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$\delta(\tau_1, \tau_2)$-Continuous Functions

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Abstract. This paper introduces a new class of functions called $\delta(\tau_1, \tau_2)$-continuous functions. Several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions are investigated. The relationships between $\delta(\tau_1, \tau_2)$-continuity and the other types of $\delta(\tau_1, \tau_2)$-continuity are also discussed.

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1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. By using various forms of open sets many authors introduced and studied various types of continuity. In 1968, Veličko [22] introduced a new class of open sets in topological spaces called δ-open sets and investigated some properties of δ-closed sets and δ-open sets. The class of open sets including the class of δ-open sets. In 1980, Noiri [17] introduced and studied the notion of δ-continuous functions. Munshi and Bassan [16] defined and developed the concept of super-continuity. The concept has been investigated further by Reilly and Vamanamurthy [21] where super-continuity is characterized in terms of the semi-regularization topology. Super-continuity is related to the concepts of δ-continuity and strong θ-continuity developed by Noiri [17]. In particular, super-continuity is strictly between strong θ-continuity and δ-continuity and strictly between complete continuity [1] and δ-continuity. Raychaudhuri and Mukherjee [20] introduced the concept of δ-preopen sets which is weaker than that of preopen sets and used this concept to define the notion of δ-almost continuous functions as a generalization of precontinuous functions due to Mashhour et al. [15]. Baker [2] introduced and investigated the notion of weakly δ-continuous functions. The class of weakly δ-continuous functions is a generalization of δ-continuous functions.

In 1997, Park et al. [19] introduced the notion of δ-semiopen sets by using δ-open sets due to Valičko [22]. In 2005, Ekici and Navalagi [13] introduced and investigated

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the concept of δ-semicontinuous functions. The class of δ-semicontinuous functions is a weaker form of the classes of perfectly continuous functions [18], strongly θ-continuous functions [14] and super-continuous functions. Ekici [12] introduced the notion of almost δ-semicontinuous functions which generalize R-maps [11] and δ-continuous functions. Yüksel et al. [25] extended the concept of δ-open sets to ideal topological spaces and defined δ-I-continuous functions. Moreover, some characterizations of θ-$\mathscr{I}$-continuous functions, $\star$-continuous functions and $\theta(\star)$-precontinuous functions were presented in [3], [5] and [6], respectively. In [9], the present authors introduced and studied the concept of (τ_1, τ_2)-continuous functions. Futhermore, several characterizations of almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were established in [8] and [7], respectively. In this paper, we introduce the notions of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. We also investigate several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. Finally, the relationships among $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [10] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [10] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [10] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [23] (resp. $(\tau_1, \tau_2)s$-*open* [4], $(\tau_1, \tau_2)p$-*open* [4], $(\tau_1, \tau_2)\beta$-*open* [4], $\alpha(\tau_1, \tau_2)$-*open*) [24]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*, $\alpha(\tau_1, \tau_2)$-*closed*). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point x of X is called a $\delta(\tau_1, \tau_2)$-*cluster point* of A if $V \cap A \neq \emptyset$ for every $(\tau_1, \tau_2)r$-open set V containing x. The set of all $\delta(\tau_1, \tau_2)$-cluster points of A is called the $\delta(\tau_1, \tau_2)$-*closure* of A and is denoted by $\delta(\tau_1, \tau_2)$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is called $\delta(\tau_1, \tau_2)$-*closed* if $A = \delta(\tau_1, \tau_2)$-Cl$(A)$. The complement of a $\delta(\tau_1, \tau_2)$-closed set is called $\delta(\tau_1, \tau_2)$-*open* ($\tau_1\tau_2$-δ-open [8]). The family of all $\delta(\tau_1, \tau_2)$-open (resp. $\delta(\tau_1, \tau_2)$-closed) sets of a bitopological space (X, τ_1, τ_2) is denoted by $\delta(\tau_1, \tau_2)O(X)$ (resp. $\delta(\tau_1, \tau_2)C(X)$). The $\delta(\tau_1, \tau_2)$-*interior* of A denoted by

$\delta(\tau_1, \tau_2)$-Int(A) is defined as follows:

$$\delta(\tau_1, \tau_2)\text{-Int}(A) = \cup\{G \subseteq X \mid G \in \delta(\tau_1, \tau_2)O(X) \text{ and } G \subseteq A\}.$$

Lemma 1. *For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$ if and only if $V \cap A \neq \emptyset$ for every $V \in \delta(\tau_1, \tau_2)O(X)$ containing x.*

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\delta(\tau_1, \tau_2)$-Int$(X - A) = X - \delta(\tau_1, \tau_2)$-Cl$(A)$.

