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Connected Co-Independent Hop Domination in the Edge Corona and Complementary Prism of Graphs

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Abstract. Let G be a connected graph. A subset S of $V(G)$ is a connected co-independent hop dominating set in G if the subgraph induced by S is connected and $V(G)\backslash S$ is an independent set where for each $v \in V(G)\backslash S$, there exists a vertex $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of such an S is called the connected co-independent hop domination number of G. Here, authors presented the characterizations of the connected co-independent hop dominating sets in the edge corona and complementary prism of graphs and determines the exact values of their corresponding connected co-independent hop domination number.

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1. Introduction

Domination in graphs was first introduced by C. Berge in 1958 [2]. There are now many studies involving domination and its variations. One of its variation is the connected co-independent domination number of graphs introduced by Gayathri and Kaspar in 2010 [3] and further studied in [1, 10]. Recently, Natarajan and Ayyaswamy [7] introduced and studied the concept of hop domination in graphs. Hop domination in graphs were also studied in [5, 6, 8, 9, 11]. In [6], the connected co-independent hop dominating sets of a graph is defined and studied under the join, corona and lexicographic product of graphs.

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2. Preliminaries

All graphs considered in this study are finite, simple, undirected and connected. Some necessary definitions are presented in this section. Readers are referred to [4] for elementary Graph Theoretic concepts.

Definition 1. An *independent set* S in a graph G is a subset of the vertex-set of G such that no two vertices in S are adjacent in G. The cardinality of a maximum independent set is called the *independence number* of G and is denoted by $\beta(G)$. An independent set $S \subseteq V(G)$ with $|S| = \beta(G)$ is called a β-set of G.

Definition 2. A *perfect matching* of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.

Definition 3. A dominating set $D \subseteq V(G)$ is called a *connected co-independent dominating set* of G if D is a connected dominating set of G and $V(G) \setminus D$ is an independent set. The cardinality of such a minimum set D is called a *connected co-independent domination number* of G denoted by $\gamma_{c,coi}(G)$. A connected co-independent dominating set D with $|D| = \gamma_{c,coi}(G)$ is called a $\gamma_{c,coi}$-set of G.

Definition 4. A set $S \subseteq V(G)$ is a *hop dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

Definition 5. A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G(u, 2) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u in G is given by $N_G[u, 2] = N_G(u, 2) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G(X, 2) = \bigcup_{u \in X} N_G(u, 2)$. The *closed hop neighborhood* of X in G is the set $N_G[X, 2] = N_G(X, 2) \cup X$.

Definition 6. A set $S \subseteq V(G)$ is a *connected co-independent set* of G if $\langle S \rangle$ is connected and $V(G)\backslash S$ is independent.

Definition 7. A subset S of $V(G)$ is a *strictly co-independent set* of G if $V(G)\backslash S$ is an independent set and $N_G(v) \cap S \neq S$ for all $v \in V(G)\backslash S$. The minimum cardinality of a strictly co-independent set in G, denoted by $sci(G)$ is called the *strictly co-independent number* of G. A strictly co-independent set S with $|S| = sci(G)$ is called an *sci*-set of G.

Definition 8. Let G be a connected graph. A hop dominating set $S \subseteq V(G)$ is a *connected co-independent hop dominating set* of G if $\langle S \rangle$ is connected and $V(G)\backslash S$ is an independent set. The minimum cardinality of a connected co-independent hop dominating set of G, denoted by $\gamma_{ch,coi}(G)$, is called the *connected co-independent hop domination number* of G. A connected co-independent hop dominating set S with $|S| = \gamma_{ch,coi}(G)$ is called a $\gamma_{ch,coi}$-set of G.

Example 1. Let $P_8 = [a, b, c, d, e, f, g, h]$. Then $S_1 = \{c, d, g, h\}$, $S_2 = \{c, d, e, f\}$ and $S_3 = \{b, c, d, e, f, g\}$ are hop dominating sets of P_8. $\langle S_1 \rangle$ is not connected while $\langle S_2 \rangle$ and $\langle S_3 \rangle$ are connected. However, S_2 is not a connected co-independent dominating set since $V(G) \backslash S_2$ is not an independent set while S_3 is a connected co-independent dominating set because $V(G) \backslash S_3$ is an independent set. Thus, S_3 is a connected co-independent hop dominating set. It can be verified that $\gamma_{ch,coi}(P_8) = 6$.

