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Inclusive Subclasses of Bi-univalent Functions Specified by Euler Polynomials

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Abstract. Our research delineates novel two subclasses $\mathcal{F}_{\Pi}(\varkappa, \varepsilon, \ell)$ and $\mathcal{L}_{\Pi}(\varphi, \ell)$ of analytical functions using Euler polynomials. Afterwards, we estimate the Fekete–Szegö functional problem and the Maclaurin coefficients for this subclasses, namely $|c_2|$ and $|c_3|$. Additionally, several new results are shown to follow after specializing the parameters employed in our main results.

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1. preliminaries

Euler polynomials, which date back to Leonhard Euler's research in the eighteenth century, are fundamental components for articulating complex functions and comprehending their geometric characteristics. They play a significant role in the characterization of conformal mappings in geometric function theory that preserve angles locally. They are also useful in the study of univalent and analytic functions.

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Because Euler polynomials are used so widely in pure mathematics, many academics have begun to work in a number of domains. The geometric properties of special functions and several other related functions are the main focus of current study in geometric function theory. For a few of these functions geometric characteristics, we refer to [4, 17] and any pertinent references.

Let $\digamma$ be the class of analytic functions p in the unit disk $\Delta = \{\beth \in \mathbb{C} : |\beth| < 1\}$ and normalized by $p(0) = p'(0) - 1 = 0$ of the form:

$$p(\beth) = \beth + \sum_{i=2}^{\infty} c_i \beth^i, \quad (\beth \in \Delta). \tag{1}$$

We also let Φ the class of univalent functions in Δ.

Every function $p \in \Phi$ has an inverse p^{-1}, defined by

$$p^{-1}(p(\beth)) = \beth \text{ and } \varpi = p(p^{-1}(\varpi))(\beth \in \Delta, \ |\varpi| < s_{10}(p) \geq \frac{1}{4})$$

where

$$p^{-1}(\varpi) = h(\varpi) = \varpi - c_2\varpi^2 + (2c_2^2 - c_3)\varpi^3 \\ - (c_4 + 5, c_2^3 - 5c_3c_2)\varpi^4 + \cdots . \tag{2}$$

Let Γ the class of bi-univalent functions in Δ given by (1) (a function p is bi-univalent in Δ if p and p^{-1} are univalent in Δ).

Example in the class Γ is $h(\beth) = \frac{\beth}{1-\beth}$ but $h(\beth) = \frac{\beth}{1-\beth^2}$ not members of Γ (see [3]).

The first differential subordination problem introduced Miller and Mocanu [10], see [11] and [12]. The function p is subordinate to h, written as $p \prec h$, if p and h are analytic in Δ and exists function $\varpi \in \digamma$ in Δ with

$$\varpi(0) = 0 \text{ and } |\varpi(\beth)| < 1, \qquad (\beth \in \Omega)$$

such that

$$p(\beth) = h(\varpi(\beth)).$$

Also, if h is univalent in Δ, then

$$p(\beth) \prec h(\beth) \quad \text{if and only if} \quad p(0) = h(0) \text{ and } p(\Delta) \subset h(\Delta).$$

Many authors have deduced multiple subordination between certain classes of analytic functions by applying a subordination theorem for analytic functions., for example, see [7] and [16].

Geometric function theory offers fascinating uses for Euler polynomials, a basic tool in mathematical analysis, especially when studying conformal mappings.

In this paper, we take a specific special function, the Euler polynomial, and we build two new and comprehensive subclasses of bi-univalent functions.

the Eulers polynomials $\Phi_i(\upsilon)$ are defined using the generating function (see, e.g., [9, 15]):

$$\begin{aligned}\mathcal{K}(\upsilon,h) &= \frac{2e^{h\upsilon}}{e^h+1}\\ &= \sum_{i=0}^{\infty}\Phi_i(\upsilon)\frac{h^i}{i!},\left(\frac{1}{2}<\upsilon\leq 1,\ |h|<\pi\right).\end{aligned}$$

