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Some New Applications of the Quantum Calculus for New Families of Sigmoid Activation Bi-Univalent Functions Connected to Horadam Polynomials

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Abstract. The study of q-calculus is becoming increasingly prominent in the field of geometric function theory, reflecting a growing interest in its applications. In this research work, we first develop a new type of modified Sigmoid-Salagean q-differential operator in the open unit disk $\mathfrak{D}$, utilizing the concepts of quantum calculus and the Sigmoid activation function. Using this newly defined q-analogous differential operator and Horadam polynomials, we introduce new subclasses of bi-univalent functions in $\mathfrak{D}$. We determine upper bounds on initial coefficients, as well as the Fekete-Szegö problems, for functions belonging to these special families. Additionally, we discuss several interesting consequences related to the findings presented in this study.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Holomorphic function, Bi-univalent function, Horadam polynomials, Modified Sigmoid function, q-Calculus, the q-Difference operator, Fekete-Szegö problem

1. Introduction

Let $\mathcal{A}$ be the set of normalized analytic functions that have the form

$$g(z) = z + d_2z^2 + d_3z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1}$$

in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that, an analytic function g, which is a function of a single-value in some domain $\Delta \subset \mathbb{C}$. If g does not take the same value twice in Δ, we say that it is a univalent function, that is, if $g(z_1) \neq g(z_2)$ for $z_1 \neq z_2$, (see ([10], page 26). The class of all univalent functions represented by $\mathcal{S}$. The theorem of Koebe one-quarter

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([10], page 31) prove that the range of each function $g \in \mathcal{S}$ contains the open disk of radius $\frac{1}{4}$. Every function $g \in \mathcal{S}$ has an inverse g^{-1} satisfies

$$z = g^{-1}(g(z)),\ z \in \mathfrak{D}$$

and

$$\omega = g(g^{-1}(\omega)), \quad |\omega| < r_0(g), \quad r_0(g) \geq 1/4.$$

The inverse function $f = g^{-1}$ for each $g \in \mathcal{S}$ has Taylor series expansion as follows (see ([3], page 185)):

$$\begin{aligned} g^{-1}(\omega) &= f(\omega) = \omega + \sum_{j=2}^{\infty} \frac{1}{j} K_{j-1}^{-j}(d_2, d_3, d_4, ...d_n)\,\omega^j \\ &= \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots, \end{aligned} \tag{2}$$

where the coefficients of j parametric function $K_j^p(d_2, d_3, d_4, ...d_n)$ are given by

$$\begin{aligned} K_1^p &= pd_2, \\ K_2^p &= \frac{p(p-1)}{2} d_2^2 + pd_3, \\ K_3^p &= p(p-1)d_2d_3 + pd_4 + \frac{p(p-1)(p-2)}{3!} d_2^3. \end{aligned}$$

An analytic function $g \in \mathcal{A}$ will be the bi-univalent in $\mathfrak{D}$, if both g and g^{-1} are univalent in $\mathfrak{D}$ and the family of bi-univalent functions of the form (1) is represented by the symbol Σ. The subject has gained renewed attention in the last ten years, with several studies published on the subject since 2011, for instance [16, 30]. There were intriguing findings about the estimation of coefficients for certain types of univalent functions, (see [32, 40, 45–47]).

Lemma 1. *(Schwarz lemma ([10], page 3)). Let $\psi(z)$ is analytic in $\mathfrak{D}$ with $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathfrak{D}$, then we have*

$$|\psi(z)| < |z| \text{ and } |\psi'(0)| < 1 \text{ in } \mathfrak{D}.$$

Definition 1. *The analytic function $\zeta_1(z)$ is subordinate to the analytic function $\zeta_2(z)$, (written as $\zeta_1(z) \prec \zeta_2(z)$, $z \in \mathfrak{D}$), if there exist a Schwarz function $\psi(z)$ in $\mathfrak{D}$ such that*

$$\zeta_1(z) = \zeta_2(\psi(z)), \quad z \in \mathfrak{D}.$$

To be specific, if ζ_2 is univalent in $\mathfrak{D}$, then (see also [10]):

$$\zeta_1(z) \prec \zeta_2(z) \Leftrightarrow \zeta_1(0) = \zeta_2(0) \text{ and } \zeta_1(\mathfrak{D}) \subset \zeta_2(\mathfrak{D}).$$

