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On Fuzzy Soft α-open Sets, α-continuity, and α-compactness: Some Novel Results

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Abstract. In this paper, we defined the notions of fuzzy soft α-interior (α-closure) operators via fuzzy soft topologies based on the sense of Šostak and studied some topological properties of them. Also, the notion of r-fuzzy soft α-connected sets was introduced and investigated. Thereafter, we defined and characterized the notions of fuzzy soft weakly (almost) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings. Moreover, we showed that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity, but the converse may not be true. In addition, we investigated some properties of fuzzy soft α-continuity. Finally, several types of fuzzy soft compactness via r-fuzzy soft α-open sets were given and the relationships between them were studied with the help of some examples.

2020 Mathematics Subject Classifications: 54A05, 54A40, 54C05, 54C10, 54D30

Key Words and Phrases: Fuzzy soft topology, r-fuzzy soft α-open (α-closed) set, fuzzy soft α-interior (α-closure) operator, connectedness, fuzzy soft weakly (almost) α-continuity, compactness

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1. Introduction and preliminaries

The theory of soft sets was first introduced by Molodtsov [24], which is a completely new approach for vagueness and modeling uncertainty. He demonstrated many applications of this theory in solving several practical problems in mathematics, engineering, economics, social science, etc. In [28], the notion of soft sets was used to introduced soft topologies. Moreover, the study in [28] was particularly important in the development of the field of soft topology, see [10, 18, 33, 38]. Generalizations of soft open subsets play an effective role in soft topologies through their use to improve on some known results or to open the door to reintroduce and establish many of the soft topological notions such as soft separation axioms [7, 20], soft continuity [25], soft connectedness [34, 36], etc. Akdag

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and Ozkan [3] defined the notion of soft α-open sets on soft topological spaces and some properties are specified. The notion of soft β-open sets was defined and studied by the authors of [2, 17]. Also, the concepts of soft semi-open, somewhere dense and Q-sets were studied by the authors of [4, 6]. Moreover, Al-shami et al. [8] initiated the notion of weakly soft β-open sets and examined weakly soft β-continuity. Kaur et al. [21] introduced a new approach to studying soft continuous mappings using an induced mapping based on soft sets. Al Ghour and Al-Mufarrij [5] defined two new notions of mappings over soft topological spaces: soft somewhat-r-continuity and soft somewhat-r-openness.

The concept of fuzzy soft sets was defined by Maji et al. [22], which combines soft sets [24] and fuzzy sets [37]. The concept of fuzzy soft topology was introduced and some characterized such as fuzzy soft interior (closure) set, fuzzy soft continuity, and fuzzy soft subspace were studied in [16, 19] based on fuzzy topologies in the sense of Šostak [35]. A new approach to studying separation and regularity axioms via fuzzy soft sets was introduced by the author of [29, 31] based on the paper by Aygünoğlu et al. [19]. The concept of r-fuzzy soft regularly open sets was introduced by Çetkin and Aygün [15]. Also, the concepts of r-fuzzy soft pre-open (resp. β-open) sets were defined by Taha [30]. In 2024, Alshammari and Taha [12] introduced and studied the notions of fuzzy soft almost (weakly) β-continuous mappings, which are weaker forms of a fuzzy soft β-continuity in fuzzy soft topological spaces. In addition, many authors have contributed to fuzzy soft set theory in the different fields such as topology, see e.g. [9, 26, 27].

In our study, the layout is designed as follows.

• In Section 2, we introduce the concepts of fuzzy soft α-closure (α-interior) operators in fuzzy soft topological space (W, τ_N) based on the paper by Aygünoğlu et al. [19] and examine some of its properties. Also, the concept of r-fuzzy soft α-connected sets is introduced and studied.

• In Section 3, we are going to investigate some properties of fuzzy soft α-continuous mappings between two fuzzy soft topological spaces (W, τ_N) and (V, η_F). Moreover, we define and study the concepts of fuzzy soft weakly (almost) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings. Also, the relationships between these classes of mappings are investigated with the help of some examples.

• In Section 4, several types of fuzzy soft compactness via r-fuzzy soft α-open sets are defined, and the relationships between them are specified.

• Finally, we close this manuscript with some conclusions and proposed some future works in Section 5.

In this work, nonempty sets will be denoted by W, V, etc. N is the set of all parameters for W and $C \subseteq N$. The family of all fuzzy sets on W is denoted by I^W (where $I_\circ = (0, 1]$, $I = [0, 1]$), and for $s \in I$, $\underline{s}(w) = s$, for all $w \in W$.

The following concepts and results will be used in the next sections.

Definition 1. *[1, 14, 19]* A fuzzy soft set h_C on W is a mapping from N to I^W, such that $h_C(n)$ is a fuzzy set on W, for each $n \in C$ and $h_C(n) = \underline{0}$, if $n \notin C$. The family of all fuzzy soft sets on W is denoted by $\widetilde{(W, N)}$.

Definition 2. *[32]* The difference between two fuzzy soft sets h_C and g_B is a fuzzy soft set, defined as follows, for each $n \in N$:

$$(h_C \overline{\sqcap}\, g_B)(n) = \begin{cases} \underline{0}, & \text{if } h_C(n) \le g_B(n), \\ h_C(n) \wedge (g_B(n))^c, & \text{otherwise.} \end{cases}$$

Definition 3. *[23]* A fuzzy soft point n_{w_s} on W is a fuzzy soft set, defined as follows:

$$n_{w_s}(k) = \begin{cases} w_s, & \text{if} \quad k = n, \\ \underline{0}, & \text{if} \quad k \in N - \{n\}, \end{cases}$$

where w_s is a fuzzy point on W. A fuzzy soft point n_{w_s} is called belong to a fuzzy soft set f_A, denoted by $n_{w_s} \tilde{\in} f_A$, if $s \le f_A(n)(w)$. The family of all fuzzy soft points on W is denoted by $\widetilde{P_s(W)}$.

Definition 4. *[13]* A fuzzy soft point $n_{w_s} \in \widetilde{P_s(W)}$ is called a soft quasi-coincident with $h_C \in \widetilde{(W, N)}$ and denoted by $n_{w_s} \widetilde{q} h_C$, if $s + h_C(n)(w) > 1$. A fuzzy soft set $h_C \in \widetilde{(W, N)}$ is called a soft quasi-coincident with $g_B \in \widetilde{(W, N)}$ and denoted by $h_C \widetilde{q} g_B$, if there is $n \in N$ and $w \in W$, such that $h_C(n)(w) + g_B(n)(w) > 1$, if h_C is not soft quasi-coincident with g_B, $h_C \widetilde{\not q} g_B$.

Definition 5. *[19]* A mapping $\tau : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ is called a fuzzy soft topology on W if it satisfies the following, for each $n \in N$:

(1) $\tau_n(\Phi) = \tau_n(\widetilde{N}) = 1$,

(2) $\tau_n(h_C \sqcap g_B) \ge \tau_n(h_C) \wedge \tau_n(g_B)$, for each $h_C, g_B \in \widetilde{(W, N)}$,

(3) $\tau_n(\sqcup_{\delta \in \Delta}(h_C)_\delta) \ge \wedge_{\delta \in \Delta} \tau_n((h_C)_\delta)$, for each $(h_C)_\delta \in \widetilde{(W, N)}, \delta \in \Delta$.

Thus, (W, τ_N) is called a fuzzy soft topological space (briefly, FSTS) in the sense of Šostak [35].

Definition 6. *[19]* Let (W, τ_N) and (V, η_F) be an FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft continuous if $\tau_n(\varphi_\psi^{-1}(h_C)) \geq \eta_k(h_C)$ for each $h_C \in \widetilde{(V,F)}$, $n \in N$, and $(k = \psi(n)) \in F$.

