operator is nonexpansive. This brings us to our next set of Theorems:

Theorem 6. *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{A} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be a $\beta-$cocoercive operator, with $\beta > 0$, such that $Zer(\mathcal{A}+\mathcal{B}) \neq \emptyset$. Let $\gamma \in (0, 2\beta)$. Then starting from $a_0, a_1 \in \mathcal{H}$, the sequence $\{a_n\}$ generated from the iterative scheme*

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1 - \alpha_n)b_n + \alpha_n J_{\mathcal{A}}^{\gamma}(c_n - \gamma\mathcal{B}c_n), \ n \geq 1 \end{cases} \tag{14}$$

where $\{\alpha_n\}, \{r_n\}$ and $\{t_n\}$ are real sequences in $(0, 1)$ satisfying:

(a) $\liminf \alpha_n(1-\alpha_n) > 0$

(b) $t_n \le d$ *for some real constant* $d \in (0,1)$,

(c) $r_n \le d_1$ *for some real constant* $d_1 \in (0,1)$.

converges weakly to an element of $Zer(\mathcal{A}+\mathcal{B})$.

Proof. Set $\mathcal{T} = J_{\mathcal{A}}^{\gamma} \circ (I-\gamma\mathcal{B})$, where I is the identity operator, so that (14) can be re-written as

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1-\alpha_n)b_n + \alpha_n \mathcal{T} c_n, \ n \ge 1. \end{cases} \tag{15}$$

Recall that $J_{\mathcal{A}}^{\gamma}$ is nonexpansive (see for example [20]). Since $\mathcal{B}$ is $\beta-$cocoercive, then $I-\gamma\mathcal{B}$ is $\frac{\gamma}{2\beta}-$averaged (see for example [3, Proposition 4.33]) and hence nonexpansive. Therefore $\mathcal{T} = J_{\mathcal{A}}^{\gamma} \circ (I-\gamma\mathcal{B})$ is nonexpansive (the composition of two nonexpansive mappings is easily verifiable to be nonexpansive). The results now follow from Theorem 2, since $F(\mathcal{T}) = Zer(\mathcal{A}+\mathcal{B})$ (see [3, Proposition 25.1(iv)]).

Theorem 6 can be applied in solving convex optimization problems of the form

$$\min_{a\in\mathcal{H}}\{f(a)+g(a)\},$$

where $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function and $g : \mathcal{H} \to \mathbb{R}$ is a convex and Frechet differentiable function which is such that ∇g is $\frac{1}{\beta} - Lipschitzian$, for some $\beta > 0$. To do this, we recall the following:
If $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, then its (convex) subdifferential at $a \in \mathcal{H}$ is defined by

$$\partial f(a) = \{b \in \mathcal{H} : f(z) \ge f(a) + \langle b, z-a\rangle \forall z \in \mathcal{H}\},$$

for all $a \in \mathcal{H}$, with $f(a) = +\infty$ and $\partial f(a) = \emptyset$ otherwise. When the convex subdifferential is seen as a set-valued mapping, then, it is maximally monotone (see [28]) and its resolvent is given by $J_{\partial f} = prox_f$ (see [3]), where $Prox_f : \mathcal{H} \to \mathcal{H}$ is defined by

$$prox_f(a) = argmin_{b\in\mathcal{H}}\{f(b) + \frac{1}{2}\|b-a\|^2\}$$

and is called the proximal operator of f. We now have the following:

Corollary 1. *Let* $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a set-valued, proper, convex and lower semicontinuous function and* $g : \mathcal{H} \to \mathbb{R}$ *be a convex and Frechet differentiable function which is such that* ∇g *is* $\frac{1}{\beta}-$*Lipschitzian, for some* $\beta > 0$ *and* $argmin_{a\in H}\{f(a)+g(a)\} \neq \emptyset$. *Let*

$\gamma \in (0, 2\beta)$ and starting from $a_0, a_1 \in \mathcal{H}$, generate the sequence $\{a_n\}$ from the iterative scheme

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1 - \alpha_n)b_n + \alpha_n prox_f^{\gamma}(c_n - \gamma \nabla g(c_n)) \end{cases} \tag{16}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{r_n\}$ and $\{t_n\}$ are real sequences in $(0, 1)$ satisfying:

(a) $\liminf \alpha_n(1 - \alpha_n) > 0$

(b) $t_n \leq d$ *for some real constant* $d \in (0, 1)$,

(c) $r_n \leq d_1$ *for some real constant* $d_1 \in (0, 1)$.

Then $\{a_n\}$ converges weakly to an element of $argmin\{f(a) + g(a)\}$.