The study of coefficients for functions in particular classes has been a cornerstone of univalent function research from its earliest beginnings. The Gronwall Area Theorem, established in 1914, is a significant discovery in the theory of univalent functions, used to determine bounds on the coefficients of the class of meromorphic functions. Different approaches in the geometric theory of functions of a complex variable have been inspired by Bieberbach's famous hypothesis, presented in 1916 but only verified in 1984, which he used to solve similar problems for the class $\mathcal{S}$. When examining bi-univalent functions, as in the classes investigated by Gronwall and Bieberbach, it is common practice to provide estimates for the first two Taylor-Maclaurin coefficients. Obtaining comparable estimates for different types of functions is known as the Fekete-Szegö problem. In 1933, it was shown by Fekete and Szegö [14] that

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} 3 - 4\delta & \text{if} \quad \delta < 0, \\ 1 + 2\exp\left(\frac{2\delta}{\delta - 1}\right) & \text{if} \quad 0 \leq \delta < 1, \\ 4\delta - 3 & \text{if} \quad \delta \geq 1, \end{cases}$$

is sharp and valid for every normalized univalent function. The Fekete-Szegö problem is the one where the objective is to maximize the absolute value of the functional $\left|d_3 - \delta d_2^2\right|$. According to many writers, Fekete-Szegö inequalities have been shown for several types of functions (see to references [9, 11]).

Geometric function theory have built and studied new classes of analytic functions using the q-calculus and the fractional q-calculus. To construct a class of q-starlike functions in $\mathfrak{D}$, Ismail et al. [23] first used the q-calculus (∂_q) operator, which was created by Jackson [24] in 1909. References such as [6, 26, 31, 35, 36] provide more information on q-calculus.

Definition 2. *Jackson [24, 25] developed the following q-derivative operator ∂_q for an analytic function g as follows:*

$$\partial_q g(z) = \frac{g(z) - g(qz)}{(1-q)\, z}, \quad 0 < q < 1; \quad z \neq 0.$$

It is evident that there is a limit relationship:

$$\lim_{q \to 1-} \partial_q g(z) = g^{'}(z)$$

and

$$\partial_q g(0) = g^{'}(0).$$

For the function $g \in \mathcal{A}$, defined by (1), we deduce the following series

$$\partial_q g(z) = 1 + \sum_{j=2}^{\infty} [j]_q \, d_n z^{j-1},$$

where $[j]_q$, called the q-analogue of $j \in \mathbb{N}$ is given by

$$[j]_q = \frac{1 - q^j}{1 - q}, \ j \in \mathbb{N}.$$

As $q \to 1-$, *we have* $[j]_q \to j$ *and* $[0]_q \to 0$.

Fadipe-Joseph et al. [12] recently (2013) defined a modified Sigmoid function, $\Psi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, and showed that it has a positive real part and belongs to the class $\mathcal{P}$ of Caratheodory functions.

Definition 3. *Let* $\mathcal{A}_\Psi$ *denote the family of functions of the form*

$$g_\Psi(z) = z + \sum_{j=2}^{\infty} \frac{2}{1+e^{-s}} d_j z^j = z + \sum_{j=2}^{\infty} \Psi(s) d_j z^j, \tag{3}$$

where $\Psi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, *is a modified Sigmoid function. Clearly* $\Psi(0) = 1$ *and hence* $\mathcal{A}_1 = \mathcal{A}$ *(see also [13]).*

Now we use the definitions of Modified Sigmoid function $g_\Psi(z)$ and q-difference operator ∂_q, we define a Modified Sigmoid Sălăgean q-differential operator $D_q^k : \mathcal{A}_\Psi \to \mathcal{A}_\Psi$ as follows:

Definition 4. *For* $g_\Psi \in \mathcal{A}_\Psi$, $k \in \mathbb{N} \cup \{0\}$, *the Sălăgean q-differential operator* $D_q^k : \mathcal{A}_\Psi \to \mathcal{A}_\Psi$, *is defined by*

$$D_q^0 g_\Psi(z) = g_\Psi(z), \ D_q^1 g_\Psi(z) = z\partial_q g_\Psi(z), ..., D_q^k g_\Psi(z) = \partial_q(D_q^{k-1} g_\Psi(z)), \ z \in \mathfrak{D}.$$

For $g_\Psi \in \mathcal{A}_\Psi$, *defined by (3), we deduce the following series:*

$$D_q^k g_\Psi(z) = z + \sum_{j=2}^{\infty} [j]_q^k \, \Psi(s) d_j z^j. \tag{4}$$

Remark 1. *When* $s = 0$ *then* $\Psi(s) = 1$, *then we have the Sălăgean q-differential operator [19].*

Remark 2. *When* $q \to 1-$, *and* $\Psi(s) = 1$, *then we have the Sălăgean differential operator [33].*

The Horadam polynomials $\Upsilon_j(y)$ were used in a comparable setting by Srivastava et al. [41]. The well-known Horadam polynomials $\Upsilon_j(y)$, as defined in Definition 5 in the field of Geometric Function Theory of Complex Analysis, were recently examined by Hörçum and Koçer [22], see also [21].