An explicit formula for $\Phi_i(\upsilon)$ is given by

$$\Phi_j(\upsilon)=\sum_{i=0}^{j}\frac{1}{2^i}\sum_{u=0}^{i}(-1)^u\binom{i}{u}(\upsilon+u)^j. \tag{3}$$

Now $\Phi_i(\upsilon)$ in terms of Φ_u, obtained from (3) as:

$$\Phi_i(\upsilon)=\sum_{u=0}^{i}\frac{\Phi_u}{2^u}\binom{i}{u}(\upsilon-\frac{1}{2})^{i-u}.$$

Initial Euler polynomial values are:

$$\begin{aligned}\Phi_0(\upsilon) &= 1;\\ \Phi_1(\upsilon) &= \frac{2\upsilon-1}{2};\\ \Phi_2(\upsilon) &= \upsilon^2-\upsilon;\\ \Phi_3(\upsilon) &= \frac{4\upsilon^3-6\upsilon^2+1}{4};\\ \Phi_4(\upsilon) &= \upsilon^4-2\upsilon^3+\upsilon.\end{aligned} \tag{4}$$

Several subclasses of the class Γ were introduced and non-sharp estimates on the coefficients $|c_2|$ and $|c_3|$ in the Taylor series expansion (1). For example, Al-Hawary et al. [18] defined the novel subclass $\mathcal{K}_{\Sigma}^{\gamma}(\sigma,\delta,\mu,x)$ using Gegenbauer polynomials. Amourah et al. [1] defined the class $\mathcal{K}(\vartheta,\delta)$ by means of $(p,h)-$Lucas polynomials. Amourah et al. [2] defined the class $\mathrm{s}_1(\alpha,\beta,t)$ by means of Chebyshev polynomials. Peng et al. [19] defined the class $\mathcal{S}_{\Sigma}^{a,p,c}(\gamma,\lambda,\phi)$ using Hohlov operator. Yousef, et al. [5] defined some subclasses by Frasin differentia operator. Bulut et al. [14] introduced a subclass $\mathcal{K}_{\Sigma}^{\mu}(\lambda,t)$ using the Chebyshev polynomials. Srivastava et al. [8] investigated two interesting subclasses $\mathcal{H}_{\Sigma}^{\alpha}$ and $\mathcal{H}_{\Sigma}(\beta)$.

In this paper, we define new two subclasses of Γ utilizing Euler polynomials which are denote by $\mathcal{F}_{\Gamma}(\varkappa,\epsilon,\upsilon)$ and $\mathcal{L}_{\Gamma}(\psi,\upsilon)$, and derive bounds for the coefficients $|c_2|$ and $|c_3|$ and Fekete–Szegö problems. Additionally, several new results are shown.

2. Bounds of the classes $\mathcal{F}_\Gamma(\varkappa,\epsilon,\upsilon)$ and $\mathcal{L}_\Gamma(\psi,\upsilon)$

A definitions of the new subclasses $\mathcal{F}_\Gamma(\varkappa,\epsilon,\upsilon)$ and $\mathcal{L}_\Gamma(\psi,\upsilon)$ connected to Euler polynomials is provided at the beginning of this section.

Definition 1. *If the next subordinations are satisfied for a function $p \in \Delta$ given by (1), then $p \in \mathcal{F}_\Gamma(\varkappa,\epsilon,\upsilon)$:*

$$(1-\varkappa)\frac{p(\daleth)}{\daleth} + \varkappa p'(\daleth) + \epsilon\daleth p''(\daleth) \prec \mathcal{K}(\upsilon,\daleth) = \sum_{i=0}^{\infty}\Phi_i(\upsilon)\frac{\daleth^i}{i!} \tag{5}$$

and

$$(1-\varkappa)\frac{h(\varpi)}{\varpi} + \varkappa h'(\varpi) + \epsilon\varpi h''(\varpi) \prec \mathcal{K}(\upsilon,\varpi) = \sum_{i=0}^{\infty}\Phi_i(\upsilon)\frac{\varpi^i}{i!}, \tag{6}$$

where $\varkappa \geq 1$, $\epsilon \geq 0$, $\frac{1}{2} < \upsilon \leq 1$ $\daleth, \varpi \ \in \ \Delta$ and $h = p^{-1}$.

