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Convex Accessibility Number of the Complements and Some Binary Operations of Graphs

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Abstract. This study explores various aspects of the Convex Accessibility Number in graph theory, focusing on some binary operations namely Cartesian Product and Strong Product and Complements of graphs. The computation of the Convex Accessibility Number of Cartesian Product and Strong Product of graphs is examined. Also, the Convex Accessibility Number of the Complement of some known graphs is explored. Through these investigations, this study contributes to a deeper understanding of the Convex Accessibility Number in graph theory, offering insights into its behavior under different graph operations and Complementation scenarios.

2020 Mathematics Subject Classifications: 05C12

Key Words and Phrases: H-Convex Accessibility Number, Convex Subgraph, Strong Product, Cartesian Product, Complement of a Graph, Accessibility Number

1. Introduction

The concept of H-Convex accessibility number was introduced by R. G. Artes, Jr. and M.J. F. Luga [3] [2] in 2014, it was about the H-Convex accessibility number of some graphs and graphs under binary operations join, corona and composition.

This paper presents the H-convex accessibility number for various graph operations such as Cartesian products, strong products, and complements was determined by analyzing how the proper convex subgraphs influence the accessibility number. As the size of these proper convex subgraphs increases, the Convex Accessibility Number tends to approach 1. Therefore, by starting with smaller convex subgraphs and progressively expanding their size, the study aimed to derive a general formula by comparing the Convex Accessibility Numbers across different graph configurations. The distance from a vertex

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u to a subgraph H is defined as the shortest path between u and any vertex $v \in H$. In this paper, the subgraph H is required to be a proper convex subgraph of a graph G. The accessibility number is defined as the minimum k for which G is H-convex k- accessible. The Convex Accessibility Number of a graph helps in covering all points with the minimum number of surveillance cameras, which is essential for secure network design. It also aids in placing key facilities like hospitals or fire stations to improve emergency response times. In wireless sensor networks, it determines the optimal sensor placement for full coverage, ensuring efficient resource use.

All the graphs considered in this study are finite, undirected and connected. Most of the definitions are from [1]. Those that are not from the said source are so indicated. The symbols $V(G)$ and $E(G)$ denote the *vertex set* and *edge set* of G. An edge joining vertices $u, v \in G$ is denoted by $[u, v]$. In this case, u and v are *adjacent.* A graph H is a *subgraph* of a graph G, denoted by $H \preceq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $H = \langle V(H) \rangle$ is an *induced subgraph* of a graph G if $H \preceq G$ and two vertices in H are adjacent whenever they are adjacent in G. A graph H is a *proper subgraph* of G if $E(G) \setminus E(H) \neq \varnothing$.

Given a connected graph G, the distance between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of the shortest path joining u and v[1]. The *distance between a vertex* $u \in V(G)$ *and a subgraph* H of G is defined as $d_G(u, H) = \ min\ \{d_G(u, v) : v \in V(H)\}$. For vertices u and v of a graph G, a *u-v geodesic* is any shortest path in G joining u and v. The closed interval $I_G[u, v]$ is the set of vertices lying in any u-v geodesics of G and the set $I_G[u, v]$ consist all the vertices in any u-v geodesic including u and v. A subset C of $V(G)$ is *convex* if for every $u, v \in C$, the vertex set of every *u-v geodesic* is contained in C. Equivalently, C is convex if for every $u, v \in C$, the closed interval $I_G[u, v]$ is a subset of C.

A *convex subgraph* H of a graph G is a subgraph of G induced by a convex subset of $V(G)$. A proper convex subgraph H of G. Subgraph H is said to be the *maximum proper convex subgraph* of G if for any proper convex subgraph H^* with $H \preceq H^* \preceq G$, then $H = H^*$. A graph G is *H-convex k-accessible* if there exists a proper convex subgraph H of G such that for every $v \in V(G) \setminus V(H)$, there exists $u \in V(H)$ satisfying $d_G(u, v) \leq k$, $k \in \mathbb{N}$. For a proper convex subgraph H of G, we define the *H-Convex accessbility number* of G as $\Gamma_H(G) = min\{k :\ G \text{ is } H \text{ convex } k \text{ accessible}\}$.

The *complement of a graph* G is a graph $\bar{G}$, with vertex set same as G and two vertices in $\bar{G}$ are adjacent if and only if they are not adjacent in G. The *Cartesian Product* $G\square H$ of graphs G and H is a graph such that the vertex set $G\square H$ is the cartesian product $V(G) \times V(H)$ and vertices (u, v) and (u', v') are adjacent in $G\square H$ if and only if u is adjacent to u' in G **or**, v is adjacent to v' in H. The *Strong product* $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the cartesian product $V(G) \times V(H)$ and distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if $u = v$ and u' is adjacent to v' in H **or**, $u' = v'$ and $u$is adjacent to v in G **or**, u is adjacent to v in G and u' is adjacent to v' in H.