Definition 5. *([21, 22]). The following recurrence relation gives the Horadam polynomials* $\Upsilon_j(y, a, b; p, t)$ *(or briefly* $\Upsilon_j(y)$*):*

$$\Upsilon_j(y) = py\Upsilon_{j-1}(y) + t\Upsilon_{j-2}(y) \tag{5}$$

with

$$\Upsilon_1(y) = a, \quad \Upsilon_2(y) = by,$$

where $j \in \mathbb{N} = \{1, 2, ...\}$, $y \in \mathbb{R}$, a, b, p and t are real constants. From (5) we have

$$\Upsilon_3(y) = pby^2 + ta.$$

In addition, the characteristic equation for the recurrence relation (5) is provided by

$$s^2 - pys - t = 0,$$

where

$$\beta_1 = \frac{py + \sqrt{p^2y^2 + 4t}}{2} \quad and \quad \beta_2 = \frac{py - \sqrt{p^2y^2 + 4t}}{2}$$

are two real roots. The $\digamma(y, z)$ is the generating function of $\Upsilon_j(y)$ is given below (see [22]):

$$\digamma(y, z) := \sum_{j=1}^{\infty} \Upsilon_j(y) z^{j-1} = \frac{a + (b - ap)yz}{1 - pyz - tz^2}, \tag{6}$$

where $y \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$, that is $y \neq \Re(z)$.

Remark 3. *By selecting the parameters a, b, p, and t properly, we present here a few unique instances of $\Upsilon_j(y, a, b; p, t)$.*
(i): $\Upsilon_j(y, 2, 2; 2, 1) = \mathcal{Q}_j(y)$, the Pell-Lucas polynomials.
(ii): $\Upsilon_j(y, 1, 1; 2, -1) = \mathcal{T}_j(y)$, the first kind Chebyshev polynomials.
(iii): $\Upsilon_j(y, 1, 2; 2, -1) = \mathcal{U}_j(y)$, the second kind Chebyshev polynomials.
(iv): $\Upsilon_j(y, 1, 1; 1, 1) = \mathcal{F}_j(y)$, the Fibonacci polynomials.
(v): $\Upsilon_j(y, 2, 1; 1, 1) = \mathcal{L}_j(y)$, the Lucas polynomials.
(vi): $\Upsilon_j(y, 1, 2; 2, 1) = \mathcal{P}_j(y)$, the Pell polynomials.

Applications: The Horadam polynomial is a mathematical series used in texture analysis and picture processing. The Horadam polynomial, a distinct kind of polynomial sequence, is used for tasks such as filtering and resampling. Scale-space representations of pictures can be generated and adjusted in image processing and computer vision by using the Horadam polynomial. This polynomial is applicable for edge detection, texture examination, and multi-scale picture analysis. The Horadam polynomial has been used for texture analysis to extract features, segment images, and denoise pictures. This method may be used to examine the statistical characteristics of textures, such as the distribution of gray levels and the geographical arrangement of textures.

Abirami et al. [1] examined the initial coefficient estimates of Taylor-Maclaurin series for bi-Mocanu-convex and bi-μ-starlike functions related to Horadam polynomials. Additionally, Abirami et al. [2] discussed coefficient estimates for λ-bi-pseudo-starlike and bi-Bazilevic functions. Alamoush [4, 5] introduced subclasses of bi-starlike and bi-convex functions, utilizing the Poisson distribution series and Horadam polynomials, and also explored a class of bi-univalent functions associated with Horadam polynomials. These studies yielded initial coefficient estimates for the respective subclasses. Recent studies [7, 8, 29, 37–39] have explored various classes of bi-univalent functions associated with Horadam polynomials, Chebyshev polynomials, and other special functions. These works have established initial coefficient estimates, Fekete-Szegö bounds, and coefficient estimates for different classes of bi-univalent functions. We also taken notice of the fact that unique polynomials like the ones mentioned above might play a significant role in the fields of engineering, mathematics, statistics, and physical science. References [15, 21, 26] provide more information on these polynomials. In the works of [44] and [43], you can find further information on the Fekete-Szegö problem as it relates to Haradam polynomials. See references [17, 18, 20, 27, 28, 34] for a discussion of the many uses and applications of orthogonal polynomial families as well as other special functions and specialized polynomials. Illustrating from present trends in bi-univalent functions associated with various polynomials, we establish the following unique families of the class Σ using the Horadam polynomials $\Upsilon_j(y)$ linked to the Modified Sigmoid function (3) and its Sălăgean q-differential operator.