(2) $\delta(\tau_1, \tau_2)$-Cl$(X - A) = X - \delta(\tau_1, \tau_2)$-Int$(A)$.

3. On $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the notion of $\delta(\tau_1, \tau_2)$-continuous functions. We also discuss several characterizations of $\delta(\tau_1, \tau_2)$-continuous functions.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $\delta(\tau_1, \tau_2)$-continuous if f is $\delta(\tau_1, \tau_2)$-continuous at each point of X.*

Example 1. *Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3\}$ with topologies $\sigma_1 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, Y\}$ and $\sigma_2 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, Y\}$. Define a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as follows: $f(a) = f(c) = 2$ and $f(b) = 3$. Then, f is $\delta(\tau_1, \tau_2)$-continuous.*

Theorem 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\delta(\tau_1, \tau_2)$-continuous at x;

(2) $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(3) $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(f(A)))$ for every $A \subseteq X$ such that $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$;

(4) $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every $B \subseteq Y$ such that $x \in \delta(\tau_1, \tau_2)$-Cl$(f^{-1}(B))$;

(5) $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(B))$ for every $B \subseteq Y$ such that $x \in f^{-1}(\sigma_1\sigma_2$-Int$(B))$;

(6) $x \in f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y such that $x \in \delta(\tau_1, \tau_2)$-Cl$(f^{-1}(F))$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), There exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. Hence, $x \in U \subseteq f^{-1}(V)$. Therefore, $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ such that $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open

set U of X containing x such that $U \subseteq f^{-1}(V)$. By Lemma 1, we have $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. Thus, $f(x) \in \sigma_1\sigma_2\text{-Cl}(f(A))$ and so $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(f(A)))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y and $x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B))$. Then by (3), we have $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y and $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$. Suppose that $x \notin \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$. Then, $x \in X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$. Since

$$X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B)) = \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - B))$$

and by (4), we obtain that $x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) = X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$ and hence $x \notin f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$, which is a contradiction that $x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(B))$. Therefore, $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$.

(5) $\Rightarrow$ (6): Let F be any $\sigma_1\sigma_2$-closed set of Y and $x \in \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$. Suppose that $x \notin f^{-1}(F)$. Since $Y - F$ is $\sigma_1\sigma_2$-open in Y, $x \in X - f^{-1}(F) = f^{-1}(Y - F) = f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - F))$. By (5) and Lemma 2 (1), we have

$$x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - F)) = X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$$

and hence $x \notin \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$. This is a contradiction. Therefore, $x \in f^{-1}(F)$.

(6) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, $x \in f^{-1}(V)$. Suppose that $x \notin \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. By Lemma 2 (2), $x \in X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V)) = \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - V))$. Since $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (6), we have

$$x \in f^{-1}(Y - V) = X - f^{-1}(V).$$

This implies that $x \notin f^{-1}(V)$, which is a contradiction. Thus, $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$.

(2) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have $x \in \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U such that $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$. This shows that f is $\delta(\tau_1, \tau_2)$-continuous at x.

Theorem 2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\delta(\tau_1, \tau_2)$-continuous;

(2) $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f(\delta(\tau_1, \tau_2)\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(f(A))$ for every subset A of X;

(4) $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$ for every subset B of Y;

(6) $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X for every $\sigma_1\sigma_2$-closed set F of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open of Y and $x \in f^{-1}(V)$. By (1), there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$ and hence $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. This implies that $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X.

(2) $\Rightarrow$ (3): Let A be any subset of X and let $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$. Then, we have $f(x) \in f(\delta(\tau_1, \tau_2)\text{-Cl}(A))$. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X. Therefore, $x \in f^{-1}(V) = \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Since $x \in \delta(\tau_1, \tau_2)$-Cl$(A)$, then $U \cap A \neq \emptyset$. Hence, $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. Thus, $f(x) \in \sigma_1\sigma_2$-Cl$(f(A))$ and so $f(\delta(\tau_1, \tau_2)\text{-Cl}(A)) \subseteq \sigma_1\sigma_2$-Cl$(f(A))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then by (3), $f(\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B))) \subseteq \sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B))) \subseteq \sigma_1\sigma_2$-Cl$(B)$. Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4) and Lemma 2 (2), we have

$$\begin{aligned} X - \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B)) &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and hence $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(B))$.