For a set $C \subset V(G \times H)$, we denote, $C_G = \{u : (u, v) \in C\ for\ some\ v \in V(H)\}$ and $C_H = \{v : (u, v) \in C\ for\ some\ u \in V(G)\}$. A set $C \in V(G\square H)$ is a convex set in $G\square H$ if and only if $C = C_G\square C_H$, where C_G and C_H are convex sets in G and H respectively, where G and H are connected graphs [5].

The distance between vertices (g, h) and (g', h') in the Cartesian product $G\Box H$ is equal to $d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$ [4]. The distance between vertices (u, v) and (u', v') in the Strong product $G \boxtimes H$ is equal to $d_{G\boxtimes H}((u,v),(u',v')) = max\{d_G(u,u'), d_H(v,v')\}$ [4].

2. H Convex accessibility Number of the Complement of Some Known Graphs

In this section, we established the H-Convex accessibility number of the complement of some known graphs.

Theorem 1. *Let G be a graph such that $G = P_n = [x_1, x_2, \ldots, x_n]$ for $n \geq 4$ and H be a proper convex subgraph of G. Then*

$$\Gamma_H(\overline{G}) = \begin{cases} 2, & \text{if } \ H = K_1 \ \text{or} \ \ H = P_2 = [x_i, x_{i+2}] \\ 1, & \text{otherwise.} \end{cases}$$

Proof.

Let $G = P_n$ be a connected path graph where the vertices are $x_1, x_2, x_3, \ldots, x_n$ and the edges are $[x_1, x_2][x_2, x_3], [x_3, x_4], \ldots, [x_{n-1}, x_n] \in E(G)$. Consider the following cases for the graph G and its complement $\overline{G}$.

Case 1: $H = K_1$. Let $H = K_1$ where $V(H) = \{x_i\}$ and x_i is not an end vertex of G. This means that x_i is connected to x_{i-1} and x_{i+1} in G. Consequently, in the complement $\overline{G}$, x_i is adjacent to all vertices except x_{i-1} and x_{i+1}. Therefore, the distance from x_i to any vertex $u \in V(\overline{G}) \setminus \{x_{i-1}, x_{i+1}\}$ is 1. For x_{i-1} and x_{i+1}, the distance is 2. Thus, $d_{\overline{G}}(u, x_i) \leq 2$ for all $u \in V(\overline{G}) \setminus \{x_i\}$. Hence, $\overline{G}$ is K_1-convex 2-accessible, i.e., $\Gamma_{K_1}(\overline{G}) = 2$.

Case 2: $H = P_2$. Let $H = P_2$ in $\overline{G}$. Without loss of generality, assume that $V(H) = \{x_i, x_{i+2}\}$. Since $[x_i, x_{i+2}] \in E(\overline{G})$, the distance $d_{\overline{G}}(u, P_2) = 1$ if and only if $u \neq x_{i-1}$ and $u \neq x_{i+1}$. However, $d_{\overline{G}}(x_{i-1}, P_2) = 2$ and $d_{\overline{G}}(x_{i+1}, P_2) = 2$. Therefore, for any $u \in V(\overline{G}) \setminus V(P_2)$, $d_{\overline{G}}(u, P_2) \leq 2$. Therefore, $\overline{G}$ is P_2-convex 2-accessible, i.e.,$\Gamma_{P_2}(\overline{G}) = 2$.

Case 3: The degree of x_1 and x_n in G are both 1. In this case, x_1 and x_n are the start and end vertices of the path G, respectively, meaning they are not directly connected in G. Therefore, there exists and edge $[x_1, x_n] \in E(\overline{G})$ connecting x_1 and x_n. Considering this path as the proper convex subgraph in $\overline{G}$, $\overline{G}$ is P_2-convex 1-accessible, i.e., $\Gamma_{P_2}(\overline{G}) = 1$. ■

Consider the Complement of P_5, that is $\overline{P_5}$. If $H_1 = K_1$, then $\Gamma_{H_1}(\overline{P_5}) = 2$. If $H_2 = [a, c]$, then $\Gamma_{H_2}(\overline{P_5}) = 2$. If $H_3 = [a, e]$, then $\Gamma_{H_3}(\overline{P_5}) = 1$. If $H_4 = \{a, c, e\}$, then $\Gamma_{H_4}(\overline{P_5}) = 1$.