Definition 7. *[15, 16]* In an FSTS (W, τ_N), for each $h_C \in \widetilde{(W,N)}$, $n \in N$, and $r \in I_0$, we define the fuzzy soft operators C_τ and $I_\tau : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ as follows:

$$C_\tau(n, h_C, r) = \sqcap \{g_B \in \widetilde{(W,N)} : h_C \sqsubseteq g_B,\ \tau_n(g_B^c) \geq r\},$$

$$I_\tau(n, h_C, r) = \sqcup \{g_B \in \widetilde{(W,N)} : g_B \sqsubseteq h_C,\ \tau_n(g_B) \geq r\}.$$

Definition 8. Let (W, τ_N) be an FSTS and $r \in I_0$. A fuzzy soft set $h_C \in \widetilde{(W,N)}$ is called r-fuzzy soft regularly open [15] (resp., β-open [30], pre-open [30], α-open [11], and semi-open [11]) if $h_C = I_\tau(n, C_\tau(n, h_C, r), r)$ (resp., $h_C \sqsubseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$, $h_C \sqsubseteq I_\tau(n, C_\tau(n, h_C, r), r)$, $h_C \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, h_C, r), r), r)$, and $h_C \sqsubseteq C_\tau(n, I_\tau(n, h_C, r), r)$) for each $n \in N$.

Definition 9. *[15]* Let (W, τ_N) be an FSTS and $r \in I_0$. A fuzzy soft set $h_C \in \widetilde{(W,N)}$ is called r-fuzzy soft regularly closed if $h_C = C_\tau(n, I_\tau(n, h_C, r), r)$ for each $n \in N$.

Definition 10. *[11]* Let (W, τ_N) and (V, η_F) be an FSTSs and $r \in I_0$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft almost (resp., weakly) continuous if for any $n_{w_s} \in \widetilde{P_s(W)}$ and any $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ containing $\varphi_\psi(n_{w_s})$, there is $h_C \in \widetilde{(W,N)}$ with $\tau_n(h_C) \geq r$ containing n_{w_s}, such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$ (resp., $\varphi_\psi(h_C) \sqsubseteq C_\eta(k, f_A, r)$).

Remark 1. *[11]* From Definitions 6 and 10, we have: fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity, but the converse may not be true.

Lemma 1. Let (W, τ_N) and (V, η_F) be an FSTSs and $r \in I_0$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is fuzzy soft almost continuous if $\tau_n(\varphi_\psi^{-1}(h_C)) \geq r$ for each $h_C \in \widetilde{(V,F)}$ is r-fuzzy soft regularly open, $n \in N$, and $(k = \psi(n)) \in F$.

Proof. Easily proved from Definition 10.

Definition 11. Let (W, τ_N) and (V, η_F) be an FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft open if $\eta_k(\varphi_\psi(h_C)) \geq \tau_n(h_C)$ for each $h_C \in \widetilde{(W,N)}$, $n \in N$, and $(k = \psi(n)) \in F$.

The basic concepts and results that we need in the next sections are found in [16, 19].

2. On r-fuzzy soft α-open sets

Here, we introduce and discuss the notions of fuzzy soft α-closure (α-interior) operators in an FSTSs based on the paper by Aygünoğlu et al. [19]. Also, the notion of r-fuzzy soft α-connected sets has been defined and studied with help of fuzzy soft α-closure operators.

Definition 12. Let (W, τ_N) be an FSTS and $r \in I_0$. A fuzzy soft set h_C is called r-fuzzy soft α-closed (resp., semi-closed, β-closed, and pre-closed) if $C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r) \sqsubseteq h_C$ (resp., $I_\tau(n, C_\tau(n, h_C, r), r) \sqsubseteq h_C$, $I_\tau(n, C_\tau(n, I_\tau(n, h_C, r), r), r) \sqsubseteq h_C$, and $C_\tau(n, I_\tau(n, h_C, r), r) \sqsubseteq h_C$) for each $n \in N$.

Remark 2. The complement of r-fuzzy soft α-closed (resp., semi-closed, β-closed, and pre-closed) set is r-fuzzy soft α-open [11] (resp., semi-open [11], β-open [30], and pre-open [30]) set.

Lemma 2. Let (W, τ_N) be an FSTS and $r \in I_0$, then any intersection (resp., union) of r-fuzzy soft α-closed (resp., α-open) sets is an r-fuzzy soft α-closed (resp., α-open) set.

Proof. Easily proved from Definitions 8 and 12.

Proposition 1. Let (W, τ_N) be an FSTS, $h_C \in \widetilde{(W, N)}$, $n \in N$, and $r \in I_0$, then the following statements are equivalent.

(1) h_C is r-fuzzy soft α-closed.

(2) h_C is r-fuzzy soft semi-closed and r-fuzzy soft pre-closed.

Proof. (1) $\Rightarrow$ (2) Let h_C be an r-fuzzy soft α-closed,

$$h_C \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r) \sqsupseteq I_\tau(n, C_\tau(n, h_C, r), r).$$

This shows that h_C is r-fuzzy soft semi-closed.

Since $h_C \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$ and $C_\tau(n, h_C, r) \sqsupseteq h_C$, then

$$h_C \sqsupseteq C_\tau(n, I_\tau(n, h_C, r), r).$$

Therefore, h_C is r-fuzzy soft pre-closed

(2) $\Rightarrow$ (1) Let h_C be an r-fuzzy soft semi-closed and r-fuzzy soft pre-closed, then $h_C \sqsupseteq C_\tau(n, I_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r), r) = C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r)$. This shows that h_C is r-fuzzy soft α-closed.

Proposition 2. Let (W, τ_N) be an FSTS, $g_B, h_C \in \widetilde{(W, N)}$, $n \in N$, and $r \in I_0$. If g_B is an r-fuzzy soft semi-closed set, such that $g_B \sqsupseteq h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, then h_C is r-fuzzy soft α-closed.

Proof. Let g_B be an r-fuzzy soft semi-closed and $g_B \sqsupseteq h_C$, then

$$g_B \sqsupseteq I_\tau(n, C_\tau(n, g_B, r), r) \sqsupseteq I_\tau(n, C_\tau(n, h_C, r), r).$$

Let $h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, then

$$h_C \sqsupseteq C_\tau(n, I_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r), r) = C_\tau(n, I_\tau(n, C_\tau(n, h_C, r), r), r).$$

Therefore, h_C is r-fuzzy soft α-closed.

Lemma 3. Let (W, τ_N) be an FSTS, $g_B, h_C \in \widetilde{(W,N)}$, $n \in N$, and $r \in I_0$. If g_B is an r-fuzzy soft α-closed set, such that $g_B \sqsupseteq h_C \sqsupseteq C_\tau(n, I_\tau(n, g_B, r), r)$, then h_C is r-fuzzy soft α-closed.

Proof. It is easily proved from every r-fuzzy soft α-closed set that is an r-fuzzy soft semi-closed set.

Remark 3. From the previous definition, we can summarize the relationships among different types of fuzzy soft sets as in the next diagram.

$$\begin{array}{ccc}
 & fuzzy\ \ soft\ \ \alpha-closed\ \ set & \\
 & \Downarrow \qquad\qquad\qquad \Downarrow & \\
fuzzy\ \ soft\ \ semi{-}closed\ \ set & & fuzzy\ \ soft\ \ pre{-}closed\ \ set \\
 & \Downarrow \qquad\qquad\qquad \Downarrow & \\
 & fuzzy\ \ soft\ \ \beta-closed\ \ set &
\end{array}$$

Remark 4. The converses of the above relationships may not be true, as shown by Examples 1 and 2.