Proof. Set $\mathcal{T} = prox_f^{\gamma} \circ (I - \gamma \nabla g)$, where I is the identity operator. From the Baillon-Haddad Theorem (see [3], Corollary 16), ∇g is $\beta-$ cocoercive. From Theorem 6, by setting $\mathcal{A} := \partial f$ and $\mathcal{B} = \nabla g$ and considering the fact that $Zer(\partial f + \nabla g) = argmin_{a \in H}\{f(a) + g(a)\}$, the result follows.

Theorem 7. *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{A} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be a $\beta-$cocoercive operator, with $\beta > 0$, such that $Zer(\mathcal{A}+\mathcal{B}) \neq \emptyset$. Let $\gamma \in (0, 2\beta)$ and starting from $a_0, a_1 \in \mathcal{H}$, generate the sequence $\{a_n\}$ from the iterative scheme*

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1 - \alpha_n)b_n + \alpha_n J_{\mathcal{A}}^{\gamma}(c_n - \gamma \mathcal{B} c_n) \end{cases} \tag{17}$$

where $\{\alpha_n\}, \{r_n\}$ and $\{t_n\}$ are real sequences in $(0, 1)$ satisfying:

(a) $0 < a \leq \alpha_n \leq b < 1$ *for some real constants* $a, b \in (0, 1)$,

(b) $t_n \leq d$ *for some real constant* $d \in (0, 1)$,

(c) $r_n \leq d_1$ *for some real constant* $d_1 \in (0, 1)$

Then $\{x_n\}$ converges weakly to an element of $Zer(A + B)$.

Proof. Set $\mathcal{T} = J_{\mathcal{A}}^{\gamma} \circ (I - \gamma \mathcal{B})$, where I is the identity operator, so that (7) can be re-written as

$$\begin{cases} b_n = a_n + t_n(a_{n-1} - a_n) \\ c_n = a_n + r_n(a_{n-1} - a_n) \\ a_{n+1} = (1 - \alpha_n)b_n + \alpha_n \mathcal{T} c_n \end{cases} \tag{18}$$

Recall that $J_{\mathcal{A}}^{\gamma}$ is nonexpansive (see for example [20]). Since $\mathcal{B}$ is $\beta-$cocoercive, then $I - \gamma \mathcal{B}$ is $\frac{\gamma}{2\beta}-$ averaged (see for example [3, Proposition 4.33]) and hence nonexpansive. Therefore

$\mathcal{T} = J^{\gamma}_{\mathcal{A}} \circ (I - \gamma\mathcal{B})$ is nonexpansive (the composition of two nonexpansive mappings is easily verifiable to be nonexpansive). The results now follow from Theorem 3, since $F(\mathcal{T}) = Zer(\mathcal{A} + \mathcal{B})$ (see [3, Proposition 25.1(iv)])

Remark 2. *The equivalence of Theorem 6 and Corollary 1 easily follow from Theorem 7 with the conditions imposed on the iteration parameters in Theorem 7.*

5. Numerical Examples

In this section, we demonstrate the efficiency of our Algorithm 4 with the aid of numerical experiments. Furthermore, we compare our iterative method with the methods of Dong at al. [7] (Qial-Li et al. 3) and Mainge. [18] (Mainge 2). In all the numerical implementations, we choose the control sequence for the algorithm in $(0, 1)$.

Example 1. *Let $\mathcal{H} = \mathbb{R}^4$, endowed with the inner product $\langle a, b\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ and the norm $||a|| = \left(\sum_{i=1}^{4} |a_i|^2\right)^{\frac{1}{2}}$ for all $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$. Define $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^4$ as follows:*

$$T\dashv = \left(a_1, 1 + \frac{a_2}{2}, 1 + \frac{a_3}{3}, \frac{a_4}{2}\right), \quad \forall\ a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4.$$

Then, clearly $Fix(T) = \{(0, 2, \frac{3}{2}, 0)\}$ and for all $a, b \in \mathbb{R}^4$, it is easy to see that T is nonexpansive mapping. We test the algorithm using the following initial points:

Case I: $a_0 = (2, 2, 2, 2)', a_1 = (5, 5, 5, 5)'$;

Case II: $a_0 = (1, 3, 3, 1)', a_1 = (0.5, 1, 1.5, 3)'$;

Case III: $a_0 = (2, 0, 0, 2)', a_1 = (8, 3, 3, 8)'$;

Case IV: $a_0 = (3, 3, 3, 4)', a_1 = (9, 9, 9, 8)'$.

We use $||a_{n+1} - a_n|| < 10^{-4}$ as the stopping criterion. The numerical results are shown in Table 1 and Figure 1.

Table 1: Comparison of Algorithm 4, Algorithm 3, and Algorithm 2.