Definition 2. *If the nex subordinations are satisfied for a function $p \in \Delta$ given by (1), then $p \in \mathcal{L}_\Gamma(\psi,\upsilon)$:*

$$p'(\daleth) + \daleth\frac{e^{i\psi}+1}{2}p''(\daleth) \prec \mathcal{K}(\upsilon,\daleth) = \sum_{i=0}^{\infty}\Phi_i(\upsilon)\frac{\daleth^i}{i!} \tag{7}$$

and

$$h'(\varpi) + \varpi\frac{e^{i\psi}+1}{2}h''(\varpi) \prec \mathcal{K}(\upsilon,\varpi) = \sum_{i=0}^{\infty}\Phi_i(\upsilon)\frac{\varpi^i}{i!}, \tag{8}$$

where $-\pi < \psi \leq \pi$, $\frac{1}{2} < \upsilon \leq 1$ $\daleth, \varpi \ \in \ \Delta$ and $h = p^{-1}$.

Remark 1. *Many subclasses can be found by taking special values for the parameters $\varkappa, \lambda$ and υ in Definition 1, and for the parameters ψ and υ in Definition 2.*

Lemma 1. *([13]) If $g \in \mathcal{G}$, then $|m_n| \leq 2$ for each $n \in \mathbb{N}$, where $\mathcal{G}$ is the family of analytic functions in Δ such that*

$$Re\,(g(\daleth)) > 0,\ g(\daleth) = 1 + m_1\daleth + m_2^2\daleth + \cdots \quad (\daleth \in \Delta).$$

For a function $p \in \Delta$, we solve Fekete-Szegö and provide the coefficient estimations (see [6]) for the classes $\mathcal{F}_\Gamma(\varkappa,\epsilon,\upsilon)$ and $\mathcal{L}_\Gamma(\psi,\upsilon)$, respectively.

Theorem 1. *Let $p \in \Gamma$ given by (1) in the class $\mathcal{F}_\Gamma(\varkappa,\epsilon,\upsilon)$ where $\varkappa \geq 1$, $\epsilon \geq 0$, $\frac{1}{2} < \upsilon \leq 1$ $\daleth, \varpi \ \in \ \Delta$ and $h = p^{-1}$. Then*

$$|c_2| \leq \sqrt{\digamma(\epsilon,\varkappa,\upsilon)},$$

$$|c_3| \le \frac{(2\upsilon-1)^2}{4(2\epsilon+\varkappa+1)^2} + \frac{2\upsilon-1}{2(6\epsilon+2\varkappa+1)}.$$

and

$$\left|c_3-\zeta c_2^2\right| \le \begin{cases} \frac{2\upsilon-1}{6\epsilon+2\varkappa+1} \ \textit{if} \ 0 \le |1-\zeta| F(\epsilon,\varkappa,\upsilon) < \frac{2\upsilon-1}{2(6\epsilon+2\varkappa+1)}, \\ 2|1-\zeta| F(\epsilon,\varkappa,\upsilon) \ \textit{if} \ |1-\zeta| F(\epsilon,\varkappa,\upsilon) \ge \frac{2\upsilon-1}{2(6\epsilon+2\varkappa+1)}. \end{cases}$$

where

$$F(\epsilon,\varkappa,\upsilon) = \frac{(2\upsilon-1)^3}{2\left|(6\epsilon+2\varkappa+1)(2\upsilon-1)^2 - 2(2\epsilon+\varkappa+1)^2(\upsilon^2-3\upsilon+1)\right|}.$$