Example 1. Let $W = \{w_1, w_2\}$, $N = \{n_1, n_2\}$, and define $f_N, g_N, h_N \in \widetilde{(W,N)}$ as follows: $f_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}\})\}$, $g_N = \{(n_1, \{\frac{w_1}{0.6}, \frac{w_2}{0.2}\}), (n_2, \{\frac{w_1}{0.6}, \frac{w_2}{0.2}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.5}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.5}\})\}$. Define fuzzy soft topology $\tau_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows:

$$\tau_{n_1}(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ \frac{2}{3}, & \text{if} \quad t_N = g_N, \\ \frac{2}{3}, & \text{if} \quad t_N = f_N \sqcap g_N, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N \sqcup g_N, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_{n_2}(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{4}, & \text{if} \quad t_N = f_N, \\ \frac{1}{2}, & \text{if} \quad t_N = g_N, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N \sqcap g_N, \\ \frac{1}{4}, & \text{if} \quad t_N = f_N \sqcup g_N, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, h_N is $\frac{1}{4}$-fuzzy soft semi-closed and $\frac{1}{4}$-fuzzy soft β-closed, but it is neither $\frac{1}{4}$-fuzzy soft α-closed nor $\frac{1}{4}$-fuzzy soft pre-closed.

Example 2. Let $W = \{w_1, w_2\}$, $N = \{n_1, n_2\}$, and define $f_N, h_N \in \widetilde{(W,N)}$ as follows: $f_N = \{(n_1, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}\}), (n_2, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.7}, \frac{w_2}{0.6}\}), (n_2, \{\frac{w_1}{0.7}, \frac{w_2}{0.6}\})\}$. Define fuzzy soft topology $\tau_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows:

$$\tau_{n_1}(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{3}, & \text{if} \quad t_N = f_N, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_{n_2}(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, h_N is $\frac{1}{3}$-fuzzy soft pre-closed and $\frac{1}{3}$-fuzzy soft β-closed, but it is neither $\frac{1}{3}$-fuzzy soft semi-closed nor $\frac{1}{3}$-fuzzy soft α-closed.

Definition 13. In an FSTS (W, τ_N), for each $h_C \in \widetilde{(W,N)}$, $n \in N$, and $r \in I_0$, we define a fuzzy soft α-closure operator $\alpha C_\tau : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ as follows: $\alpha C_\tau(n, h_C, r) = \sqcap \{f_A \in \widetilde{(W,N)} : h_C \sqsubseteq f_A,\ f_A \text{ is } r\text{-fuzzy soft } \alpha\text{-closed}\}$.

Theorem 1. In an FSTS (W, τ_N), for each $g_B, h_C \in \widetilde{(W,N)}$, $n \in N$, and $r \in I_0$, the operator $\alpha C_\tau : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ satisfies the following properties.

(1) $\alpha C_\tau(n, \Phi, r) = \Phi$.

(2) $h_C \sqsubseteq \alpha C_\tau(n, h_C, r) \sqsubseteq C_\tau(n, h_C, r)$.

(3) $\alpha C_\tau(n, h_C, r) \sqsubseteq \alpha C_\tau(n, g_B, r)$ if, $h_C \sqsubseteq g_B$.

(4) $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r) = \alpha C_\tau(n, h_C, r)$.

(5) $\alpha C_\tau(n, h_C \sqcup g_B, r) \sqsupseteq \alpha C_\tau(n, h_C, r) \sqcup \alpha C_\tau(n, g_B, r)$.

(6) $\alpha C_\tau(n, h_C, r) = h_C$ iff h_C is r-fuzzy soft α-closed.

(7) $\alpha C_\tau(n, C_\tau(n, h_C, r), r) = C_\tau(n, h_C, r)$.

Proof. (1), (2), (3), and (6) are easily proved from Definition 13.

(4) From (2) and (3), $\alpha C_\tau(n, h_C, r) \sqsubseteq \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)$. Now, we show that $\alpha C_\tau(n, h_C, r) \sqsupseteq \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)$. Suppose that $\alpha C_\tau(n, h_C, r)$ does not contain $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)$, then there is $w \in W$ and $s \in (0,1)$, such that

$$\alpha C_\tau(n, h_C, r)(n)(w) < s < \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)(n)(w). \qquad (A)$$

Since $\alpha C_\tau(n, h_C, r)(n)(w) < s$, by the definition of αC_τ, there is g_B as an r-fuzzy soft α-closed with $h_C \sqsubseteq g_B$, such that $\alpha C_\tau(n, h_C, r)(n)(w) \leq g_B(n)(w) < s$. Since

$h_C \sqsubseteq g_B$, we have $\alpha C_\tau(n, h_C, r) \sqsubseteq g_B$. Again, by the definition of αC_τ, we have $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r) \sqsubseteq g_B$. Hence, $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)(n)(w) \leq g_B(n)(w) < s$, which is a contradiction for (A). Thus, $\alpha C_\tau(n, h_C, r) \sqsupseteq \alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r)$, then $\alpha C_\tau(n, \alpha C_\tau(n, h_C, r), r) = \alpha C_\tau(n, h_C, r)$.

(5) Since h_C and $g_B \sqsubseteq h_C \sqcup g_B$, hence by (3), $\alpha C_\tau(n, h_C, r) \sqsubseteq \alpha C_\tau(n, h_C \sqcup g_B, r)$ and $\alpha C_\tau(n, g_B, r) \sqsubseteq \alpha C_\tau(n, h_C \sqcup g_B, r)$. Thus, $\alpha C_\tau(n, h_C \sqcup g_B, r) \sqsupseteq \alpha C_\tau(n, h_C, r) \sqcup \alpha C_\tau(n, g_B, r)$.

(7) From (6) and $C_\tau(n, h_C, r)$ is r-fuzzy soft α-closed set, then $\alpha C_\tau(n, C_\tau(n, h_C, r), r) = C_\tau(n, h_C, r)$.

Theorem 2. In an FSTS (W, τ_N), for each $h_C \in \widetilde{(W, N)}$, $n \in N$, and $r \in I_0$, we define a fuzzy soft α-interior operator $\alpha I_\tau : N \times \widetilde{(W, N)} \times I_\circ \to \widetilde{(W, N)}$ as follows: $\alpha I_\tau(n, h_C, r) = \sqcup \{f_A \in \widetilde{(W, N)} : f_A \sqsubseteq h_C,\ f_A$ is r-fuzzy soft α-open$\}$. For each g_B and $h_C \in \widetilde{(W, N)}$, the operator αI_τ satisfies the following properties.

(1) $\alpha I_\tau(n, \widetilde{N}, r) = \widetilde{N}$.

(2) $I_\tau(n, h_C, r) \sqsubseteq \alpha I_\tau(n, h_C, r) \sqsubseteq h_C$.

(3) $\alpha I_\tau(n, h_C, r) \sqsubseteq \alpha I_\tau(n, g_B, r)$ if, $h_C \sqsubseteq g_B$.

(4) $\alpha I_\tau(n, \alpha I_\tau(n, h_C, r), r) = \alpha I_\tau(n, h_C, r)$.

(5) $\alpha I_\tau(n, h_C, r) \sqcap \alpha I_\tau(n, g_B, r) \sqsupseteq \alpha I_\tau(n, h_C \sqcap g_B, r)$.

(6) $\alpha I_\tau(n, h_C, r) = h_C$ iff h_C is r-fuzzy soft α-open.

(7) $\alpha I_\tau(n, h_C^c, r) = (\alpha C_\tau(n, h_C, r))^c$.