Definition 9. For every $u, v \in V(G)$ such that $uv \in E(G)$, denote by H^{uv} the copy of H whose vertices are attached one by one to the end vertices u and v of each edge uv of G and a set $S_{uv} \subseteq H^{uv}$.

3. Results

3.1. Preliminary Results

It is worth mentioning that every connected graph G admits a connected co-independent hop dominating set. Indeed, the vertex-set $V(G)$ of G is a connected co-independent hop dominating set. As a simple observation, we state the following.

Remark 1. Every connected co-independent hop dominating set in a connected graph G is a hop dominating set. Hence, $\gamma_h(G) \leq \gamma_{ch,coi}(G)$.

Remark 2. Let G be a connected graph of order n. Then $1 \leq \gamma_{ch,coi}(G) \leq n$. Moreover, $\gamma_{ch,coi}(G) = 1$ if and only if $G = K_1$.

Example 2. The formulas below give the connected co-independent hop domination number of the path P_n and cycle C_n.

$$\gamma_{ch,coi}(P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2, 3 \\ n - 2 & \text{if } n \geq 4 \end{cases}$$

$$\gamma_{ch,coi}(C_n) = \begin{cases} 3 & \text{if } n = 3 \\ n - 1 & \text{if } n \geq 4 \end{cases}$$

Remark 3. If G is a complete graph, then $\gamma_{ch,coi}(G) = n$.

Theorem 1. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{ch,coi}(G) = 2$ if and only if there exist adjacent vertices x and y of G such that for each $z \in V(G) \backslash \{x, y\}$, $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$ and $z \notin N_G(x) \cap N_G(y)$.

Proof: Suppose $\gamma_{ch,coi}(G) = 2$. Let $S = \{x, y\}$ be $\gamma_{ch,coi}$-set of G. Since S is connected, $xy \in E(G)$. Let $z \in V(G) \backslash \{x, y\}$. Then $z \notin S$. Since S is a hop dominating set of G, $z \in N_G(x, 2) \cup N_G(y, 2)$. Suppose $z \in N_G(x, 2)$. Then there exist $w \in N_G(z) \cap N_G(x)$. Since $V(G) \backslash S$ is an independent set, $w \in S$. Thus, $w = y$, that is, $N_G(z) = \{y\}$. Similarly, if $z \in N_G(y,2)$, then $N_G(z) = \{x\}$. Since $z \in N_G(x,2) \cup N_G(y,2)$, $z \notin N_G(x) \cap N_G(y)$.

Conversely, suppose that there exist adjacent vertices x and y of G satisfying the given condition. Let $S = \{x,y\}$. Since $xy \in E(G)$, S is connected. Let $z \in V(G)\backslash S$. If $N_G(z) = \{x\}$, then since $xy \in E(G)$, $d_G(y,z) = 2$. While on the other hand, if $N_G(z) = \{y\}$, then $d_G(x,z) = 2$. Thus, S is a hop dominating set of G. Since $N_G(z) = \{x\}$ or $N_G(z) = \{y\}$, $V(G)\backslash S$ is an independent set. Therefore, S is a connected co-independent hop dominating set of G. So, $\gamma_{ch,coi}(G) \leq |S| = 2$. But G is nontrivial. Hence, $\gamma_{ch,coi}(G) \neq 1$ and so $\gamma_{ch,coi}(G) = 2$. □

Example 3. The graph $P_2 \circ \overline{K}_n$ in Figure 1 has $\gamma_{ch,coi}(P_2 \circ \overline{K}_n) = 2$.

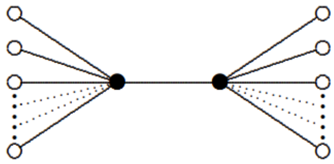


Figure 1: $P_2 \circ \overline{K}_n$

Theorem 2. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{ch,coi}(G) = n$ if and only if G is complete.

Proof: Suppose $\gamma_{ch,coi}(G) = n$. Suppose that G is not complete. Then there exist distinct vertices $u, v \in V(G)$ such that $d_G(u,v) = 2$. Let $S = V(G)\backslash\{u\}$. Then S is a connected co-independent hop dominating set of G. Therefore, $\gamma_{ch,coi}(G) \leq |S| = n-1$, a contradiction. Thus, G is a complete graph.

Conversely, by Remark 3, $\gamma_{ch,coi}(K_n) = n$. □

3.2. Connected Co-Independent Hop Domination in the Edge Corona of Graphs

The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H.

