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Every Uniquely Remotal Set in a Hilbert Space is a Singleton

Roshdi Khalil[1,*], Abdelrahman Yousef[2], Waseem Ghazi Alshanti[3], Ma′mon Abu Hammad[3]

[1] *Department of Mathematics, The University of Jordan, Queen Rania St, Amman, 11942, Jordan*
[2] *Department of Mathematics and Statistics, American University of Sharjah, P.O.Box 26666, Sharjah, United Arab Emirates*
[3] *Department of Mathematics, Al Zaytoonah University of Jordan, Queen Alia Airport St 594, Amman, 11733, Jordan*

Abstract. In this paper we prove that every uniquely remotal set in a Hilbert space is a singleton.

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1. Introduction

Let H be a Hilbert space, and let $E \subseteq H$ be a closed and bounded subset of H. The real-valued function $D(\cdot, E) : H \to \mathbb{R}$ defined by

$$D(x, E) := \sup_{e \in E} \|x - e\|, \quad \text{for } x \in H, \tag{1}$$

is referred to as the farthest distance function. The set E is termed remotal if, for every $x \in H$, there exists an $e \in E$ such that $D(x, E) = \|x - e\|$. In this case, the set

$$P(x, E) := \{e \in E : D(x, E) = \|x - e\|\}$$

is denoted as $P(x, E)$. Clearly, $P(\cdot, E) : H \to 2^E$ is a multi-valued function. However, if $P(\cdot, E) : H \to 2^E$ is single-valued, meaning $P(x, E)$ is a singleton for all $x \in H$, then E is called uniquely remotal.

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*Corresponding author.
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Email addresses: `roshdi@ju.edu.jo` (R. Khalil), `afyousef@aus.edu` (A. Yousef), `w.alshanti@zuj.edu.jo` (W. G. Alshanti), `m.abuhammad@zuj.edu.jo` (M. Abu Hammad)

The study of remotal and uniquely remotal sets has garnered significant interest over the past few decades due to their connections with the geometry of Hilbert and Banach spaces [2–15]. Among these, uniquely remotal sets hold particular significance. Indeed, one of the most intriguing open problems in functional analysis is the well-known farthest point problem, which conjectures: "Every uniquely remotal set in H is a singleton". This is the central focus of this paper.

This problem has persisted for over 70 years. Its significance was further highlighted when Klee [9] established the equivalence of the following two statements:

(i) Every uniquely remotal set is a singleton.

(ii) Every uniquely proximinal set in a Hilbert space H is convex.

Since Klee's result, considerable progress has been made toward resolving this question, with many partial results supporting the conjecture.

In [1], the following theorem was established:

Theorem 1 ([1]). *Every uniquely remotal set that is also uniquely distant is a singleton.*

This result represents the closest attempt to date to resolve the farthest point problem.

2. Preliminaries

In [1], the following definition and result were presented:

Definition 1 ([1]). *Let H be a Hilbert space, and let $E \subseteq H$ be a closed and bounded subset. Then E is said to be a* ***uniquely distant set*** *in H if the following two conditions are satisfied:*

(i) E is uniquely remotal,

(ii) If $x \in H$ and y is the farthest point from $x \in E$, then for every $\epsilon > 0$, there exists a $\delta > 0$ such that $D(x, E \setminus B(y, \delta)) \leq D(x, E) - \epsilon$.

The main result in [1] was as follows:

Theorem 2 ([1]). *Every uniquely distant set in H is a singleton.*

In this paper, we prove that Theorem 2 holds true for uniquely remotal sets in H, without requiring condition (ii) in Definition 1. This effectively proves the farthest point conjecture.

3. Main Result

In this section, using the result from [1], we prove the following: **Every uniquely remotal set in H is a singleton**. Consequently, by Klee's result [14], we also obtain the result: **Every uniquely proximinal set in a Hilbert space is convex.**

Theorem 3. *Every convex uniquely remotal set in a Hilbert space H is uniquely distant.*

Proof. Consider the statement Q:

"For every $\epsilon > 0$, there exists a $\delta > 0$ such that $D(x, E \setminus B(y, \delta)) \leq D(x, E) - \epsilon$."

Here, $x \in H$ and $y = P(x, E)$.

To prove the theorem, we show that Q holds for any uniquely remotal set $E \subset H$. This is achieved by demonstrating that the negation of Q, denoted $\sim Q$, is false for any uniquely remotal set in H.

Let $E \subseteq H$ be a closed, convex, and bounded subset that is uniquely remotal. By Definition 1, uniquely distant implies that if $x \in H$, and y is the farthest point from x in E, then

$$D(x, E) = r = \|x - y\| . \tag{2}$$

Moreover, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $D(x, E \setminus B(y, \delta)) \leq D(x, E) - \epsilon = r - \epsilon$.

Now, $\sim Q$ states that there exists an ϵ, with $0 < \epsilon < r$, such that for every $\delta > 0$, one has $D(x, E \setminus B(y, \delta)) > r - \epsilon$.

Assume without loss of generality that x belongs to the boundary of E, and let $D(y, E) = s$. Then, clearly $s \geq r$. Define $\delta_n = s - \frac{1}{n}$, and set $B_n = B(y, \delta_n)$. Let $E_n = E \setminus B_n$. Then, it is evident that:

$$B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subseteq \ldots$$

and

$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots$$

Further, we have:

$$\cup B_n = B(y, s) \text{ and } \cap E_n = \emptyset \tag{3}$$

Now, $D(x, E_n) \geq D(x, E_{n+1})$ for all n. Thus, by (3), we have $D(x, E_n) \to 0$ as $n \to \infty$. It follows that

$$D(x, E \setminus B_n(y, \delta)) = D(x, E_n) > r - \epsilon > 0, \quad \text{for all } n. \tag{4}$$

Hence,

$$\lim_{n \to \infty} D(x, E_n) > r - \epsilon > 0$$

So, we obtain:

$$\lim D(x, E_n) = 0 > r - \epsilon > 0$$

Thus, $0 > 0$, which is a contradiction. Therefore, $\sim Q$ is false, and Q is true. This implies that every uniquely remotal set in H is a uniquely distant set.

A key corollary of this result is:

Corollary 1. *Every uniquely remotal set in H is a singleton.*

Proof. The corollary follows directly from Theorem 2 and Theorem 3.

Remark 1. *It is evident that constructing a concrete example where a uniquely remotal set in a Hilbert space is a singleton is challenging. However, we provide a straightforward example to help the reader grasp the concept more easily and to illustrate the underlying principles.*

The following example illustrates that a subset of $\mathbb{R}^2$ containing two elements cannot be uniquely remotal.

Example 1. *Let $M = \{x, y\}$, where $x \neq y$, be any arbitrary subset of the Hilbert space $\mathbb{R}^2$. If M is uniquely remotal, then for $z = \frac{x+y}{2}$ we obtain the following*

$$||x - z|| = ||\frac{x-y}{2}||$$

and

$$||y - z|| = ||\frac{x-y}{2}||$$

This implies that $D(z, M) = ||z - x|| = ||z - y||$, which contradicts the fact that M is uniquely remotal.

4. Conclusion

This paper presents a proof for one of the well-known longstanding open problems, known as the Farthest Point Problem. We have shown that every uniquely remotal set in a Hilbert space is a singleton.

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