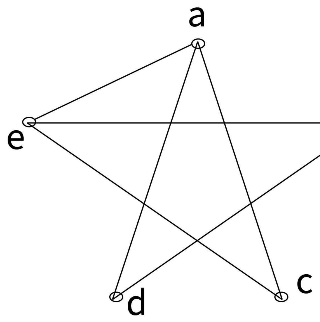

Figure 1: $\overline{P_5}$

Theorem 2. *Let G be a graph such that $G = C_n$ for $n \geq 5$ and H be a proper convex subgraph of G. Then*

$$\Gamma_H(\overline{G}) = \begin{cases} 2, & \text{if } H = P_2 \text{ or } H = K_1 \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let C_n be a cycle graph defined by the sequence of vertices $[u_1, u_2, \ldots, u_n, u_1]$ where the edges are $[u_1, u_2], [u_2, u_3], \ldots, [u_{n-1}, u_n], [u_n, u_1] \in E(C_n)$.

In this cycle graph, observe that $[u_{i-1}, u_i]$ and $[u_i, u_{i+1}]$ are edges of C_n. This implies that $[u_{i-1}, u_i]$ and $[u_i, u_{i+1}]$ cannot be edges in the complement graph $\overline{C_n}$.

Without loss of generality, let $P_2 = \{u_i, u_{i+2}\}$ where $[u_i, u_{i+2}] \in E_{\overline{C_n}}$. For any vertex v in $\overline{C_n}$, the distance $d_{\overline{C_n}}(v, P_2)$ is defined as the minimum distance from v to either u_i or u_{i-2}. This means that for $v = u_{i-1}$ or $v = u_{i+1}$, $d_{\overline{C_n}}(v, P_2) = 2$ and for any other vertex v, which is neither $v = u_{i-1}$ nor $v = u_{i+1}$, $d_{\overline{C_n}}(v, P_2) = 2$ as well. Thus, the distance from any vertex to P_2 is atmost 2, showing that $\overline{C_n}$ is P_2-convex 2-accessible. ■

Remark 1. *For the star, wheel, fan, complete graph, complete bipartite and join, the complement of these graphs have isolated vertices. This means that it is not possible to get the H convex accessibility number of these graphs.*

3. H convex Accessibility Number of the Cartesian Product of Graphs

In this section, we established the H-Convex accessibility number of the Cartesian product of graphs.

Theorem 3. *Let G_1 and G_2 be connected graphs and $H = H_1 \square H_2$ be a proper convex subgraph of $V(G_1 \square G_2)$, where H_1 and H_2 are proper convex subgraphs of G_1 and G_2 respectively. Then,*

$$\Gamma_H(G_1 \square G_2) = \Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2)$$

Proof. Suppose that G_1 and G_2 are connected graphs and $H = H_1 \square H_2$ is a convex set in $G_1 \square G_2$. By [5], H_1 and H_2 are convex sets in G_1 and G_2 respectively.

Consider any vertex $(u, v) \in V(G_1 \square G_2) \setminus V(H_1 \square H_2)$ and any vertex $(x, y) \in V(H_1 \square H_2)$. Then the distance between these vertices in $G_1 \square G_2$ is given by,

$$d_{G_1 \square G_2}((u, v), (x, y)) = d_{G_1}(u, x) + d_{G_2}(v, y).$$

Since H_1 is a proper convex subgraph of G_1 and H_2 is also a proper convex subgraph of G_2, we have

$$\Gamma_{H_1}(G_1) \leq d_{G_1}(u, H_1)$$
$$\Gamma_{H_2}(G_2) \leq d_{G_2}(v, H_2).$$

Adding these inequalities, we have

$$\Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2) \leq d_{G_1}(u, H_1) + d_{G_2}(v, H_2)$$
$$\Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2) \leq d_{G_1 \square G_2}((u, v), H_1 \square H_2).$$

Since (u, v) and (x, y) are arbitrarily chosen vertices in $G_1 \square G_2$ and $H_1 \square H_2$, respectively, the distance $d_{G_1 \square G_2} = ((u, v), (x, y))$ represents the shortest path distance between (u, v) and (x, y). Therefore,

$$d_{G_1 \square G_2}((u, v), H_1 \square H_2) = \Gamma_H(G_1 \square G_2).$$

Substituting this result, we have

$$\Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2) \leq \Gamma_H(G_1 \square G_2).$$

By the definition of convex accessibility number, we also have

$$\Gamma_H(G_1 \square G_2) \leq \Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2).$$

Combining these inequalities, we obtain,

$$\Gamma_H(G_1 \square G_2) = \Gamma_{H_1}(G_1) + \Gamma_{H_2}(G_2).$$

■

Consider the Cartesian Product of P_6 and P_6, that is $P_6 \square P_6$ is as shown in Figure 2 and a proper convex subgraph $H = P_2 \square P_2$, where P_2 is a convex subgraph of P_6. For this graph, $\Gamma_H(P_6 \square P_6) = \Gamma_{P_2}(P_6) + \Gamma_{P_2}(P_6) = 2 + 2 = 4$.