Proof. Since $p(\sqsupset) = \sqsupset + \sum_{i=2}^{\infty} c_i \sqsupset^i \in \mathcal{F}_\Gamma(\varkappa,\lambda,\upsilon)$, So from Definition 1, we have

$$(1-\varkappa)\frac{p(\sqsupset)}{\sqsupset} + \varkappa p\prime(\sqsupset) + \epsilon\sqsupset p''(\sqsupset) \prec \mathcal{K}(\upsilon,\sqsupset) \tag{9}$$

and

$$(1-\varkappa)\frac{h(\varpi)}{\varpi} + \varkappa h'(\varpi) + \epsilon\varpi h''(\varpi) \prec \mathcal{K}(\upsilon,\varpi). \tag{10}$$

We may think of two functions r_1, $r_2 : \Delta \to \Delta$, with $r_1(0) = r_2(0) = 0$ and $|r_1(\sqsupset)| < 1$, $|r_2(\varpi)| < 1$ for all $\sqsupset, \varpi \in \Delta$. So we can define γ, $\lambda \in \mathcal{D}$ as:

$$\gamma(\sqsupset) = \frac{s_1(\sqsupset)+1}{1-s_1(\sqsupset)} = 1 + \gamma_1\sqsupset + \gamma_2\sqsupset^2 + \gamma_3\sqsupset^3 + \cdots, \ |\gamma_i| \le 2, \ i \in \mathbb{N}.$$

$$\Rightarrow s_1(\sqsupset) = \frac{\gamma(\sqsupset)-1}{\gamma(\sqsupset)+1} = \frac{\gamma_1}{2}\sqsupset + \left(\frac{\gamma_2}{2} - \frac{\gamma_1^2}{4}\right)\sqsupset^2 + \frac{1}{2}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{4}\right)\sqsupset^3 + \cdots \tag{11}$$

and

$$\lambda(\varpi) = \frac{r_2(\varpi)+1}{1-r_2(\varpi)} = 1 + \lambda_1\varpi + \lambda_2\varpi^2 + \lambda_3\varpi^3 + \cdots, \ |\lambda_i| \le 2, \ i \in \mathbb{N}.$$

$$\Rightarrow r_2(\varpi) = \frac{\lambda(\varpi)-1}{\lambda(\varpi)+1} = \frac{\lambda_1}{2}\varpi + \left(\frac{\lambda_2}{2} - \frac{\lambda_1^2}{4}\right)\varpi^2 + \frac{1}{2}\left(\lambda_3 - \lambda_1\lambda_2 + \frac{\lambda_1^3}{4}\right)\varpi^3 + \cdots. \tag{12}$$

Using (11) and (12), we get

$$\mathcal{K}(\upsilon, s_1(\beth)) = \Phi_0(\upsilon) + \frac{\Phi_1(\upsilon)}{2}\gamma_1\beth + \left(\frac{\Phi_1(\upsilon)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Phi_2(\upsilon)}{8}\gamma_1^2\right)\beth^2 + \left(\begin{array}{c}\frac{\Phi_1(\upsilon)}{2}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{4}\right)\\ +\frac{\Phi_2(\upsilon)}{4}\left(\gamma_1\gamma_2 - \frac{\gamma_1^3}{2}\right) + \frac{\Phi_3(\upsilon)}{48}\gamma_1^3\end{array}\right)\beth^3 + \cdots \tag{13}$$

and

$$\mathcal{K}(\upsilon, s(\varpi)) = \Phi_0(\upsilon) + \frac{\Phi_1(\upsilon)}{2}\lambda_1\varpi + \left(\frac{\Phi_1(\upsilon)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Phi_2(\upsilon)}{8}\lambda_1^2\right)\varpi^2 + \left(\begin{array}{c}\frac{\Phi_1(\upsilon)}{2}\left(\lambda_3 - \lambda_1\lambda_2 + \frac{\lambda_1^3}{4}\right)\\ +\frac{\Phi_2(\upsilon)}{4}\left(\lambda_1\lambda_2 - \frac{\lambda_1^3}{2}\right) + \frac{\Phi_3(\upsilon)}{48}\lambda_1^3\end{array}\right)\varpi^3 + \cdots \tag{14}$$