Here we give the value of all parameters, which will be used in this article

$$\mu \geq 0,\; q \in (0,1)\,,\; \mu \geq \gamma,\; 0 \leq \gamma \leq 1,\; k \in \mathbb{N} \cup \{0\},\; \xi \geq 1, \tau \geq 1$$

and

$$\Psi(s) = \frac{2}{1+e^{-s}}, \quad s \geq 0,$$

also

$$f_\Psi(\omega) = g_\Psi^{-1}(\omega)$$

which is an extension of g^{-1} to $\mathfrak{D}$ given by (2), a, b, p and t are as in (5) and $\digamma$ is as in (6).

Definition 6. *A function $g(z)$ in Σ is expressed as (1), then it is belong to the family $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s))$, if*

$$\frac{z\partial_q(D_q^k g_\Psi(z)) + \mu z^2 \partial_q^2(D_q^k g_\Psi(z))}{(1-\gamma)D_q^k g_\Psi(z) + \gamma z \partial_q(D_q^k g_\Psi(z))} \prec \digamma(y,z) + 1 - \alpha, \quad z \in \mathfrak{D}$$

and

$$\frac{\omega\partial_q(D_q^k f_\Psi(\omega)) + \mu \omega^2 \partial_q^2(D_q^k f_\Psi(\omega))}{(1-\gamma)D^k f_\Psi(\omega) + \gamma \omega \partial_q(D_q^k f_\Psi(\omega))} \prec \digamma(y,\omega) + 1 - \alpha, \quad \omega \in \mathfrak{D}.$$

Remark 4. *For the special values of γ and μ, the family $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s))$ reduces to the following new subfamilies.*
(i): For $\gamma = \mu = \frac{1}{2}$, we have $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s)) = J_\Sigma(y,k,q,\Psi(s))$, a new family of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.
(ii): For $\gamma = 0$, and $\mu = \frac{1}{2}$, we obtain a new family $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s)) = K_\Sigma(y,k,q,\Psi(s))$ of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.
(iii): For $\gamma = \frac{1}{2}$, and $\mu = 1$, we obtain a new family $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s)) = L_\Sigma(y,k,q,\Psi(s))$ of

bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.
(iv): For $\gamma = 0$*, we obtain a new family* $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s)) = M_{\Sigma}(y,\mu,k,q,\Psi(s))$ *of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.*

The Class $L_{\Sigma}(y,\gamma,\mu,k,q,\Psi(s))$

Definition 7. *A function* $g(z)$ *in* Σ *is given in (1), then it is belong to the family* $\mathcal{L}_{\Sigma}(y,\gamma,\mu,k,q,\Psi(s))$*, and* $\Psi(s) = \frac{2}{1+e^{-s}}$*,* $s \geq 0$*, if*

$$\frac{z\partial_q(D^k g_{\Psi}(z)) + \mu z^2\partial_q^2(D_q^k g_{\Psi}(z))}{(1-\gamma)\,z + \gamma z\partial_q(D_q^k g_{\Psi}(z))} \prec F(y,z) + 1 - \alpha,\ z \in \mathfrak{D}$$

and

$$\frac{\omega\partial_q(D_q^k f_{\Psi}(\omega)) + \mu\omega^2\partial_q^2(D_q^k f_{\Psi}(\omega))}{(1-\gamma)\,\omega + \gamma\omega\partial_q(D_q^k f_{\Psi}(\omega))} \prec F(y,\omega) + 1 - \alpha,\ \omega \in \mathfrak{D}.$$