(5) $\Rightarrow$ (6): Let F be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - F$ is $\sigma_1\sigma_2$-open in Y. By (5) and Lemma 2 (1), we obtain that $X - f^{-1}(F) = f^{-1}(Y - F) = f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - F)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - F)) = X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F))$. Therefore, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)) \subseteq f^{-1}(F)$. This shows that $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X.

(6) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (6), we have $X - f^{-1}(V) = f^{-1}(Y - V)$ is $\delta(\tau_1, \tau_2)$-closed in X. Thus, $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X.

(2) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X. Hence, $x \in f^{-1}(V) = \delta(\tau_1, \tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U such that $x \in U \subseteq f^{-1}(V)$. Therefore, $f(U) \subseteq V$. Thus, f is $\delta(\tau_1, \tau_2)$-continuous at x. This shows that f is $\delta(\tau_1, \tau_2)$-continuous.

4. On almost $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the notion of almost $\delta(\tau_1, \tau_2)$-continuous functions and investigate some characterizations of almost $\delta(\tau_1, \tau_2)$-continuous functions. Moreover, the relationships between $\delta(\tau_1, \tau_2)$-continuous functions and almost $\delta(\tau_1, \tau_2)$-continuous functions are considered.

Definition 2. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called almost $\delta(\tau_1, \tau_2)$-continuous if f is almost $\delta(\tau_1, \tau_2)$-continuous at each point of X.*

Remark 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

$$\delta(\tau_1, \tau_2)\text{-}continuity \Rightarrow almost\ \delta(\tau_1, \tau_2)\text{-}continuity.$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 2. *Let $X = \{1, 2, 3, 4\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and*

$$\tau_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X\}.$$

Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and

$$\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}.$$

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $f(1) = a$, $f(2) = b$ and $f(3) = f(4) = c$. Then f is almost $\delta(\tau_1, \tau_2)$-continuous, but f is not $\delta(\tau_1, \tau_2)$-continuous.

Theorem 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous at x;

(2) $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)))) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(3) $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(V)$) for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$;

(4) for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is almost $\delta(\tau_1, \tau_2)$-continuous at $x \in X$. There exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). Thus, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$))). Therefore, $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)))).

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then, V is $\sigma_1\sigma_2$-open in Y. By (2), $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)))) $= \delta(\tau_1, \tau_2)$-Int($f^{-1}(V)$).

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Thus by (3), we have $x \in \delta(\tau_1, \tau_2)$-Int($f^{-1}(V)$). Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Therefore, $f(U) \subseteq V$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) is a $(\sigma_1, \sigma_2)r$-open set and by (4), there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). Consequently, f is almost $\delta(\tau_1, \tau_2)$-continuous at x.

Theorem 4. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\delta(\tau_1,\tau_2)$-continuous;

(2) $f^{-1}(V) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\delta(\tau_1,\tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(F)))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed set F of Y;

(4) $\delta(\tau_1,\tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B))))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B)))))$ for every subset B of Y;

(6) $f^{-1}(V)$ is $\delta(\tau_1,\tau_2)$-open in X for every $(\sigma_1,\sigma_2)r$-open set V of Y;

(7) $f^{-1}(F)$ is $\delta(\tau_1,\tau_2)$-closed in X for every $(\sigma_1,\sigma_2)r$-closed set F of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Since f is almost $\delta(\tau_1,\tau_2)$-continuous, there exists a $\delta(\tau_1,\tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Then, we have $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ and hence $x \in \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$. This implies that

$$f^{-1}(V) \subseteq \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(2) $\Rightarrow$ (3): Let F be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y-F$ is $\sigma_1\sigma_2$-open in Y. Thus by (2), we have $X - f^{-1}(F) = f^{-1}(Y-F) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(Y-F)))) = \delta(\tau_1,\tau_2)$-Int$(f^{-1}(Y-\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(F)))) = X-\delta(\tau_1,\tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(F))))$ and hence $\delta(\tau_1,\tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y and by (3), $\delta(\tau_1,\tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B))))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then by (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y-B)) \\ &\subseteq X - \delta(\tau_1,\tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y-B))))) \\ &= \delta(\tau_1,\tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

Therefore, $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B)))))$.