From (9), (10) and the previous two equations, we have

$$(2\epsilon + \varkappa + 1)\, c_2 = \frac{\Phi_1(\upsilon)}{2}\gamma_1, \tag{15}$$

$$(6\epsilon + 2\varkappa + 1)\, c_3 = \frac{\Phi_1(\upsilon)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Phi_2(\upsilon)}{8}\gamma_1^2, \tag{16}$$

$$-(2\epsilon + \varkappa + 1)\, c_2 = \frac{\Phi_1(\upsilon)}{2}\lambda_1, \tag{17}$$

and

$$(6\epsilon + 2\varkappa + 1)\left(2c_2^2 - c_3\right) = \frac{\Phi_1(\upsilon)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Phi_2(\upsilon)}{8}\lambda_1^2. \tag{18}$$

Adding two equations (15) and (17) and some simplifying, we obtain

$$\gamma_1 = -\lambda_1 \text{ and } \gamma_1^2 = \lambda_1^2 \tag{19}$$

and

$$8\left(2\epsilon + \varkappa + 1\right)^2 c_2^2 = \Phi_1^2(\upsilon)(\gamma_1^2 + \lambda_1^2). \tag{20}$$

$$\Rightarrow c_2^2 = \frac{\Phi_1^2(\upsilon)(\gamma_1^2 + \lambda_1^2)}{8\left(2\epsilon + \varkappa + 1\right)^2} \tag{21}$$

Adding (16) to (18) gives

$$8\left(6\epsilon + 2\varkappa + 1\right) c_2^2$$

$$= 2\Phi_1(\upsilon)(\gamma_2 + \lambda_2) + (\gamma_1^2 + \lambda_1^2)\left(\frac{1}{2}\Phi_2(\upsilon) - \Phi_1(\upsilon)\right).$$

By (19), we have

$$\begin{aligned} &8\left(6\epsilon + 2\varkappa + 1\right)c_2^2 \\ &= 2\Phi_1(\upsilon)(\gamma_2 + \lambda_2) + \gamma_1^2\left(\Phi_2(\upsilon) - 2\Phi_1(\upsilon)\right) \end{aligned} \tag{22}$$

Also, appling (19) in (20)

$$\gamma_1^2 = \frac{4\left(2\epsilon + \varkappa + 1\right)^2 c_2^2}{\Phi_1^2(\upsilon)} \tag{23}$$

Replacing γ_1^2 in (22)

$$c_2^2 = \frac{\Phi_1^3(\upsilon)\left(\gamma_2 + \lambda_2\right)}{2\left[\begin{array}{c} 2\left(6\epsilon + 2\varkappa + 1\right)\Phi_1^2(\upsilon) \\ -\left(2\epsilon + \varkappa + 1\right)^2\left(\Phi_2(\upsilon) - 2\Phi_1(\upsilon)\right) \end{array}\right]} \tag{24}$$

$$\Rightarrow |c_2|^2 = \frac{\Phi_1^3(\upsilon)\left(|\gamma_2| + |\lambda_2|\right)}{2\left|\begin{array}{c} 2\left(6\epsilon + 2\varkappa + 1\right)\Phi_1^2(\upsilon) \\ -\left(2\epsilon + \varkappa + 1\right)^2\left(\Phi_2(\upsilon) - 2\Phi_1(\upsilon)\right) \end{array}\right|}$$

Applying (4) and Lemma 1, we obtain

$$\begin{aligned} |c_2| &\leq \sqrt{\frac{\left(2\upsilon - 1\right)^3}{2\left|\begin{array}{c} \left(6\epsilon + 2\varkappa + 1\right)\left(2\upsilon - 1\right)^2 \\ -2\left(2\epsilon + \varkappa + 1\right)^2\left(\upsilon^2 - 3\upsilon + 1\right) \end{array}\right|}} \\ &= \sqrt{F\left(\epsilon, \varkappa, \upsilon\right)}. \end{aligned}$$

Subtracting (18) from (16), then view (19) and after doing some calculations, we arrive at

$$c_3 = c_2^2 + \frac{\Phi_1(\upsilon)\left(\gamma_2 - \lambda_2\right)}{4\left(6\epsilon + 2\varkappa + 1\right)} \tag{25}$$