Remark 5. *It is easy to observe that the special values of* γ *lead the family* $N_{\Sigma}(y,\gamma,\mu,k,q,\Psi(s))$ *to the following various subfamilies:*
(i): For $\gamma = 0$*, we obtain a new family* $\mathcal{L}_{\Sigma}(y,\gamma,\mu,k,q,\Psi(s)) = N_{\Sigma}(y,\mu,k,q,\Psi(s))$ *of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.*
(ii): For $\gamma = 1$*, we obtain a new family* $\mathcal{L}_{\Sigma}(y,\gamma,\mu,k,q,\Psi(s)) = O_{\Sigma}(y,\mu,k,q,\Psi(s))$ *of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.*

The Class $B_{\Sigma}(y,\xi,\tau,k,q,\Psi(s))$

Definition 8. *A function* $g(z)$ *in* Σ *is expressed as (1), then it is belong to the family* $B_{\Sigma}(y,\xi,\tau,k,q,\Psi(s))$*, and* $\Psi(s) = \frac{2}{1+e^{-s}}$*,* $s \geq 0$*, if*

$$\frac{(1-\xi) + \xi\left[\partial_q(z\partial_q\left(D_q^k g_{\Psi}(z)\right))\right]^{\tau}}{\partial_q(D_q^k g_{\Psi}(z))} \prec F(y,z) + 1 - \alpha,\ z \in \mathfrak{D},$$

and

$$\frac{(1-\xi) + \xi\left[\partial_q(\omega\partial_q\left(D_q^k f_{\Psi}(\omega)\right))\right]^{\tau}}{\partial_q(D_q^k f_{\Psi}(\omega))} \prec F(y,\omega) + 1 - \alpha,\ \omega \in \mathfrak{D}.$$

Remark 6. *It is easy to observe that the special values of* γ *lead the family* $B_{\Sigma}(y,\xi,\tau,k,q,\Psi(s))$ *to the following various subfamilies:*
(i): For $\tau = 1$*, we obtain a new family* $B_{\Sigma}(y,\xi,\tau,k,q,\Psi(s)) = \mathcal{M}_{\Sigma}(y,\xi,k,q,\Psi(s))$ *of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.*
(ii): For $\xi = 1$*, we obtain a new family* $B_{\Sigma}(y,\xi,\tau,k,q,\Psi(s)) = \mathfrak{N}_{\Sigma}(y,\tau,k,q,\Psi(s))$ *of bi-univalent functions connected with Sigmoid activation functions and Horadam polynomials.*

2. Main Results

2.1. Coefficient estimates and Fekete-Szegö problem for the class $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s))$

Theorem 1. *Let $g(z)$ is of the form (1) belong to $\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s))$. Then*

$$|d_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)-Q(\Upsilon(y),\gamma,q,\mu)\right\}\right|}}, \tag{7}$$

$$|d_3| \leq \frac{(by)^2}{[2]_q^{2k}\Psi^2(s)(q-\gamma+[2]_q\mu)^2}+\frac{|by|}{[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)}, \tag{8}$$

where

$$Q(\Upsilon(y),\gamma,q,\mu)=[2]_q^{2k}\Psi^2(s)(q-\gamma+[2]_q\mu)\left\{(\Upsilon_2(y))^2(1+\gamma)+\Upsilon_3(y)(q-\gamma+[2]_q\mu)\right\}.$$

For $\delta\in\mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)}, & |1-\delta|\leq J, \\ \frac{|by|^3|1-\delta|}{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)-Q(\Upsilon(y),\gamma,q,\mu)\right\}\right|}, & |1-\delta|\geq J, \end{cases} \tag{9}$$

where

$$J=\frac{\left|\left\{[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)-(\Upsilon_2(y))^2Q(\Upsilon(y),\gamma,q,\mu)\right\}\right|}{[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)}.$$

Proof. Let $g(z)\in\mathfrak{S}^{\mu,k}_{\Sigma,y,\gamma}(q,\Psi(s))$. Then, for the analytic functions $m(z)$ and $n(z)$ such that

$$m(0)=n(0)=0$$

and

$$|m(z)|<1 \text{ and } |n(\omega)|<1, \quad z,\omega\in\mathfrak{D}.$$

By Definition 6, we can write

$$\frac{z\partial_q(D_q^k g_\Psi(z))+\mu z^2\partial_q^2(D_q^k g_\Psi(z))}{(1-\gamma)D_q^k g_\Psi(z)+\gamma z\partial_q(D_q^k g_\Psi(z))}=F(y,m(z))+1-\alpha$$

and

$$\frac{\omega\partial_q(D_q^k f_\Psi(\omega))+\mu\omega^2\partial_q^2(D_q^k f_\Psi(\omega))}{(1-\gamma)D^k f_\Psi(\omega)+\gamma\omega\partial_q(D_q^k f_\Psi(\omega))}=F(y,n(\omega))+1-\alpha.$$