Example 4. Let $G = C_3$ and $H = K_2$. The edge corona of $G \diamond H$ and $H \diamond G$ are shown in Figure 2.

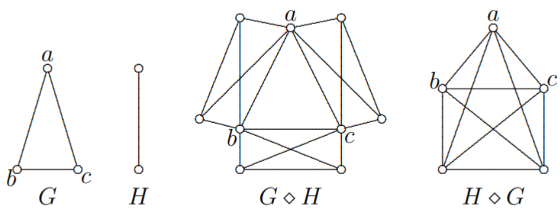


Figure 2: Edge corona $C_4 \diamond K_2$ and $K_2 \diamond C_4$

Remark 4. If G is a connected graph of order 2 and H is any graph, then $G \diamond H = G + H$.

Theorem 3. Let G be a connected graph of order $n \geq 3$ and H be any graph. Then $C \subseteq V(G \diamond H)$ is a connected co-independent hop dominating set of $G \diamond H$ if and only if $C = A \cup (\bigcup_{uv \in E(G)} S_{uv})$ where

(i) $A \subseteq V(G)$ is a connected co-independent set of G containing all vertices incident to all the edges of G.

(ii) $S_{uv} = V(H^{uv})$ if $uv \in E(G)$ such that $u \in V(G) \backslash A$ or $v \in V(G) \backslash A$.

(iii) For every $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H^{ab})$, $V(H^{ab}) \backslash S_{ab}$ is an independent set in H^{ab}.

Proof: Suppose that C is a connected co-independent hop dominating set of $G \diamond H$. Let $A = C \cap V(G)$ and let $S_{uv} = C \cap V(H^{uv})$ for each $uv \in E(G)$. Then $C = A \cup (\bigcup_{uv \in V(G)} S_{uv})$ where $A \subseteq V(G)$. First, we show that $\langle A \rangle$ is connected. Let $x, y \in A$ with $x \neq y$. If $xy \in E(G)$, then we are done. Suppose that $xy \notin E(G)$. Since $\langle C \rangle$ is connected and $x, y \in C$, there exists an x-y path $[x_1, x_2, ..., x_n]$ in $\langle C \rangle$ where $x = x_1$, $y = x_n$ and $n > 2$. If $x_i \in A$ for all $i \in \{1, 2, ..., n\}$, then the path $[x_1, x_2, ..., x_n]$ is in A. Suppose there exists $x_i \notin A$. Then $x_i \in S_{uv}$ for some edge $uv \in E(G)$. By definition of $G \diamond H$, $u, v \in A$. Hence, $[x_1, ..., u, v, ..., x_n]$ is a path in A, showing that $\langle A \rangle$ is connected. Next, let $u, v \in V(G) \backslash A$ with $u \neq v$. Then $u, v \in V(G \diamond H) \backslash C$. Since $V(G \diamond H) \backslash C$ is independent, $uv \notin E(G \diamond H)$. Since $u, v \in V(G)$, $uv \notin E(G)$ implying that $V(G) \backslash A$ is independent. Now, suppose v is a vertex incident to all the edges of G and $v \notin A$. Then $v \in N_G(w) \cap N_{G \diamond H}(p)$ for all $w \in V(G)$ and for all $p \in V(H^{vw})$. Thus, $N_{G \diamond H}(v, 2) \cap C = \varnothing$, a contradiction since C is a hop dominating set. Hence, A is a connected co-independent set of G containing all vertices incident to all edges of G, showing that (i) holds. Let $uv \in E(G)$ with $u \notin A$. Suppose $S_{uv} \neq V(H^{uv})$. Then there exists $x \in V(H^{uv}) \backslash S_{uv}$. Hence, $x, u \in V(G \diamond H) \backslash C$ and $xu \in E(G \diamond H)$, a contradiction to the independence of $V(G \diamond H) \backslash C$. Thus, $S_{uv} = V(H^{uv})$ and (ii) holds. Lastly, let $a, b \in A$ such that $ab \in E(G)$ and $S_{ab} \neq V(H^{ab})$. Since $V(G \diamond H) \backslash C$ is independent and $(V(H^{ab}) \backslash S_{ab}) \subseteq V(G \diamond H) \backslash C$, $V(H^{ab}) \backslash S_{ab}$ is an independent set in H^{ab}. Hence, (iii) holds.