Proof. (1), (2), (3), and (6) are easily proved from the definition of αI_τ.

(4) and (5) are easily proved by a similar way in Theorem 1.

(7) For each $h_C \in \widetilde{(W, N)}$, $n \in N$, and $r \in I_0$, we have $\alpha I_\tau(n, h_C^c, r) = \sqcup\{f_A \in \widetilde{(W, N)} : f_A \sqsubseteq h_C^c,\ f_A$ is r-fuzzy soft α-open$\} = [\sqcap\{f_A^c \in \widetilde{(W, N)} : h_C \sqsubseteq f_A^c,\ f_A^c$ is r-fuzzy soft α-closed$\}]^c$ $= (\alpha C_\tau(n, h_C, r))^c$.

Definition 14. Let (W, τ_N) be an FSTS, $r \in I_0$, and $g_B, h_C \in \widetilde{(W, N)}$, then we have

(1) Two fuzzy soft sets g_B and h_C are called r-fuzzy soft α-separated iff $g_B \,\widetilde{\not q}\, \alpha C_\tau(n, h_C, r)$ and $h_C \,\widetilde{\not q}\, \alpha C_\tau(n, g_B, r)$ for each $n \in N$.

(2) Any fuzzy soft set which cannot be expressed as the union of two r-fuzzy soft α-separated sets is called an r-fuzzy soft α-connected.

Theorem 3. In an FSTS (W, τ_N), we have:

(1) If f_A and $g_B \in \widetilde{(W, N)}$ are r-fuzzy soft α-separated and h_C, $t_D \in \widetilde{(W, N)}$ such that $h_C \sqsubseteq f_A$ and $t_D \sqsubseteq g_B$, then h_C and t_D are r-fuzzy soft α-separated.

(2) If $f_A \,\widetilde{\not q}\, g_B$ and either both are r-fuzzy soft α-open or both are r-fuzzy soft α-closed, then f_A and g_B are r-fuzzy soft α-separated.

(3) If f_A and g_B are either both r-fuzzy soft α-open or both r-fuzzy soft α-closed, then $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r-fuzzy soft α-separated.

Proof. (1) and (2) are obvious.

(3) Let f_A and g_B be an r-fuzzy soft α-open. Since $f_A \sqcap g_B^c \sqsubseteq g_B^c$, $\alpha C_\tau(n, f_A \sqcap g_B^c, r) \sqsubseteq g_B^c$ and hence $\alpha C_\tau(n, f_A \sqcap g_B^c, r) \,\widetilde{\not q}\, g_B$. Then, $\alpha C_\tau(n, f_A \sqcap g_B^c\,, r) \,\widetilde{\not q}\, (g_B \sqcap f_A^c)$.
Again, since $g_B \sqcap f_A^c \sqsubseteq f_A^c$, $\alpha C_\tau(n, g_B \sqcap f_A^c, r) \sqsubseteq f_A^c$ and hence $\alpha C_\tau(n, g_B \sqcap f_A^c, r) \,\widetilde{\not q}\, f_A$. Then, $\alpha C_\tau(n, g_B \sqcap f_A^c, r) \,\widetilde{\not q}\, (f_A \sqcap g_B^c)$. Thus, $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r-fuzzy soft α-separated. The other case follows similar lines.

Theorem 4. In an FSTS (W, τ_N), then f_A, $g_B \in \widetilde{(W, N)}$ are r-fuzzy soft α-separated iff there exist two r-fuzzy soft α-open sets h_C and t_D such that $f_A \sqsubseteq h_C$, $g_B \sqsubseteq t_D$, $f_A \,\widetilde{\not q}\, t_D$, and $g_B \,\widetilde{\not q}\, h_C$.

Proof. ($\Rightarrow$) Let f_A and $g_B \in \widetilde{(W, N)}$ be an r-fuzzy soft α-separated, $f_A \sqsubseteq (\alpha C_\tau(n, g_B, r))^c = h_C$ and $g_B \sqsubseteq (\alpha C_\tau(n, f_A, r))^c = t_D$, where t_D and h_C are r-fuzzy soft α-open, then $t_D \,\widetilde{\not q}\alpha C_\tau(n, f_A, r)$ and $h_C \,\widetilde{\not q}\alpha C_\tau(n, g_B, r)$. Thus, $g_B \,\widetilde{\not q}\, h_C$ and $f_A \,\widetilde{\not q}\, t_D$. Hence, we obtain the required result.

($\Leftarrow$) Let h_C and t_D be an r-fuzzy soft α-open such that $g_B \sqsubseteq t_D$, $f_A \sqsubseteq h_C$, $g_B \,\widetilde{\not q}\, h_C$ and $f_A \,\widetilde{\not q}\, t_D$. Then, $g_B \sqsubseteq h_C^c$ and $f_A \sqsubseteq t_D^c$. Hence, $\alpha C_\tau(n, g_B, r) \sqsubseteq h_C^c$ and $\alpha C_\tau(n, f_A, r) \sqsubseteq t_D^c$. Then, $\alpha C_\tau(n, g_B, r) \,\widetilde{\not q}\, f_A$ and $\alpha C_\tau(n, f_A, r) \,\widetilde{\not q}\, g_B$. Thus, g_B and f_A are r- fuzzy soft α-separated. Hence, we obtain the required result.

Theorem 5. In an FSTS (W, τ_N), if $g_B \in \widetilde{(W, N)}$ is r-fuzzy soft α-connected such that $g_B \sqsubseteq f_A \sqsubseteq \alpha C_\tau(n, g_B, r)$, then f_A is r-fuzzy soft α-connected.

Proof. Suppose that f_A is not r-fuzzy soft α-connected, then there is r-fuzzy soft α-separated sets h^*_C and $t^*_D \in \widetilde{(W,N)}$ such that $f_A = h^*_C \sqcup t^*_D$. Let $h_C = g_B \sqcap h^*_C$ and $t_D = g_B \sqcap t^*_D$, then $g_B = t_D \sqcup h_C$. Since $h_C \sqsubseteq h^*_C$ and $t_D \sqsubseteq t^*_D$, by Theorem 3(1), h_C and t_D are r-fuzzy soft α-separated, which is a contradiction. Thus, f_A is r-fuzzy soft α-connected, as required.

3. On fuzzy soft α-continuity

Here, we investigate some properties of fuzzy soft α-continuous mappings. Additionally, we introduce and study the notions of fuzzy soft almost (weakly) α-continuous mappings, which are weaker forms of fuzzy soft α-continuous mappings. Also, we show that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity.

Definition 15. *[11]* Let (W,τ_N) and (V,η_F) be an FSTSs and $r \in I_o$. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft α-continuous if $\varphi^{-1}_\psi(h_C)$ is r-fuzzy soft α-open set for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$, $n \in N$, and $(k = \psi(n)) \in F$.

Theorem 6. Let (W,τ_N) and (V,η_F) be an FSTSs, and $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A \in \widetilde{(V,F)}$, $n \in N$, $(k = \psi(n)) \in F$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft α-continuous.

(2) For each f_A with $\eta_k(f^c_A) \geq r$, $\varphi^{-1}_\psi(f_A)$ is r-fuzzy soft α-closed.

(3) $\alpha C_\tau(n, \varphi^{-1}_\psi(f_A), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.

(4) $\varphi^{-1}_\psi(I_\eta(k, f_A, r)) \sqsubseteq \alpha I_\tau(n, \varphi^{-1}_\psi(f_A), r)$.

(5) $C_\tau(n, I_\tau(n, C_\tau(n, \varphi^{-1}_\psi(f_A), r), r), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.