By (21) and (19)

$$c_3 = \frac{\Phi_1^2(\upsilon)\gamma_1^2}{4\left(2\epsilon + \varkappa + 1\right)^2} + \frac{\Phi_1(\upsilon)\left(\gamma_2 - \lambda_2\right)}{4\left(6\epsilon + 2\varkappa + 1\right)}. \tag{26}$$

Applying (4) and Lemma 1, we have:

$$|c_3| \leq \frac{\left(2\upsilon - 1\right)^2}{4\left(2\epsilon + \varkappa + 1\right)^2} + \frac{2\upsilon - 1}{2\left(6\epsilon + 2\varkappa + 1\right)}.$$

From (25), we obtain

$$c_3 - \zeta c_2^2 = \frac{\Phi_1(\upsilon)\left(\gamma_2 - \lambda_2\right)}{4\left(6\epsilon + 2\varkappa + 1\right)} + (1 - \zeta)c_2^2$$

By using assist (4) in conjunction with the triangular inequality, we arrive at:

$$\left|c_3 - \zeta c_2^2\right| \leq \frac{2\upsilon - 1}{2\left(6\epsilon + 2\varkappa + 1\right)} + \left|1 - \zeta\right| F\left(\epsilon, \varkappa, \upsilon\right)$$

If

$$\left|1 - \zeta\right| F\left(\epsilon, \varkappa, \upsilon\right) \leq \frac{2\upsilon - 1}{2\left(6\epsilon + 2\varkappa + 1\right)}$$

we obtain

$$\left|c_3 - \zeta c_2^2\right| \leq \frac{2\upsilon - 1}{6\epsilon + 2\varkappa + 1}$$

and if:

$$\left|1 - \zeta\right| F\left(\epsilon, \varkappa, \upsilon\right) \geq \frac{2\upsilon - 1}{2\left(6\epsilon + 2\varkappa + 1\right)}$$

we obtain

$$\left|c_3 - \zeta c_2^2\right| \leq 2\left|1 - \zeta\right| F\left(\epsilon, \varkappa, \upsilon\right)$$

Which the Theorem 1 asserts.

Theorem 2. *Let* $p \in \Gamma$ *of the form (1) in the class* $\mathcal{L}_\Gamma(\psi, \upsilon)$ *where* $-\pi < \psi \leq \pi$, $\frac{1}{2} < \upsilon \leq 1$ $\beth, \varpi \in \Delta$ *and* $h = p^{-1}$. *Then*

$$|c_2| \leq \sqrt{\Phi(\psi, \upsilon)},$$

$$|c_3| \leq \frac{(2\upsilon - 1)^2}{4\left(e^{i\psi} + 3\right)^2} + \frac{2\upsilon - 1}{6\left(e^{i\psi} + 2\right)}.$$

and

$$\left|c_3 - \zeta c_2^2\right| \leq \begin{cases} \frac{2\upsilon-1}{3\left(e^{i\psi}+2\right)} \text{ if } 0 \leq |1-\zeta|\,\Phi(\psi,\upsilon) < \frac{2\upsilon-1}{6\left(e^{i\psi}+2\right)}, \\ 2\,|1-\zeta|\,\Phi(\psi,\upsilon) \text{ if } |1-\zeta|\,\Phi(\psi,\upsilon) \geq \frac{2\upsilon-1}{6\left(e^{i\psi}+2\right)}. \end{cases}$$

where

$$\Phi(\psi, \upsilon) = \frac{(2\upsilon - 1)^3}{2\left|3\left(e^{i\psi} + 2\right)(2\upsilon - 1)^2 - 2\left(e^{i\psi} + 3\right)^2\left(\upsilon^2 - 3\upsilon + 1\right)\right|}.$$

Proof. Since $p(\beth) = \beth + \sum_{i=2}^{\infty} c_i \beth^i \in \mathcal{L}_\Gamma(\psi, \upsilon)$, So from equations (13), (14) and Definition 2, we are able to write

$$p'(\beth) + \beth\frac{e^{i\psi} + 1}{2}p''(\beth) \prec \mathcal{K}(\upsilon, \beth) \tag{27}$$

and

$$h'(\varpi)+\varpi\frac{e^{i\psi}+1}{2}h''(\varpi)\prec\mathcal{K}(\upsilon,\varpi). \tag{28}$$