For the converse, suppose $C = A \cup (\bigcup_{uv \in E(G)} S_{uv})$ where $(i), (ii)$ and (iii) hold. First, we show that C is connected. Let $u, v \in C$ with $u \neq v$. If $uv \in E(G \diamond H)$, then we are done. So, suppose that $uv \notin E(G \diamond H)$. Consider the following cases.

Case 1. $u, v \in A$

By (i), $\langle A \rangle$ is connected. Hence, there exists a u-v path $P[u, v]$ in A. Since $A \subseteq C$, the path $P[u, v]$ is in C.

Case 2. $u \in A$ and $v \in S_{xy}$ for some $xy \in E(G)$

Since $uv \notin E(G \diamond H)$, $u \neq x$ and $u \neq y$. Since $V(G) \backslash A$ is independent by (i), $x \in A$ or $y \in A$, say $x \in A$. If $ux \in E(G)$, then the path $[u, x, v]$ is a u-v path in C. Suppose $ux \notin E(G)$. Since $\langle A \rangle$ is connected by (i) and $u, x \in A$, there exists u-x path $[y_1, y_2, ..., y_k]$

in A where $u = y_1$, $x = y_k$ and $k > 2$. Hence, the path $[y_1, y_2, ..., y_k, v]$ is a u-v path in C.

Case 3. $u, v \in S_{pq}$ for some edge $pq \in E(G)$.
Since $V(G)\backslash A$ is independent by (i), $p \in A$ or $q \in A$. Hence, the path $[u, p, v]$ or $[u, q, v]$ is in C.

In any case, $\langle C\rangle$ is connected. Next, we show that $V(G \diamond H)\backslash C$ is independent. Let $p, q \in V(G \diamond H)\backslash C$ with $p \neq q$. Consider the following cases.

Case 1. $p \in V(G)\backslash A$ and $q \in V(G)\backslash A$
Since $V(G)\backslash A$ is independent by (i), $pq \notin E(G)$. Thus, $pq \notin E(G \diamond H)$.

Case 2. $p \in V(G)\backslash A$, $q \in V(H^{xy})\backslash S_{xy}$ for some $xy \in E(G)$
Since $S_{xy} \neq V(H^{xy})$, $x, y \in A$ by (ii). Hence, $p \neq x$ and $p \neq y$. By definition of $G \diamond H$, $pq \notin E(G \diamond H)$.

Case 3. $p \in V(H^{xy})\backslash S_{xy}$ and $q \in V(H^{rs})\backslash S_{rs}$ for some distinct edges $xy, rs \in E(G)$
Then, by definition of $G \diamond H$, $pq \notin E(G \diamond H)$.

Case 4. $p, q \in V(H^{zt})\backslash S_{zt}$ for some edge $zt \in E(G)$
Since $V(H^{zt})\backslash S_{zt}$ is independent by (iii), $pq \notin E(G \diamond H)$.

Therefore, in any case, $V(G \diamond H)\backslash C$ is an independent set in $G \diamond H$. Lastly, we show that C is a hop dominating set of $G \diamond H$. Let $u \in V(G \diamond H)\backslash C$. Consider the following cases.

Case 1. $u \in V(G)\backslash A$
Let $deg_G(u) = 1$. Since $|V(G)| \geq 3$, there exists $vw \in E(G)$ with $u \in N_G(v)\backslash N_G(w)$ or $u \in N_G(w)\backslash N_G(v)$. If $w \in A$, then $w \in N_G(u, 2) \cap A$. If $w \notin A$, then $S_{vw} = V(H^{vw})$ by (ii). Thus, a vertex $p \in N_{G\diamond H}(u, 2) \cap S_{vw}$ exists. Hence, $p \in N_{G\diamond H}(u, 2) \cap C$.

Case 2. $u \in V(H^{xy})\backslash S_{xy}$ for some $xy \in E(G)$
By (ii), $x, y \in A$. Since $|V(G)| \geq 3$, there exist $z \in V(G) \cap N_G(x)$ or $z \in V(G) \cap N_G(y)$. If $z \in A$, then $z \in N_{G\diamond H}(u, 2) \cap C$. If $z \notin A$, then $S_{yz} = V(H^{yz})$. Hence, a vertex $w \in N_{G\diamond H}(u, 2) \cap S_{yz}$ or $w \in N_{G\diamond H}(u, 2) \cap S_{xz}$.