(5) $\Rightarrow$ (6): Let V be any $(\sigma_1,\sigma_2)r$-open set of Y. Then, V is a $\sigma_1\sigma_2$-open set of Y and $V = \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(V)))$. By (5), $f^{-1}(V) = f^{-1}(\sigma_1\sigma_2$-Int$(V)) \subseteq \delta(\tau_1,\tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(V))))) = \delta(\tau_1,\tau_2)$-Int$(f^{-1}(V))$ and so $f^{-1}(V)$ is $\delta(\tau_1,\tau_2)$-open in X.

(6) $\Rightarrow$ (7): The proof is obvious.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any $(\sigma_1,\sigma_2)r$-open set of Y containing $f(x)$. Then, $Y-V$ is $(\sigma_1,\sigma_2)r$-closed in Y. Thus by (7), we have $X - f^{-1}(V) = f^{-1}(Y-V) = \delta(\tau_1,\tau_2)$-Cl$(f^{-1}(Y-V)) = X-\delta(\tau_1,\tau_2)$-Int$(f^{-1}(V))$ and hence $x \in \delta(\tau_1,\tau_2)$-Int$(f^{-1}(V))$. Then, there exists a $\delta(\tau_1,\tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$. By Theorem 3 (4), f is almost $\delta(\tau_1,\tau_2)$-continuous at x. This shows that f is almost $\delta(\tau_1,\tau_2)$-continuous.

Theorem 5. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\delta(\tau_1, \tau_2)$-continuous;

(2) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y. By Theorem 4 (7), $f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ is $\delta(\tau_1, \tau_2)$-closed in X. Hence, $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V)) \subseteq \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) $\Rightarrow$ (3): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (1): Let F be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, F is $(\sigma_1, \sigma_2)s$-open in Y. By (3), we have $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(F)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is $\delta(\tau_1, \tau_2)$-closed in X. By Theorem 4 (7), f is almost $\delta(\tau_1, \tau_2)$-continuous.

(1) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open in Y. By Theorem 4 (6), $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\delta(\tau_1, \tau_2)$-open in X. Thus, $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ and hence $f^{-1}(V) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(4) $\Rightarrow$ (1) : Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and V is $(\sigma_1, \sigma_2)p$-open in Y. By (4), $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\delta(\tau_1, \tau_2)$-open in X. By Theorem 4 (6), f is almost $\delta(\tau_1, \tau_2)$-continuous.

5. On weakly $\delta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce and investigate the concept of weakly $\delta(\tau_1, \tau_2)$-continuous functions. Furthermore, we discuss the relationships between almost $\delta(\tau_1, \tau_2)$-continuous functions and weakly $\delta(\tau_1, \tau_2)$-continuous functions.

Definition 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called weakly $\delta(\tau_1, \tau_2)$-continuous if f is weakly $\delta(\tau_1, \tau_2)$-continuous at each point of X.*

Remark 2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

almost $\delta(\tau_1, \tau_2)$-continuity $\Rightarrow$ weak $\delta(\tau_1, \tau_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3. *Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3\}$ with topologies $\sigma_1 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, Y\}$ and $\sigma_2 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, Y\}$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $f(a) = f(b) = 2$ and $f(c) = 1$. Then f is weakly $\delta(\tau_1, \tau_2)$-continuous, but f is not almost $\delta(\tau_1, \tau_2)$-continuous.*

Theorem 6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is weakly $\delta(\tau_1, \tau_2)$-continuous at x, so there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Therefore, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$. This implies that $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

Conversely let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By the hypothesis, we obtain that $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set U of X containing x such that $U \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$. Thus, $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. This shows that f is weakly $\delta(\tau_1, \tau_2)$-continuous at x.

Theorem 7. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly $\delta(\tau_1, \tau_2)$-continuous if and only if $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Since f is weakly $\delta(\tau_1, \tau_2)$-continuous, by Theorem 6, $x \in \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. Therefore, $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

Conversely, let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By the hypothesis, $x \in f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. By Theorem 6, f is weakly $\delta(\tau_1, \tau_2)$-continuous at x. This shows that f is weakly $\delta(\tau_1, \tau_2)$-continuous.