Cases		**Algorithm 4**	**Algorithm 3**	**Algorithm 2**
I	Iter.	66	134	272
	CPU (time)	0.0048	0.0050	0.0062
II	Iter.	63	128	254
	CPU (time)	0.0040	0.0044	0.0060
III	Iter.	64	129	254
	CPU (time)	0.0041	0.0058	0.0062
IV	Iter.	69	141	285
	CPU (time)	0.0054	0.0066	0.0059

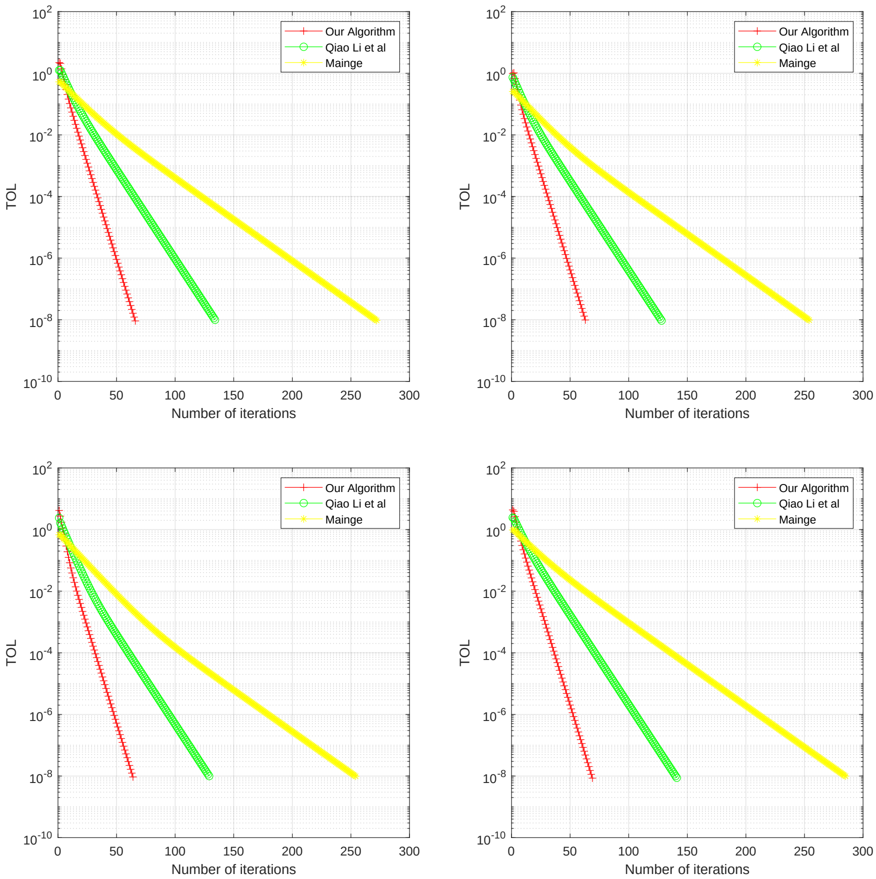

Figure 1: Example 1. Top left: Case I; top right: Case II; bottom left: Case III; bottom right: Case IV.

Example 2. *Let* $\mathcal{H} = L_2[0,1]$ *and* $\mathcal{K} = \{a \in L_2[0,1] : \langle x, a\rangle \leq y\}$*, where* $x = t^2 + 1$ *and* $y = 1$*, with norm* $||a|| = \sqrt{\int_0^1 |a(t)|^2 dt}$ *and inner product* $\langle a, b\rangle = \int_0^t a(t)b(t)dt$*, for all*

Table 2: Comparison of Algorithm 4, Algorithm 3, and Algorithm 2.

Cases		**Algorithm 4**	**Algorithm 3**	**Algorithm 2**
1	Iter.	9	47	45
	CPU (time)	2.4041	5.9585	5.2037
2	Iter.	9	48	77
	CPU (time)	2.1578	4.9429	4.2328
3	Iter.	10	54	51
	CPU (time)	5.9832	12.9400	12.7028
4	Iter.	9	50	48
	CPU (time)	3.3125	8.8200	8.0401

$a, b \in L_2([0,1])$, $t \in [0,1]$. *Define metric projection* $P_{\mathcal{K}}$ *as follows:*

$$P_{\mathcal{K}}(a) = \begin{cases} x, & \text{if } x \in \mathcal{K} \\ \frac{y-\langle x,a\rangle}{||x||_{L_2}} x + a, & \text{otherwise.} \end{cases} \tag{19}$$

Since every projection mapping is nonexpansive, then $P_{\mathcal{K}}$ *is nonexpansive mapping. We define the sequence* $TOL_n := ||a_{n+1} - a_n||^2$ *and apply the stopping criterion* $TOL_n < \varepsilon$ *for the iterative processes because the solution to the problem is unknown.* ε *is the predetermined error. Here, the terminating condition is set to* $\varepsilon = 10^{-5}$. *For the numerical experiments illustrated in Figure 2 and Table 2 below, we take into consideration the resulting cases.*

Case 1: $a_0 = \sin t$ *and* $a_1 = t^2 + t$.