Therefore, in any case C is a hop dominating set of $G\diamond H$. Accordingly, C is a connected co-independent hop dominating set of $G \diamond H$. □

Corollary 1. Let G be a connected graph of order $n \geq 3$ and H be any graph of size p and of order m. Then $\gamma_{ch,coi}(G \diamond H) = n + p(m - \beta(H))$.

Proof: Let $C_o = A \cup (\bigcup_{uv\in V(G)} S_{uv})$ be a $\gamma_{ch,coi}$-set of $G \diamond H$. Then conditions $(i), (ii)$ and (iii) of Theorem 3 hold where $A = V(G)$ and $S_{uv} = V(H^{uv})\backslash S^*$ where S^* is any independent set of H^{uv}. Thus,

$$\begin{aligned}\gamma_{ch,coi}(G \diamond H) &= |A| + p|S_{uv}| \\ &= n + p(|V(H^{uv})| - |S^*|) \\ &\geq n + p(m - \beta(H)).\end{aligned}$$

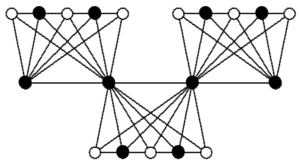
Let T be a β-set of H and $S_{uv} = V(H^{uv})\backslash T$ for each $uv \in E(G)$. Then $C = V(G) \cup (\bigcup_{uv\in E(G)} S_{uv})$ is a connected co-independent hop dominating set of $G \diamond H$ by Theorem 3.

Hence,

$$\begin{aligned}\gamma_{ch,coi}(G \diamond H) &\leq |C| \\ &= |V(G)| + p|S_{uv}| \\ &= n + p|V(H^{uv})\backslash T| \\ &= n + p(m - \beta(H)).\end{aligned}$$

Therefore, $\gamma_{ch,coi}(G \diamond H) = n + p(m - \beta(H))$. □

Example 5. The set of shaded vertices in the graph of $P_4 \diamond P_5$ represents a connected co-independent hop dominating set of $P_4 \diamond P_5$. By Corollary 1, $\gamma_{ch,coi}(P_4 \diamond P_5) = 10$.



3.3. Connected Co-Independent Hop Domination in the Complementary Prism

For a graph G, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$.

Example 6. Consider the graphs C_4, $\overline{C}_4$ in Figure 3. In the same figure is an illustration of complementary prism $C_4\overline{C}_4$.

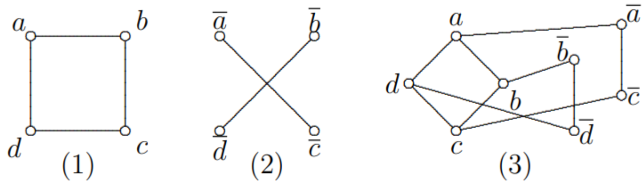


Figure 3: (1)cycle C_4 (2)complement $\overline{C}_4$ of C_4 (3)complementary prism $C_4\overline{C}_4$

Theorem 4. Let G be either a complete graph or an empty graph of order $n \geq 2$. Then $S \subseteq V(G\overline{G})$ is a connected co-independent hop dominating set of $G\overline{G}$ if and only if $S = S_G \cup S_{\overline{G}}$ and the following hold:

(i) $S_G = V(G)$ and $S_{\overline{G}} \subseteq V(\overline{G})$ if G is complete and

(ii) $S_{\overline{G}} = V(\overline{G})$ and $S_G \subseteq V(G)$ if G is an empty graph.

Proof: Suppose that $S \subseteq V(G\overline{G})$ is a connected co-independent hop dominating set of $G\overline{G}$. Let $S_G = S \cap V(G)$ and $S_{\overline{G}} = S \cap V(\overline{G})$. Then $S = S_G \cup S_{\overline{G}}$. Let G be a complete graph and suppose that $S_G \neq V(G)$. Then there exists $x \in V(G)\backslash S_G$. Since $V(G\overline{G})\backslash S$ is independent and $x\overline{x} \in E(G\overline{G})$, $\overline{x} \in V(\overline{G}) \cap S_{\overline{G}}$. Since G is complete, $\overline{G}$ is an empty graph. Thus, $\overline{x}$ is an isolated vertex of $\overline{G}$. This contradicts the connectedness of S. Hence, $S_G = V(G)$ and (i) holds. For (ii), if G is an empty graph, then $\overline{G}$ is a complete graph. By (i), $S_{\overline{G}} = V(\overline{G})$ and $S_G \subseteq V(G)$.