Proof.
(1) $\Leftrightarrow$ (2) Follows from Remark 2 and $\varphi^{-1}_\psi(f^c_A) = (\varphi^{-1}_\psi(f_A))^c$.

(2) $\Rightarrow$ (3) Let $f_A \in \widetilde{(V,F)}$, hence by (2), $\varphi^{-1}_\psi(C_\eta(k, f_A, r))$ is r-fuzzy soft α-closed. Then, we obtain $\alpha C_\tau(n, \varphi^{-1}_\psi(f_A), r) \sqsubseteq \varphi^{-1}_\psi(C_\eta(k, f_A, r))$.

(3) $\Leftrightarrow$ (4) Follows from Theorem 2(7).

(3) $\Rightarrow$ (5) Let $f_A \in \widetilde{(V,F)}$, hence by (3), we obtain $C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r) \sqsubseteq \alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$.

(5) $\Rightarrow$ (1) Let $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$, hence by (3), we obtain $(\varphi_\psi^{-1}(f_A))^c = \varphi_\psi^{-1}(f_A^c) \sqsupseteq C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(f_A^c), r), r), r) = (I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r))^c$. Then, $\varphi_\psi^{-1}(f_A) \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r)$, so $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open. Hence, φ_ψ is fuzzy soft α-continuous.

Lemma 4. Every fuzzy soft continuous mapping [19] is fuzzy soft α-continuous.

Proof. Follows from Definitions 6 and 15.

Remark 5. The converse of Lemma 4 is not true, as shown by Example 3.

Example 3. Let $W = \{w_1, w_2, w_3\}$, $N = \{n_1, n_2\}$, and define $f_N, g_N, h_N \in \widetilde{(W,N)}$ as: $f_N = \{(n_1, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\}), (n_2, \{\frac{w_1}{0.4}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\})\}$, $g_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.4}\})\}$. Define fuzzy soft topologies $\tau_N, \eta_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n \in N$,

$$\tau_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ \frac{2}{3}, & \text{if} \quad t_N = g_N, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = f_N, \\ \frac{1}{3}, & \text{if} \quad t_N = h_N, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft mapping $\varphi_\psi : (W, \tau_N) \longrightarrow (W, \eta_N)$ is fuzzy soft α-continuous, but it is not fuzzy soft continuous.

Definition 16. Let (W, τ_N) and (V, η_F) be an FSTSs. A fuzzy soft mapping $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is called fuzzy soft almost (resp., weakly) α-continuous if for each $n_{w_s} \in \widetilde{P_s(W)}$ and each $g_B \in \widetilde{(V,F)}$ with $\eta_k(g_B) \geq r$ containing $\varphi_\psi(n_{w_s})$, there is $h_C \in \widetilde{(W,N)}$ that is an r-fuzzy soft α-open set containing n_{w_s}, such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r)$ (resp., $\varphi_\psi(h_C) \sqsubseteq C_\eta(k, g_B, r)$), $n \in N$, $(k = \psi(n)) \in F$, and $r \in I_\circ$.

Lemma 5. (1) Every fuzzy soft α-continuous mapping is fuzzy soft almost α-continuous.

(2) Every fuzzy soft almost α-continuous mapping is fuzzy soft weakly α-continuous.

Proof. Follows from Definitions 15 and 16.

Remark 6. The converse of Lemma 5 is not true, as shown by Examples 4 and 5.

Example 4. Let $W = \{w_1, w_2, w_3\}$, $N = \{n_1, n_2\}$, and define $g_N, h_N \in \widetilde{(W, N)}$ as follows: $g_N = \{(n_1, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.4}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.4}\})\}$. Define fuzzy soft topologies $\tau_N, \eta_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n \in N$,

$$\tau_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = g_N, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{2}, & \text{if} \quad t_N = g_N, \\ \frac{1}{3}, & \text{if} \quad t_N = h_N, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft mapping $\varphi_\psi : (W, \tau_N) \longrightarrow (W, \eta_N)$ is fuzzy soft almost α-continuous, but it is not fuzzy soft α-continuous.

Example 5. Let $W = \{w_1, w_2, w_3\}$, $N = \{n_1, n_2\}$, and define $g_N, h_N \in \widetilde{(W, N)}$ as follows: $g_N = \{(n_1, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\}), (n_2, \{\frac{w_1}{0.5}, \frac{w_2}{0.5}, \frac{w_3}{0.5}\})\}$, $h_N = \{(n_1, \{\frac{w_1}{0.3}, \frac{w_2}{0}, \frac{w_3}{0.5}\}), (n_2, \{\frac{w_1}{0.3}, \frac{w_2}{0}, \frac{w_3}{0.5}\})\}$. Define fuzzy soft topologies $\tau_N, \eta_N : N \longrightarrow [0,1]^{\widetilde{(W,N)}}$ as follows: $\forall n \in N$,

$$\tau_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{2}{3}, & \text{if} \quad t_N = g_N, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta_n(t_N) = \begin{cases} 1, & \text{if} \quad t_N \in \{\Phi, \widetilde{N}\}, \\ \frac{1}{3}, & \text{if} \quad t_N = h_N, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft mapping $\varphi_\psi : (W, \tau_N) \longrightarrow (W, \eta_N)$ is fuzzy soft weakly α-continuous, but it is not fuzzy soft almost α-continuous.

Theorem 7. Let (W, τ_N) and (V, η_F) be an FSTSs, and $\varphi_\psi : \widetilde{(W, N)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A \in \widetilde{(V, F)}$, $n \in N$, $(k = \psi(n)) \in F$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft almost α-continuous.

(2) $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open, for each f_A is r-fuzzy soft regularly open.

(3) $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed, for each f_A is r-fuzzy soft regularly closed.

(4) $\alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$, for each f_A is r-fuzzy soft β-open.

(5) $\alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$, for each f_A is r-fuzzy soft semi-open.

(6) $\alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r) \sqsupseteq \varphi_\psi^{-1}(f_A)$, for each f_A with $\eta_k(f_A) \geq r$.

Proof. (1) $\Rightarrow$ (2) Let $n_{w_s} \in \widetilde{P_s(W)}$ and $f_A \in \widetilde{(V,F)}$ be an r-fuzzy soft regularly open set containing $\varphi_\psi(n_{w_s})$, hence by (1), there is $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-open set containing n_{w_s} such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$.

Thus, $h_C \sqsubseteq \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)) = \varphi_\psi^{-1}(f_A)$ and $n_{w_s} \tilde{\in} h_C \sqsubseteq \varphi_\psi^{-1}(f_A)$. Then, $n_{w_s} \tilde{\in} I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r)$ and $\varphi_\psi^{-1}(f_A) \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(f_A), r), r), r)$. Therefore, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-open set.

(2) $\Rightarrow$ (3) Let f_A be an r-fuzzy soft regularly closed set, hence by (2), $\varphi_\psi^{-1}(f_A^c) = (\varphi_\psi^{-1}(f_A))^c$ is r-fuzzy soft α-open set. Then, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed set.

(3) $\Rightarrow$ (4) Let f_A be an r-fuzzy soft β-open set. Since $C_\eta(k, f_A, r)$ is r-fuzzy soft regularly closed set, hence by (3), $\varphi_\psi^{-1}(C_\eta(k, f_A, r))$ is r-fuzzy soft α-closed set. Since $\varphi_\psi^{-1}(f_A) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$, then we have $\alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$.

(4) $\Rightarrow$ (5) This is obvious from each r-fuzzy soft semi-open set that is an r-fuzzy soft β-open.