Case 2: $a_0 = \cos t$ *and* $a_1 = t^3 + 2t$.

Case 3: $a_0 = \sin(2t+1)$ *and* $a_1 = 5t^4 + 3t^2 + 1$.

Case 4: $a_0 = \sin t$ *and* $a_1 = e^t$.

Remark 3.

As can be seen from Tables 1 and 2, and Figure 1 and 2, the numerical outcomes of the examples listed above (both finite and infinite dimension) demonstrate how quickly, simply, and efficiently our suggested Algorithm (4) *performs.*

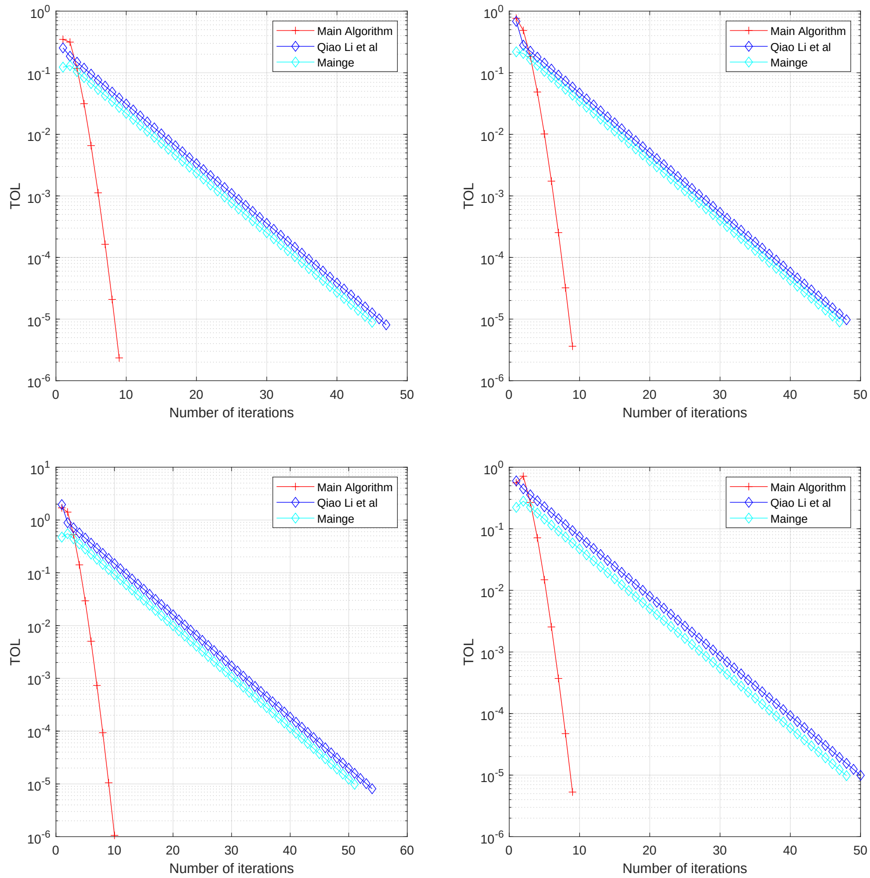

Figure 2: (**Top Left**): Case 1; (**Top Right**): Case 2; (**Bottom Left**):Case 3; (**Bottom Right**): Case 4, the error plotting of comparison of Algorithm 4, Algorithm 3, and Algorithm 2 for Example 2.

6. Conclusion and recommendation

We propose a double inertial variant of the Krasnosel'skii-Mann-type method for solving fixed point problems associated with nonexpansive mappings. In contrast to existing methods, our approach relaxes the conditions on the choice of inertial factors. Specifically, we establish weak convergence results for our method in real Hilbert spaces, requiring simpler assumptions than those previously imposed on other inertial Krasnosel'skii-Mann-type methods. Furthermore, we leverage our findings to address practical applications, including convex minimization and zero finding for sums of monotone operators. Preliminary numerical experiments demonstrate the efficiency and promise of our approach. For future research, we recommend investigating the convergence rate of the double inertial version of the Krasnosel'skii-Mann-type method and applying it to approximate fixed points of more generalized mappings.

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