For the converse, suppose that $S = S_G \cup S_{\overline{G}}$ and (i) and (ii) hold. Then clearly, S is connected. Suppose (i) holds. Then $V(G\overline{G})\backslash S = V(\overline{G})\backslash S_{\overline{G}}$ is independent since $\overline{G}$ is an empty graph. Let $x \in V(G\overline{G})\backslash S$. Then $x \in V(\overline{G})\backslash S_{\overline{G}}$. Hence, $x = \overline{p}$ for some $p \in V(G)$. Since G is complete, $py \in E(G)$ for each $y \in V(G)\backslash\{p\}$. Since $xp \in E(G\overline{G})$, $d_{G\overline{G}}(x, y) = 2$. Suppose (ii) holds. Then $V(G\overline{G})\backslash S = V(G)\backslash S_G$ is independent since G is an empty graph. Let $z \in V(G\overline{G})\backslash S$. Then $z \in V(G)\backslash S_G$. Thus, $\overline{z} \in V(\overline{G}) \cap S_{\overline{G}}$ since $z\overline{z} \in E(G\overline{G})$ and $V(G\overline{G})\backslash S$ is independent. Since $\overline{G}$ is complete, $\overline{qz} \in E(\overline{G})$ for all $\overline{q} \in V(\overline{G})\backslash\{\overline{z}\}$. It follows that $d_{G\overline{G}}(z, \overline{q}) = 2$ and $\overline{q} \in V(\overline{G})\backslash\{\overline{z}\}$. Therefore, in any case, S is a connected co-independent hop dominating set of $G\overline{G}$. □

Corollary 2. Let G be either a complete graph or an empty graph of order $n \geq 2$. Then $\gamma_{ch,coi}(G\overline{G}) = n$.

Proof: Let S be a $\gamma_{ch,coi}$-set of $G\overline{G}$. Then $S = S_G \cup S_{\overline{G}}$ and (i) and (ii) of Theorem 4 hold. If (i) holds, then $S_G = V(G)$ and $S_{\overline{G}} \subseteq V(\bar{G})$. Hence, $\gamma_{ch,coi}(G\overline{G}) = |S| = |V(G)| + |S_G| \geq n$. On the other hand, if (ii) holds, then $S_{\overline{G}} = V(\overline{G})$ and $S_G = V(G)$. Hence, $\gamma_{ch,coi}(G\overline{G}) = |S| = |V(\overline{G})| + |S_{\overline{G}}| \geq n$. Now, let $S_{\overline{G}} = \varnothing$ if (i) holds. Thus, $S = V(G) \cup S_{\overline{G}}$ is a connected co-independent hop dominating set of $G\overline{G}$ by Theorem 4. Hence, $\gamma_{ch,coi}(G\overline{G}) \leq |S| = |V(G)| = n$. If (ii) holds, then let $S_G = \varnothing$. By Theorem 4, $S = V(\overline{G}) \cup S_G$. Thus, $\gamma_{ch,coi}(G\overline{G}) \leq |S| = |V(\overline{G}) = n$. Therefore,in any case, $\gamma_{ch,coi}(G\overline{G}) = n$. □

Theorem 5. Let G be a nontrivial connected noncomplete graph and $\overline{G}$ be the complement of G. Then $S \subseteq V(G\overline{G})$ is a connected co-independent hop dominating set of $G\overline{G}$ if and only if $S = S_G \cup S_{\overline{G}}$ where $S_G \subseteq V(G)$ and $S_{\overline{G}} \subseteq V(\overline{G})$ and the following hold:

(i) $S_G \neq \varnothing$ and $S_{\overline{G}} \neq \varnothing$

(ii) $V(G)\backslash S_G$ and $V(\overline{G})\backslash S_{\overline{G}}$ are independent sets in G and $\overline{G}$, respectively.

(iii) For every $x \in V(G)\backslash S_G$, $\overline{x} \in S_{\overline{G}}$.

(iv) Either $\langle S_G \rangle$ is connected or for every pair of distinct vertices $x, y \in S_G$ with $xy \notin E(G)$, $\overline{x}, \overline{y} \in S_{\overline{G}}$.

(v) Either $\langle S_{\overline{G}} \rangle$ is connected or for every pair of distinct vertices $\overline{p}, \overline{q} \in S_{\overline{G}}$ with $\overline{pq} \notin E(\overline{G})$, $p, q \in S_G$.