(5) $\Rightarrow$ (3) Let f_A be an r-fuzzy soft regularly closed set, hence f_A is r-fuzzy soft semi-open. Then by (5), $\alpha C_\tau(n, \varphi_\psi^{-1}(f_A), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r)) = \varphi_\psi^{-1}(f_A)$. Therefore, $\varphi_\psi^{-1}(f_A)$ is r-fuzzy soft α-closed set.

(3) $\Rightarrow$ (6) Let $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ and $n_{w_s} \tilde{\in} \varphi_\psi^{-1}(f_A)$, then we have

$$n_{w_s} \tilde{\in} \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)).$$

Since $[I_\eta(k, C_\eta(k, f_A, r), r)]^c$ is r-fuzzy soft regularly closed set, $\varphi_\psi^{-1}([I_\eta(k, C_\eta(k, f_A, r), r)]^c)$ is r-fuzzy soft α-closed set (from (3)). Thus, $\varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r))$ is r-fuzzy soft α-open set and $n_{w_s} \tilde{\in} \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r)$. Then,

$$\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r).$$

(6) $\Rightarrow$ (1) Let $n_{w_s} \in \widetilde{P_s(W)}$ and $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ containing $\varphi_\psi(n_{w_s})$, hence by (6), $\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r)$.

Since $n_{w_s} \tilde{\in} \varphi_\psi^{-1}(f_A)$, then we obtain $n_{w_s} \tilde{\in} \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r) = h_C$ (say). Hence, there is $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft α-open set containing n_{w_s} such that $\varphi_\psi(h_C) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$. Therefore, φ_ψ is fuzzy soft almost α-continuous.

In a similar way, we can prove the following theorem.

Theorem 8. Let (W, τ_N) and (V, η_F) be an FSTSs, and $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ be a fuzzy soft mapping. The following statements are equivalent for each $f_A \in \widetilde{(V,F)}$, $n \in N$, $(k = \psi(n)) \in F$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft weakly α-continuous.

(2) $I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, f_A, r)), r), r), r) \quad \sqsupseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A) \geq r$.

(3) $C_\tau(n, I_\tau(n, C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, f_A, r)), r), r), r) \quad \sqsubseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A^c) \geq r$.

(4) $\alpha C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, f_A, r)), r) \sqsubseteq \varphi_\psi^{-1}(f_A)$, if $\eta_k(f_A^c) \geq r$.

(5) $\alpha C_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, f_A, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_\eta(k, f_A, r))$.

(6) $\alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, I_\eta(k, f_A, r), r)), r) \sqsupseteq \varphi_\psi^{-1}(I_\eta(k, f_A, r))$.

(7) $\varphi_\psi^{-1}(f_A) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, f_A, r)), r)$, if $\eta_k(f_A) \geq r$.

Remark 7. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft continuity as in the next diagram.

fuzzy soft continuity	$\Rightarrow$	fuzzy soft α-continuity
$\Downarrow$		$\Downarrow$
fuzzy soft almost continuity	$\Rightarrow$	fuzzy soft almost α-continuity
$\Downarrow$		$\Downarrow$
fuzzy soft weakly continuity	$\Rightarrow$	fuzzy soft weakly α-continuity

Proposition 3. Let (W, τ_N), (V, η_F) and (U, γ_E) be an FSTSs, and $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$, $\varphi^*_{\psi^*} : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be two fuzzy soft functions. Then, the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft almost α-continuous if φ_ψ is fuzzy soft α-continuous and $\varphi^*_{\psi^*}$ is fuzzy soft almost continuous (resp., continuous).

Proof. The proof is obvious.

Let $\mathcal{H}$ and $\mathcal{I} : N \times \widetilde{(W,N)} \times I_\circ \to \widetilde{(W,N)}$ be operators on $\widetilde{(W,N)}$, and $\mathcal{J}$ and $\mathcal{K}$ $: F \times \widetilde{(V,F)} \times I_\circ \to \widetilde{(V,F)}$ be operators on $\widetilde{(V,F)}$.

Definition 17. *[11]* Let (W, τ_N) and (V, η_F) be an FSTSs. $\varphi_\psi : \widetilde{(W,N)} \longrightarrow \widetilde{(V,F)}$ is said to be a fuzzy soft $(\mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K})$-continuous mapping if

$$\mathcal{H}[n, \varphi_\psi^{-1}(\mathcal{K}(k, h_C, r)), r] \,\overline{\sqcap}\, \mathcal{I}[n, \varphi_\psi^{-1}(\mathcal{J}(k, h_C, r)), r] = \Phi$$

for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$, $n \in N$, and $(k = \psi(n)) \in F$.

In (2023), Alshammari et al. [11] defined the notion of fuzzy soft α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq I_\tau(n, C_\tau(n, I_\tau(n, \varphi_\psi^{-1}(h_C), r), r), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. We can see that Definition 17 generalizes the concept of fuzzy soft continuous functions when we choose $\mathcal{H}$ = identity operator, $\mathcal{I}$ = interior closure interior operator, $\mathcal{J}$ = identity operator, and $\mathcal{K}$ = identity operator.

A historical justification of Definition 17:

(1) In Section 3, we obtained the notion of fuzzy soft almost α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(I_\eta(k, C_\eta(k, h_C, r), r)), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. Here, $\mathcal{H}$ = identity operator, $\mathcal{I}$ = α-interior operator, $\mathcal{J}$ = interior closure operator, and $\mathcal{K}$ = identity operator.

(2) In Section 3, we obtained the notion of fuzzy soft weakly α-continuous mappings: $\varphi_\psi^{-1}(h_C) \sqsubseteq \alpha I_\tau(n, \varphi_\psi^{-1}(C_\eta(k, h_C, r)), r)$, for each $h_C \in \widetilde{(V,F)}$ with $\eta_k(h_C) \geq r$. Here, $\mathcal{H}$ = identity operator, $\mathcal{I}$ = α-interior operator, $\mathcal{J}$ = closure operator, and $\mathcal{K}$ = identity operator.

4. Fuzzy soft α-compactness

Here, some novel types of fuzzy soft compactness via r-fuzzy soft α-open sets were introduced and the relationships between them were explored with the help of some examples.

Definition 18. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid \tau_n((g_B)_\delta) \geq r$ for each $n \in N\}_{\delta\in\Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}(g_B)_\delta$.

Definition 19. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}(g_B)_\delta$.

Lemma 6. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft α-compact, then h_C is r-fuzzy soft compact.

Proof. Follows from Definitions 18 and 19.

Theorem 9. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft α-continuous mapping. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V, F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta\in\Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W, N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}\varphi_\psi^{-1}((g_B)_\delta)$. Then, $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta_\circ}(g_B)_\delta$. Hence, the proof is completed.

Definition 20. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft almost compact iff for every family $\{(g_B)_\delta \in \widetilde{(W, N)} \mid \tau_n((g_B)_\delta) \geq r\}_{\delta\in\Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}C_\tau(n, (g_B)_\delta, r)$ for each $n \in N$.

Definition 21. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft almost α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W, N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}C_\tau(n, (g_B)_\delta, r)$ for each $n \in N$.

Lemma 7. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft almost α-compact, then h_C is r-fuzzy soft almost compact.

Proof. Follows from Definitions 20 and 21.

Lemma 8. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft compact (resp., α-compact), then h_C is r-fuzzy soft almost compact (resp., almost α-compact).

Proof. Follows from Definitions 18, 19, 20, and 21.

Remark 8. The converse of Lemma 8 may not be true, as shown by Example 6.