Theorem 8. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\delta(\tau_1, \tau_2)$-continuous;

(2) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $\sigma_1\sigma_2$-closed F of Y;

(4) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B))))$ for every subset B of Y;

(6) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(F))) \subseteq f^{-1}(F)$ for every $(\sigma_1, \sigma_2)r$-closed set F of Y;

(8) $\delta(\tau_1, \tau_2)$-*Cl*$(f^{-1}(\sigma_1\sigma_2$-*Int*$(\sigma_1\sigma_2$-*Cl*$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(9) $\delta(\tau_1, \tau_2)$-*Cl*$(f^{-1}(\sigma_1\sigma_2$-*Int*$(\sigma_1\sigma_2$-*Cl*$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*;*

(10) $\delta(\tau_1, \tau_2)$-*Cl*$(f^{-1}(\sigma_1\sigma_2$-*Int*$(\sigma_1\sigma_2$-*Cl*$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(11) $\delta(\tau_1, \tau_2)$-*Cl*$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(12) $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-*Int*$(f^{-1}(\sigma_1\sigma_2$-*Cl*$(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y. It follows from Theorem 7, we obtain that $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) $\Rightarrow$ (3): Let F be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - F$ is $\sigma_1\sigma_2$-open in Y. Thus by (2), we have

$$\begin{aligned} X - f^{-1}(F) = f^{-1}(Y - F) &\subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - F))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(F))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \\ &= X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \end{aligned}$$

and hence $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Since $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3), $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then by (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

Thus, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) $\Rightarrow$ (6): Let V be any $\sigma_1\sigma_2$-open set of Y. Suppose that $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, $f(x) \notin \sigma_1\sigma_2\text{-Cl}(V)$. There exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap V = \emptyset$. Hence, $\sigma_1\sigma_2\text{-Cl}(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) = f^{-1}(\sigma_1\sigma_2\text{-Int}(U)) \subseteq \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(U)))) = \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(U)))$. Then, there exists a $\delta(\tau_1, \tau_2)$-open set G of X such that $x \in G \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(U))$. Therefore,

$$f^{-1}(V) \cap G \subseteq f^{-1}(V) \cap f^{-1}(\sigma_1\sigma_2\text{-Cl}(U)) = f^{-1}(V \cap \sigma_1\sigma_2\text{-Cl}(U)) = \emptyset.$$

Thus, $x \notin \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V))$ and hence $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(6) $\Rightarrow$ (7): Let F be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Int}(F)$ is $\sigma_1\sigma_2$-open in Y. By (6), $\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(F))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F))) = f^{-1}(F)$.

(7) $\Rightarrow$ (8): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and so $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed. By (7), $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$.

(8) $\Rightarrow$ (9): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(9) $\Rightarrow$ (10): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ and $\sigma_1\sigma_2$-Cl$(V) \subseteq \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Therefore, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)s$-open in Y. Thus by (9),

$$\begin{aligned}
\delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(V))))) \\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(V))) \\
&= f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).
\end{aligned}$$

(10) $\Rightarrow$ (11): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (10), $\delta(\tau_1, \tau_2)$-Cl$(f^{-1}(V)) \subseteq \delta(\tau_1, \tau_2)$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$.

(11) $\Rightarrow$ (12): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (11),

$$\begin{aligned}
f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\
&= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&\subseteq X - \delta(\tau_1, \tau_2)\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&= \delta(\tau_1, \tau_2)\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).
\end{aligned}$$

(12) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (12), $f^{-1}(V) \subseteq \delta(\tau_1, \tau_2)$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. It follows from Theorem 7 that f is weakly $\delta(\tau_1, \tau_2)$-continuous.

6. Conclusion

This paper deals with the concepts of $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions. Moreover, some characterizations and several properties concerning $\delta(\tau_1, \tau_2)$-continuous functions, almost $\delta(\tau_1, \tau_2)$-continuous functions, and weakly $\delta(\tau_1, \tau_2)$-continuous functions are obtained. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implications hold:

$$\delta(\tau_1, \tau_2)\text{-continuity} \Rightarrow \text{almost } \delta(\tau_1, \tau_2)\text{-continuity} \Rightarrow \text{weak } \delta(\tau_1, \tau_2)\text{-continuity}.$$

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