By compare the coefficients in (27) and (28), where $\mathcal{K}(\upsilon,\daleth)$ and $\mathcal{K}(\upsilon,\varpi)$ respectively given by (13) and (14), we have

$$\left(e^{i\psi}+3\right)c_2=\frac{\Phi_1(\upsilon)}{2}\gamma_1, \tag{29}$$

$$3\left(e^{i\psi}+2\right)c_3=\frac{\Phi_1(\upsilon)}{2}\left(\gamma_2-\frac{\gamma_1^2}{2}\right)+\frac{\Phi_2(\upsilon)}{8}\gamma_1^2, \tag{30}$$

$$-\left(e^{i\psi}+3\right)c_2=\frac{\Phi_1(\upsilon)}{2}\lambda_1, \tag{31}$$

and

$$\begin{aligned}&3\left(e^{i\psi}+2\right)\left(2c_2^2-c_3\right)\\&=\frac{\Phi_1(\upsilon)}{2}\left(\lambda_2-\frac{\lambda_1^2}{2}\right)+\frac{\Phi_2(\upsilon)}{8}\lambda_1^2.\end{aligned} \tag{32}$$

By the same technique proving of Theorem 1, we get the results given by Theorem 2.

3. Some Corollaries

When we set $\varkappa=1$ in Theorems 1, the next corollary is revealed.

Corollary 1. *Let $p\in\Gamma$ given by (1) in the class $\mathcal{F}_\Gamma(1,\epsilon,\upsilon)$ where $\epsilon\geq0$, $\frac{1}{2}<\upsilon\leq1$ $\daleth,\varpi\in\Delta$ and $h=p^{-1}$. Then*

$$|c_2|\leq\sqrt{F(\epsilon,1,\upsilon)},$$

$$|c_3|\leq\frac{(2\upsilon-1)^2}{16(\epsilon+1)^2}+\frac{2\upsilon-1}{6(2\epsilon+1)}.$$

and

$$\begin{aligned}&\left|c_3-\zeta c_2^2\right|\\&\leq\begin{cases}\frac{2\upsilon-1}{3(2\epsilon+1)} & \text{if } 0\leq|1-\zeta|F(\epsilon,1,\upsilon)<\frac{2\upsilon-1}{6(2\epsilon+1)},\\ 2|1-\zeta|F(\epsilon,1,\upsilon) & \text{if } |1-\zeta|F(\epsilon,1,\upsilon)\geq\frac{2\upsilon-1}{6(2\epsilon+1)}.\end{cases}\end{aligned}$$

where

$$\begin{aligned}&F(\epsilon,1,\upsilon)\\&=\frac{(2\upsilon-1)^3}{2\left|3(2\epsilon+1)(2\upsilon-1)^2-8(\epsilon+1)^2(\upsilon^2-3\upsilon+1)\right|}.\end{aligned}$$

When we set $\epsilon = 0$ in Theorems 1, the next corollary is revealed.

Corollary 2. *Let $p \in \Gamma$ given by (1) in the class $\mathcal{F}_{\Gamma}(\varkappa, 0, \upsilon)$ where $\varkappa \geq 1$, $\frac{1}{2} < \upsilon \leq 1$ $\beth, \varpi \in \Delta$ and $h = p^{-1}$. Then*

$$|c_2| \leq \sqrt{\digamma(0, \varkappa, \upsilon)},$$

$$|c_3| \leq \frac{(2\upsilon - 1)^2}{4(\varkappa + 1)^2} + \frac{2\upsilon - 1}{2(2\varkappa + 1)}.$$