Example 6. Let $V = I$, $n \in N - \{1\}$, and $F = \{k_1, k_2\}$ be the parameter set of V. Define g_{F_n} and $f_{F_1} \in \widetilde{(V,F)}$ as follows $\forall\ k \in F$:

$$g_{F_n}(k)(v) = \begin{cases} 0.8, & \text{if} \quad v = 0, \\ nv, & \text{if} \quad 0 < v \leq \frac{1}{n}, \\ 1\ , & \text{if} \quad \frac{1}{n} < v \leq 1, \end{cases} \qquad f_{F_1}(k)(v) = \begin{cases} 1, & \text{if} \quad v = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Define fuzzy soft topology $\eta_F : F \longrightarrow [0,1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(t_F) = \begin{cases} \frac{4}{5}, & \text{if} \quad t_F \in \{\Phi, \widetilde{F}\}, \\ \frac{2}{3}, & \text{if} \quad t_F \leq f_{F_1}, \\ \frac{n}{n+1}, & \text{if} \quad t_F \leq g_{F_n}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, V is $\frac{1}{2}$-fuzzy soft almost compact, but it is not $\frac{1}{2}$-fuzzy soft compact.

Theorem 10. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft continuous mapping. If $h_C \in \widetilde{(W,N)}$ is r-fuzzy soft almost α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft almost compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V,F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta\in\Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W,N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta\in\Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft almost α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r)$. Since φ_ψ is fuzzy soft continuous mapping, it follows

$$\sqcup_{\delta\in\Delta_\circ}C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r) \sqsubseteq$$

$$\sqcup_{\delta\in\Delta_\circ}\varphi_\psi^{-1}(C_\eta(k, (g_B)_\delta, r)) =$$

$$\varphi_\psi^{-1}(\sqcup_{\delta\in\Delta_\circ}C_\eta(k, (g_B)_\delta, r)).$$

Then, $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta\in\Delta_\circ}C_\eta(k, (g_B)_\delta, r)$. Hence, the proof is completed.

Definition 22. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W,N)}$ is called an r-fuzzy soft nearly compact iff for every family $\{(g_B)_\delta \in \widetilde{(W,N)} \mid \tau_n((g_B)_\delta) \geq r\}_{\delta\in\Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta\in\Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that

$$h_C \sqsubseteq \sqcup_{\delta\in\Delta_\circ}I_\tau(n, C_\tau(n, (g_B)_\delta, r), r) \text{ for each } n \in N.$$

Definition 23. Let (W, τ_N) be an FSTS and $r \in I_\circ$, then $h_C \in \widetilde{(W, N)}$ is called an r-fuzzy soft nearly α-compact iff for every family $\{(g_B)_\delta \in \widetilde{(W, N)} \mid (g_B)_\delta$ is r-fuzzy soft α-open$\}_{\delta \in \Delta}$, such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$, there is a finite subset $\Delta_\circ$ of Δ, such that

$$h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ} I_\tau(n, C_\tau(n, (g_B)_\delta, r), r) \text{ for each } n \in N.$$

Lemma 9. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft nearly α-compact, then h_C is r-fuzzy soft nearly compact.

Proof. Follows from Definitions 22 and 23.

Lemma 10. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft compact (resp., α-compact), then h_C is r-fuzzy soft nearly compact (resp., nearly α-compact).

Proof. Follows from Definitions 18, 19, 22, and 23.

Remark 9. The converse of Lemma 10 may not be true, as shown by Example 7.

Example 7. Let $V = I$, $0 < n < 1$, and $F = \{k_1, k_2\}$ be the parameter set of V. Define g_{F_n}, g_F, and $f_F \in \widetilde{(V, F)}$ as follows $\forall\, k \in F$:

$$g_{F_n}(k)(v) = \begin{cases} \frac{v}{n}, & \text{if } \quad 0 \leq v \leq n, \\ \frac{1-v}{1-n}, & \text{if } \quad n < v \leq 1, \end{cases} \qquad g_F(k)(v) = \begin{cases} 1, & \text{if } \quad v = 0, \\ \frac{1}{2}, & \text{if } \quad 0 < v \leq \ 1, \end{cases}$$

$$f_F(k)(v) = \begin{cases} \frac{1}{2}, & \text{if } \quad 0 \leq v < 1, \\ 1, & \text{if } \quad v = 1. \end{cases}$$

Define fuzzy soft topology $\eta_F : F \longrightarrow [0, 1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(t_F) = \begin{cases} 1, & \text{if } \quad t_F \in \{g_F, f_F, \Phi, \widetilde{F}\}, \\ \max(\{1-n, n\}), & \text{if } \quad t_F = g_{F_n}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, V is $\frac{1}{2}$-fuzzy soft nearly compact, but it is not $\frac{1}{2}$-fuzzy soft compact.

Theorem 11. Let $\varphi_\psi : (W, \tau_N) \longrightarrow (V, \eta_F)$ be a fuzzy soft continuous and fuzzy soft open mapping. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft nearly α-compact, then $\varphi_\psi(h_C)$ is r-fuzzy soft nearly compact.

Proof. Let $\{(g_B)_\delta \in \widetilde{(V, F)} \mid \eta_k((g_B)_\delta) \geq r\}_{\delta \in \Delta}$ with $\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta \in \Delta}(g_B)_\delta$ for each $k \in F$. Then, $\{\varphi_\psi^{-1}((g_B)_\delta) \in \widetilde{(W, N)} \mid \varphi_\psi^{-1}((g_B)_\delta)$ is r-fuzzy soft α-open$\}_{\delta \in \Delta}$ (by φ_ψ is fuzzy soft α-continuous) such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta}\varphi_\psi^{-1}((g_B)_\delta)$. Since h_C is r-fuzzy soft nearly α-compact, there is a finite subset $\Delta_\circ$ of Δ such that $h_C \sqsubseteq \sqcup_{\delta \in \Delta_\circ} I_\tau(n, C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r), r)$. Since φ_ψ is fuzzy soft continuous and fuzzy soft open mapping, it follows

$$\varphi_\psi(h_C) \sqsubseteq \sqcup_{\delta \in \Delta_\circ}\varphi_\psi(I_\tau(n, C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r), r))$$

$$\sqsubseteq \sqcup_{\delta \in \Delta_\circ} I_\eta(k, \varphi_\psi(C_\tau(n, \varphi_\psi^{-1}((g_B)_\delta), r)), r)$$

$$\sqsubseteq \sqcup_{\delta \in \Delta_\circ} I_\eta(k, \varphi_\psi(\varphi_\psi^{-1}(C_\eta(k, (g_B)_\delta, r))), r)$$

$$\sqsubseteq \sqcup_{\delta \in \Delta_\circ} I_\eta(k, C_\eta(k, (g_B)_\delta, r), r).$$

Hence, the proof is completed.

Lemma 11. Let (W, τ_N) be an FSTS and $r \in I_\circ$. If $h_C \in \widetilde{(W, N)}$ is r-fuzzy soft nearly α-compact (resp., nearly compact), then h_C is r-fuzzy soft almost α-compact (resp., almost compact).

Proof. Follows from Definitions 20, 21, 22, and 23.