4. H-convex Accessibility Number of the Strong Product of Graphs

In this section, we established the H-Convex accessibility number of the Strong product of graphs.

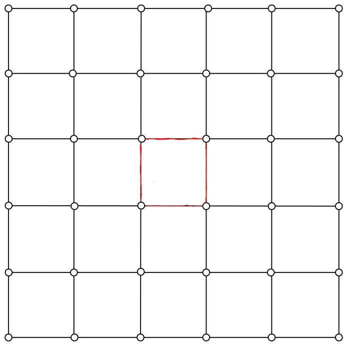

Figure 2: The Cartesian Product of P_6 and P_6

Theorem 4. *Let G and H be connected graphs. If $C = C_G \boxtimes C_H$, then a set $C \subset V(G \boxtimes H)$ is a convex set in $G \boxtimes H$, where C_G and C_H are convex sets in G and H respectively.*

Proof. Let G and H be a connected graph and let $C = C_G \boxtimes C_H$, where $C_G \subset V(G)$ and $C_H \subset V(H)$. We aim to show that C is convex in $G \boxtimes H$.

Consider any two vertices $(u, v), (u', v') \in C$. Let (x, y) be a vertex on a (u, v) - (u', v') geodesic in $G \boxtimes H$. Then, by definition of strong product, one of the following must hold, $u = x$ and v is adjacent to y in H, **or**, $v = y$ and u is adjacent to x in G **or** u is adjacent to x in G and v is adjacent to y in H.

Case 1: $u = x$ and v is adjacent to y in H. Suppose that $u = x$ and v is adjacent to y in H. By assumption, there exist the u-u' path joining vertices u and u' in G. Hence, $u = x$ must be in C_G. Similarly, y is also contained in C_H because C_H is convex.

Case 2: $v = y$ and u is adjacent to x in G. Assume that $v = y$ and u is adjacent to x in G. Then, x must be in C_G because C_G is convex. Analogously, there exist a v-v' path joining vertices v and v' in H. Thus, $y = v$ is in C_H.

Case 3: u is adjacent to x in G and v is adjacent to y in H. Let u is adjacent to x in G and v is adjacent to y in H. This must mean that x is contained in C_G since C_G is convex. In a similar fashion, y is also contained in C_H since C_H is convex.

In all cases, (x, y) is contained in $C = C_G \boxtimes C_H$. Therefore, C is convex in $G \boxtimes H$. ■

Theorem 5. *Let G and H be connected graphs. If a set $C \subset V(G \boxtimes H)$ is a convex set in $G \boxtimes H$, then $C = C_G \boxtimes C_H$, where C_G and C_H are convex sets in G and H respectively.*

Proof. Suppose a set $C \in V(G \boxtimes H)$ is a convex set in $G \boxtimes H$. Let $(u, u') \in C_G$ and x be a vertex in a u - u' geodesic in G. By definition of strong product, there exists $(v, v') \in C_H$ such that the either $u = x$ and v is adjacent to v' in H, **or**u is adjacent to x in G and v is adjacent to v' in H. In either cases, (x, v) and $(x, v') \in C$. Hence, $x \in C_G$ Thus. C_G is convex in G. Similarly, let $a, a' \in C_H$ and y be a vertex set in a a-a' geodesic in H. By definition of strong product, there exist $(b, b') \in C_G$ such that $a = y$ and b is

adjacent to b' in G, or b is adjacent to b' in G and a is adjacent to y in H. In both cases, (b, y) and $(b', y) \in C$. Thus, $y \in C_H$ and C_H is convex in H.
The assumption implies that $C \subseteq C_G \boxtimes C_H$. Assume that $(i, j) \in C_G \boxtimes C_H$. Then, there exists $m \in V(G)$ and $n \in V(H)$ such that either $i = m$ and j is adjacent to n in H, **or**, $j = n$ and i is adjacent to m in G, or i is adjacent to m in G and j is adjacent to n in H. Note that C is convex, it follows that $(i, j) \in C$.Thus, $C_G \boxtimes C_H \subseteq C$. Consequently, $C = C_G \boxtimes C_H$. ■

Corollary 1. *Let G and H be connected graphs. A set $C \in V(G \boxtimes H)$ is a convex set in $G \boxtimes H$ if and only if $C = C_G \boxtimes C_H$, where C_G and C_H are convex sets in G and H respectively.*

Proof. Notice that the preceding two theorems have established both the sufficiency and necessity conditions required for this corollary. Thus, this directly follows from Theorem 4 and Theorem 5.