(vi) For every pair of vertices $x \in S_G$ and $\overline{q} \in S_{\overline{G}}$, $q \in S_G$ if $xq \in E(G)$ or $\overline{x} \in S_{\overline{G}}$ if $xq \notin E(G)$.

(vii) For every $x \in V(G)\backslash S_G$ such that $N_G(x,2) \cap S_G = \varnothing$, either there exists $y \in V(G) \cap N_G(x)$ such that $\overline{y} \in S_{\overline{G}}$ or there exists $\overline{p} \in S_{\overline{G}} \cap N_G(\overline{x})$.

(viii) For every $\overline{q} \in V(\overline{G})\backslash S_{\overline{G}}$ such that $N_{\overline{G}}(\overline{q},2) \cap S_{\overline{G}} = \varnothing$, there exists $\overline{z} \in V(\overline{G}) \cap N_{\overline{G}} \cap N_{\overline{G}}(\overline{q})$ such that $z \in S_G$.

Proof: Suppose that S is a connected co-independent hop dominating set of $G\overline{G}$. Let $S_G = S \cap V(G)$ and $S_{\overline{G}} = S \cap V(\overline{G})$. Then $S = S_G \cup S_{\overline{G}}$. Suppose $S_G = \varnothing$. Then $S = S_{\overline{G}}$ and $V(G\overline{G})\backslash S = V(G) \cup [V(\overline{G})\backslash S_{\overline{G}}]$ is not independent since G is connected. This is a contradiction to the independence of $V(G\overline{G})\backslash S$. Thus, $S_G \neq \varnothing$. Since G is a connected noncomplete graph, there exist $x, y \in V(G)$ such that $xy \notin E(G)$. Hence, $\overline{xy} \in E(\overline{G})$. This implies that $\overline{x} \in S_{\overline{G}}$ or $\overline{y} \in S_{\overline{G}}$, showing that $S_{\overline{G}} \neq \varnothing$. Hence, (i) holds. For (ii), since $V(G\overline{G})\backslash S = (V(G)\backslash S_G)\dot{\cup}(V(\overline{G})\backslash S_{\overline{G}})$, $V(G\overline{G})\backslash S$ is independent, $V(G)\backslash S_G$ and $V(\overline{G})\backslash S_{\overline{G}}$ are independent sets of G and $\overline{G}$, respectively. Now, let $x \in V(G)\backslash S_G$. Since $x\overline{x} \in E(G\overline{G})$ and $V(G\overline{G})\backslash S$ is independent, $\overline{x} \in S_{\overline{G}}$. Hence, (iii) holds. Suppose $\langle S_G \rangle$ is not connected. Let $x, y \in S_G$ with $x \neq y$ and $xy \notin E(G)$. Since $\langle S \rangle$ is connected, an x-y path $P[x,y]$ in S exists. Since $\langle S_G \rangle$ is not connected and $\overline{xy} \in E(\overline{G})$, $\overline{x}, \overline{y} \in P[x,y]$. Hence, $\overline{x}, \overline{y} \in S_{\overline{G}}$ and (iv) holds. The proof of (v) is similar to the proof of (iv). Next, let $x \in S_G$ and $\overline{q} \in S_{\overline{G}}$. Since $x, \overline{q} \in S$ and $x \neq \overline{q}$, by connectedness of $\langle S \rangle$, there exists an x-$\overline{q}$ path $P[x,\overline{q}]$ in S. If $xq \in E(G)$, then $q \in P[x,\overline{q}]$ impyling that $q \in S_G$. On the other hand, if $xq \notin E(G)$, then $\overline{x} \in P[x,\overline{q}]$, showing that $\overline{x} \in S_{\overline{G}}$. Hence, (vi) holds. Let $x \in V(G)\backslash S_G$ such that $N_G(x,2) \cap S_G = \varnothing$. Since S is a hop dominating set of $G\overline{G}$ and $x \notin S$, there exist $z \in S$ such that $d_{G\overline{G}}(x,z) = 2$. Since$N_G(x,2) \cap S_G = \varnothing$, either $z = \overline{p} \in S_{\overline{G}}$ for some $p \in V(G) \cap N_G(x)$ or $\overline{p} \in V(\overline{G}) \cap N_{\overline{G}}(\overline{x})$. Hence, (vii) holds. Statement $(viii)$ can be shown similarly with (vii).