Remark 10. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft compactness as in the next diagram.

fuzzy soft α-compactness	$\Rightarrow$	fuzzy soft compactness
$\Downarrow$		$\Downarrow$
fuzzy soft nearly α-compactness	$\Rightarrow$	fuzzy soft nearly compactness
$\Downarrow$		$\Downarrow$
fuzzy soft almost α-compactness	$\Rightarrow$	fuzzy soft almost compactness

5. Conclusion and future work

In this study, the concepts of fuzzy soft α-closure (α-interior) operators have been introduced in an FSTSs based on the paper by Aygünoğlu et al. [19] and some of their basic properties have been investigated. Thereafter, the notion of r-fuzzy soft α-connected sets has been defined and studied. Furthermore, some properties of fuzzy soft α-continuous mappings have been obtained between two FSTSs (W, τ_N) and (V, η_F). Moreover, as a weaker form of the notion of fuzzy soft α-continuous mappings, the notions of fuzzy soft almost (weakly) α-continuous mappings have been introduced and some of their characterizations have been investigated. Also, we have shown that fuzzy soft α-continuity $\Rightarrow$ fuzzy soft almost α-continuity $\Rightarrow$ fuzzy soft weakly α-continuity and we have the following:

- Fuzzy soft $(id_W, I_\tau(C_\tau(I_\tau)), id_V, id_V)$-continuous mapping is fuzzy soft α-continuous.
- Fuzzy soft $(id_W, \alpha I_\tau, I_\eta(C_\eta), id_V)$-continuous mapping is fuzzy soft almost α-continuous.
- Fuzzy soft $(id_W, \alpha I_\tau, C_\eta, id_V)$-continuous mapping is fuzzy soft weakly α-continuous.

In the end, new types of soft compactness via r-fuzzy soft α-open sets have been explored and the relationships between them have been studied.

In upcoming papers, we will use the fuzzy soft α-closure operator to define some new separation axioms in an FSTS based on the paper by Aygünoğlu et al. [19]. Also, we shall discuss some of the notions given here in the frames of fuzzy soft r-minimal structures [30].

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References

[1] B. Ahmad and A. Kharal. On fuzzy soft sets. *Adv. Fuzzy Syst.*, page 586507, 2009.

[2] M. Akdag and A. Ozkan. On soft β-open sets and soft β-continuous functions. *Sci. World J.*, page 843456, 2014.

[3] M. Akdag and A. Ozkan. Soft α-open sets and soft α-continuous functions. *Abst. Appl. Anal.*, page 891341, 2014.

[4] S. Al-Ghour. Boolean algebra of soft q-sets in soft topological spaces. *Appl. Comput. Intell. Soft Comput.*, page 5200590, 2022.

[5] S. Al-Ghour and J. Al-Mufarrij. Between soft complete continuity and soft somewhat-continuity. *Symmetry*, 15:1–14, 2023.

[6] T. M. Al-shami. Soft somewhere dense sets on soft topological spaces. *Commun. Korean Math. Soc.*, 33:1341–1356, 2018.

[7] T. M. Al-shami. On soft separation axioms and their applications on decision-making problem. *Math. Probl. Eng.*, pages 1–12, 2021.

[8] T. M. Al-shami, M. Arar, R. Abu-Gdairi, and Z. A. Ameen. On weakly soft β-open sets and weakly soft β-continuity. *J. Inte. Fuzzy Syst.*, 45:6351–6363, 2023.

[9] T. M. Al-shami, S. Saleh, A. M. Abd El-latif, and A. Mhemdi. Novel categories of spaces in the frame of fuzzy soft topologies. *AIMS Mathematics*, 9(3):6305–6320, 2024.

[10] J. C. R. Alcantud. Soft open bases and a novel construction of soft topologies from bases for topologies. *Mathematics*, 8:672, 2020.

[11] I. Alshammari, M. H. Alqahtani, and I. M. Taha. On r-fuzzy soft δ-open sets and applications via fuzzy soft topologies. *Preprints*, page 2023121240, 2023.

[12] I. Alshammari and I. M. Taha. On fuzzy soft β-continuity and β-irresoluteness: some new results. *AIMS Mathematics*, 9(5):11304–11319, 2024.

[13] S. Atmaca and I. Zorlutuna. On fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.*, 5:377–386, 2013.

[14] N. Çağman, S. Enginoğlu, and F. Çitak. Fuzzy soft set theory and its applications. *Iran. J. Fuzzy Syst.*, 8:137–147, 2011.

[15] V. Çetkin and H. Aygün. Fuzzy soft semiregularization spaces. *Ann. Fuzzy Math. Inform.*, 7:687–697, 2014.

[16] V. Çetkin, A. Aygünoğlu, and H. Aygün. On soft fuzzy closure and interior operators. *Util. Math.*, 99:341–367, 2016.

[17] S. A. El-Sheikh, R. A. Hosny, and A. M. Abd El-latif. Characterizations of β-soft separation axioms in soft topological spaces. *Inf. Sci. Lett.*, 4:125–133, 2015.

[18] A. Aygünoğlu and H. Aygün. Some notes on soft topological spaces. *Neural Comput. Appl.*, 21:113–119, 2012.

[19] A. Aygünoğlu, V. Çetkin, and H. Aygün. An introduction to fuzzy soft topological spaces. *Hacet. J. Math. Stat.*, 43:193–208, 2014.

[20] S. Hussain and B. Ahmad. Soft separation axioms in soft topological spaces. *Hacet. J. Math. Stat.*, 44:559–568, 2015.

[21] S. Kaur, T. M. Al-shami, A. Ozkan, and M. Hosny. A new approach to soft continuity. *Mathematics*, 11:3164, 2023.

[22] P. K. Maji, R. Biswas, and A. R. Roy. Fuzzy soft sets. *J. Fuzzy Math.*, 9:589–602, 2001.

[23] S. Mishra and R. Srivastava. Hausdorff fuzzy soft topological spaces. *Ann. Fuzzy Math. Inform.*, 9:247–260, 2015.

[24] D. Molodtsov. Soft set theory-first results. *Comput. Math. Appl.*, 37:19–31, 1999.

[25] S. K. Nazmul and S. K. Samanta. Neighbourhood properties of soft topological spaces. *Ann. Fuzzy Math. Inform.*, 6:1–15, 2013.

[26] S. Saleh and J. Al-Mufarrij. On g-regularity and g-normality in fuzzy soft topological spaces. *Europ. J. Pure Appl. Math.*, 16(1):180–191, 2023.

[27] S. Saleh, T. M. Al-Shami, and A. Mhemdi. On some new types of fuzzy soft compact spaces. *J. Math.*, page 5065592, 2023.

[28] M. Shabir and M. Naz. On soft topological spaces. *Comput. Math. Appl.*, 61:1786–1799, 2011.

[29] I. M. Taha. A new approach to separation and regularity axioms via fuzzy soft sets. *Ann. Fuzzy Math. Inform.*, 20:115–123, 2020.

[30] I. M. Taha. Compactness on fuzzy soft r-minimal spaces. *Int. J. Fuzzy Logic Intell. Syst.*, 21:251–258, 2021.

[31] I. M. Taha. Some new separation axioms in fuzzy soft topological spaces. *Filomat*, 35:1775–1783, 2021.

[32] I. M. Taha. Some new results on fuzzy soft r-minimal spaces. *AIMS Mathematics*, 7:12458–12470, 2022.

[33] M. Terepeta. On separating axioms and similarity of soft topological spaces. *Soft Computing*, 23(3):1049–1057, 2019.

[34] S. S. Thakur and A. S. Rajput. Connectedness between soft sets. *New Math. Nat. Comput.*, 14:53–71, 2018.

[35] A. P. Šostak. On a fuzzy topological structure. In *In: Proceedings of the 13th winter school on abstract analysis, Section of topology, Palermo: Circolo Matematico di Palermo*, pages 89–103, 1985.

[36] H. L. Yang, X. Liao, and S. G. Li. On soft continuous mappings and soft connectedness of soft topological spaces. *Hacet. J. Math. Stat.*, 44:385–398, 2015.

[37] L. A. Zadeh. Fuzzy sets. *Inform. Control*, 8:338–353, 1965.

[38] I. Zorlutuna, M. Akdag, W. K. Min, and S. Atmaca. Remarks on soft topological spaces. *Ann. Fuzzy Math. Inform.*, 3:171–185, 2012.