Theorem 6. *Let G_1 and G_2 be connected graphs and $H = H_1 \boxtimes H_2$ be a proper convex subgraph of $V(G_1 \boxtimes G_2)$, where H_1 and H_2 are proper convex subgraphs of G_1 and G_2 respectively. Then,*

$$\Gamma_H(G_1 \boxtimes G_2) = \ max\{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\}$$

Proof. Let G_1 and G_2 be connected graphs and $H = H_1 \boxtimes H_2$ be a proper convex subgraph of G_1 and G_2. We aim to show that the graph $G_1 \boxtimes G_2$ has a certain relationship with the convexity parameters of G_1 and G_2.

Consider an arbitrary vertex $(u, v) \in V(G_1 \boxtimes G_2) \setminus V(H)$. Without loss of generality, let $(u', v') \in V(H)$. According to [4], the distance in the strong product graph $G_1 \boxtimes G_2$ is given by,

$$d_{G_1 \boxtimes G_2}((u, v), (u', v')) = \ max\ \{d_{G_1}(u, u'), d_{G_2}(v, v')\}.$$

By [3], we know that, $\Gamma_{H_1}(G_1) \leq d_{G_1}(u, u')$ and $\Gamma_{H_2}(G_2) \leq d_{G_2}(v, v')$. Thus, we have

$$max\ \{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\} \leq \ max\ \{d_{G_1}(u, u'), d_{G_2}(v, v')\}.$$

This simplifies to

$$max\ \{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\} \leq d_{G_1 \boxtimes G_2}((u, v), (u', v')).$$

Since (u, v) and (u', v') are arbitrarily chosen vertices, the distance $d_{G_1 \boxtimes G_2}((u, v), (u', v'))$ represents the shortest path between these vertices. Therefore

$$d_{G_1 \boxtimes G_2}((u, v), (u', v')) = \Gamma_H(G_1 \boxtimes G_2).$$

Substituting this to our inequality we get

$$max\{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\} \leq \Gamma_H(G_1 \boxtimes G_2).$$

From [3], we also have,

$$\Gamma_H(G_1 \boxtimes G_2) \leq \ max\{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\}.$$

Combining these results, we conclude

$$\Gamma_H(G_1 \boxtimes G_2) = \ max\{\Gamma_{H_1}(G_1), \Gamma_{H_2}(G_2)\}.\blacksquare$$

Consider the Strong Product of P_8 and P_6, that is $P_8 \boxtimes P_6$ is as shown in Figure 3 and a proper convex subgraph $H = P_2 \boxtimes P_2$, where P_2 is a convex subgraph of P_8 and P_2 is a proper convex subgraph of P_6. For this graph, $\Gamma_H(P_8 \boxtimes P_6) = max\ \{\Gamma_{P_2}(P_8), \Gamma_{P_2}(P_6)\} = max\{3, 2\} = 3$.

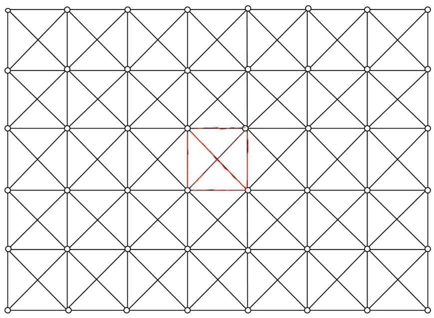

Figure 3: The Strong Product of P_8 and P_6

Conclusion

This study has advanced the understanding of the Convex accessibility number by investigating its behavior under various graph operations and complementation. The analysis of the Cartesian and Strong products revealed distinct patterns in the Convex accessibility number, offering valuable insights into how these binary operations impact graph properties. Additionally, exploring the Convex accessibility number of graph complements has provided further clarity on its interaction with graph structures. These findings not only enhance theoretical knowledge but also pave the way for future research in graph theory, particularly in understanding how different operations affect Convex accessibility. By bridging gaps in the existing literature and presenting new perspectives, this study contributes significantly to the broader field of graph theory and its applications.

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