For the converse, let $S = S_G \cup S_{\overline{G}}$ where $S_G \subseteq V(G)$ and $S_{\overline{G}} \subseteq V(G)$ and conditions (i)-$(viii)$ are satisfied. First, we show that $\langle S \rangle$ is connected. Let $x, y \in S$ with $x \neq y$. If $xy \in E(G\overline{G})$, then we are done. Suppose $xy \notin E(G\overline{G})$. Consider the following cases.

Case 1. $x, y \in S_G$

If $\langle S_G \rangle$ is connected, then an x-y path $P[x,y]$ in S_G exists. Since $S_G \subseteq S$, $P[x,y]$ is an x-y path in S. Suppose S_G is not connected. Then by (iv), $\overline{x}, \overline{y} \in S_{\overline{G}}$. Thus, $[x, \overline{x}, \overline{y}, y]$ is a path in S.

Case2. $x, y \in S_{\overline{G}}$

Same with Case 1 using (v).

Case 3.$x \in S_G$, $y \in S_{\overline{G}}$

Let $y = \overline{p}$ for some $p \in V(G)$. Then by (vi), either the path $[x, p, y]$ or $[x, \overline{x}, y]$ is in S.

Therefore, in any case, $\langle S \rangle$ is connected. Since $V(G)\backslash S_G$ and $V(\overline{G})\backslash S_{\overline{G}}$ are independent in G and $\overline{G}$, respectively by (ii) and $V(G\overline{G})\backslash S = (V(G)\backslash S_G) \cup (V(\overline{G})\backslash S_{\overline{G}})$, we have $V(G\overline{G})\backslash S$ is independent. Finally, let $x \in V(G\overline{G})\backslash S$. Consider the following cases.

Case 1. $x \in V(G)\backslash S_G$

If $N_G(x,2) \cap S_G \neq \varnothing$, then $d_{G\overline{G}}(x,y) = 2$ for some $y \in N_G(x,2) \cap S_G$. Suppose $N_G(x,2) \cap S_G = \varnothing$. Then by (vii), there exists $w \in V(G) \cap N_G(x)$ such that $\overline{w} \in S_{\overline{G}}$. Thus, $d_{G\overline{G}}(x, \overline{w}) = 2$.

Case2. $x \in V(\overline{G}) \setminus S_{\overline{G}}$
Same with Case 1 using $(viii)$.

Hence, in any case, S is a hop dominating set of $G\overline{G}$. Accordingly, S is a connected co-independent hop dominating set of $G\overline{G}$. □

The following result follows from Theorem 5.

Corollary 3. Let G be a nontrivial connected noncomplete graph. Then

$$2 \leq \gamma_{ch,coi}(G\overline{G}) \leq 2|V(G)| - \beta(G).$$

Remark 5. The strictly inequality in $\gamma_{ch,coi}(G\overline{G}) \leq 2|V(G)| - \beta(G)$ presented in Corollary 3 can be attained. However the given upper bound is sharp.

Example 7. To illustrate Remark 5, consider the path $P_6 = [v_1, v_2, v_3, v_4, v_5, v_6]$. It can be verified that the set $\{v_2, v_3, v_4, v_5, \overline{v}_1, \overline{v}_2, \overline{v}_5, \overline{v}_6\}$ is a $\gamma_{ch,coi}$-set of $P_6\overline{P}_6$, that is, $\gamma_{ch,coi}(P_6\overline{P}_6) = 8$. However, $2|V(P_6)| - \beta(P_6) = 2(6) - 3 = 9$. Hence, strict inequality is attained. On the other hand, equality is attained for $C_4 = [u_1, u_2, u_3, u_4]$, since $\{u_1, u_2, u_3, \overline{u}_2, \overline{u}_3, \overline{u}_4\}$ is a $\gamma_{ch,coi}$-set of $C_4\overline{C}_4$, that is, $\gamma_{ch,coi}(C_4\overline{C}_4) = 6 = 2|V(C_4)| - \beta(C_4)$.

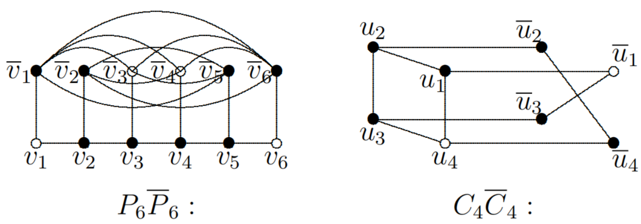


Figure 4: Connected co-independent hop dominating set of $P_6\overline{P}_6$ and $C_4\overline{C}_4$

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