and

$$\left|c_3 - \zeta c_2^2\right| \leq \begin{cases} \frac{2\upsilon-1}{2\varkappa+1} \text{ if } 0 \leq |1 - \zeta| \digamma(0, \varkappa, \upsilon) < \frac{2\upsilon-1}{2(2\varkappa+1)}, \\ 2|1 - \zeta| \digamma(0, \varkappa, \upsilon) \text{ if } |1 - \zeta| \digamma(0, \varkappa, \upsilon) \geq \frac{2\upsilon-1}{2(2\varkappa+1)}. \end{cases}$$

where

$$\digamma(0, \varkappa, \upsilon) = \frac{(2\upsilon - 1)^3}{2\left|(2\varkappa + 1)(2\upsilon - 1)^2 - 2(\varkappa + 1)^2(\upsilon^2 - 3\upsilon + 1)\right|}.$$

When $\epsilon = 0$ in Corollary 1 or $\psi = \pi$ in Theorems 2 simplifies to the following corollary.

Corollary 3. *Let $p \in \Gamma$ given by (1) in the class $\mathcal{F}_{\Gamma}(1, 0, \upsilon) \equiv \mathcal{L}_{\Gamma}(\pi, \upsilon)$ where $\frac{1}{2} < \upsilon \leq 1$ $\beth, \varpi \in \Delta$ and $h = p^{-1}$. Then*

$$|c_2| \leq \sqrt{\digamma(0, 1, \upsilon)},$$

$$|c_3| \leq \frac{(2\upsilon - 1)^2}{16} + \frac{2\upsilon - 1}{6}.$$

and

$$\left|c_3 - \zeta c_2^2\right| \leq \begin{cases} \frac{2\upsilon-1}{3} \text{ if } 0 \leq |1 - \zeta| \digamma(0, 1, \upsilon) < \frac{2\upsilon-1}{6}, \\ 2|1 - \zeta| \digamma(0, 1, \upsilon) \text{ if } |1 - \zeta| \digamma(0, 1, \upsilon) \geq \frac{2\upsilon-1}{6}. \end{cases}$$

where

$$\digamma(0, 1, \upsilon) = \frac{(2\upsilon - 1)^3}{2|4\upsilon^2 + 12\upsilon - 5|}.$$

When we set $\psi = 0$ in Theorems 2, the next corollary is revealed.

Corollary 4. *Let $p \in \Gamma$ given by (1) in the class $\mathcal{L}_{\Gamma}(0, \upsilon)$ where $\frac{1}{2} < \upsilon \leq 1$ $\beth, \varpi \in \Delta$ and $h = p^{-1}$. Then*

$$|c_2| \leq \sqrt{\Phi(0, \upsilon)},$$

$$|c_3| \leq \frac{(2\upsilon - 1)^2}{64} + \frac{2\upsilon - 1}{18}.$$

and

$$\left|c_3 - \zeta c_2^2\right|$$
$$\leq \begin{cases} \frac{2\upsilon-1}{9} \text{ } if \text{ } 0 \leq |1-\zeta| \, \Phi(0,\upsilon) < \frac{2\upsilon-1}{18}, \\ 2\,|1-\zeta| \, \Phi(0,\upsilon) \text{ } if \text{ } |1-\zeta| \, \Phi(0,\upsilon) \geq \frac{2\upsilon-1}{18}. \end{cases}$$

where

$$\Phi(0,\upsilon) = \frac{(2\upsilon-1)^3}{2\,|4\upsilon^2 + 60\upsilon - 23|}.$$

4. Conclusions

polynomials and special functions are employed in so many different mathematical and scientific domains, many eminent mathematicians have recently studied them. This paper is concerned with defining new subclasses of analytical and univalent functions by the use of Euler polynomials. For functions in these classes $\mathcal{F}_\Gamma(\varkappa, \epsilon, \upsilon)$ and $\mathcal{L}_\Gamma(\psi, \upsilon)$, We obtained an upper bound for the coefficients and solved the Fekete-Szegö problem. It remains a challenge to fined the upper bounds for $|c_2|$, $|c_3|$ and $\left|c_3 - \zeta c_2^2\right|$, and still an interesting open problem for $|c_i|$, $i \geq 3$.

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