Or

$$\frac{z\partial_q(D_q^k g_\Psi(z))+\mu z^2\partial_q^2(D_q^k g_\Psi(z))}{(1-\gamma)D_q^k g_\Psi(z)+\gamma z\partial_q(D_q^k g_\Psi(z))}$$

$$= 1+\Upsilon_1(y)-a+\Upsilon_2(y)\,m(z)+\Upsilon_3(y)\,(m(z))^2+\ldots \tag{10}$$

and

$$\frac{\omega\partial_q(D_q^k f_\Psi(\omega))+\mu\omega^2\partial_q^2(D_q^k f_\Psi(\omega))}{(1-\gamma)D^k f_\Psi(\omega)+\gamma\omega\partial_q(D_q^k f_\Psi(\omega))}$$
$$= 1+\Upsilon_1(y)-a+\Upsilon_2(y)\,n(\omega)+sc\Upsilon_3(y)\,(n(\omega))^2+\ldots. \tag{11}$$

Based on (10) and (11), in view of (5), we may deduce

$$\frac{z\partial_q(D_q^k g_\Psi(z))+\mu z^2\partial_q^2(D_q^k g_\Psi(z))}{(1-\gamma)D_q^k g_\Psi(z)+\gamma z\partial_q(D_q^k g_\Psi(z))}$$
$$= 1+\Upsilon_2(y)\,m_1 z+\left[\Upsilon_2(y)\,m_2+\Upsilon_3(y)\,m_1^2\right]z^2+\cdots \tag{12}$$

and

$$\frac{\omega\partial_q(D_q^k f_\Psi(\omega))+\mu\omega^2\partial_q^2(D_q^k f_\Psi(\omega))}{(1-\gamma)D^k f_\Psi(\omega)+\gamma\omega\partial_q(D_q^k f_\Psi(\omega))}$$
$$= 1+\Upsilon_2(y)\,n_1\omega+\left[\Upsilon_2(y)\,n_2+\Upsilon_3(y)\,n_1^2\right]\omega^2+\cdots. \tag{13}$$

It is well known that if

$$|m(z)|=|m_1z+m_2z^2+m_3z^3+...|<1, \quad z\in\mathfrak{D}$$

and

$$|n(\omega)|=|n_1\omega+n_2\omega^2+n_3\omega^3+...|<1, \quad \omega\in\mathfrak{D},$$

then

$$|m_i|\le 1 \text{ and } |n_i|\le 1, \text{ for } (i\in\mathbb{N}). \tag{14}$$

Comparing the coefficients of (12) and (13), we have

$$[2]_q^k\,\Psi(s)(q-\gamma+[2]_q\,\mu)d_2=\Upsilon_2(y)m_1, \tag{15}$$

$$\Big\{[2]_q\,[3]_q^k\,\Psi(s))(q-\gamma+[3]_q\,\mu)d_3$$
$$-[2]_q^{2k}\,\Psi^2(s)(1+\gamma)(q-\gamma+[2]_q\,\mu)d_2\Big\}$$
$$= \Upsilon_2(y)m^2+\Upsilon_3(y)m_1^2, \tag{16}$$

$$-[2]_q^k\,\Psi(s)(q-\gamma+[2]_q\,\mu)d_2=\Upsilon_2(y)n_1 \tag{17}$$

and

$$-[2]_q\,[3]_q^k\,\Psi(s))(q-\gamma+[3]_q\,\mu)d_3+\Big\{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)$$
$$-[2]_q^{2k}\,\Psi^2(s)(1+\gamma)(q-\gamma+[2]_q\,\mu)\Big\}\,d_2^2$$

$$= \Upsilon_2(y)n_2 + \Upsilon_3(y)n_1^2. \tag{18}$$

From (15) and (17), we can see that

$$m_1 = -n_1 \tag{19}$$

and also

$$2\,[2]_q^{2k}\,\Psi^2(s)(q-\gamma+[2]_q\,\mu)^2 d_2 = \left(m_1^2+n_1^2\right)\left(\Upsilon_2\left(y\right)\right)^2. \tag{20}$$

Adding (16) and (18), then we obtain

$$\begin{aligned}&\Big\{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu) - \\ &[2]_q^{2k}\,\Psi^2(s)(1+\gamma)(q-\gamma+[2]_q\,\mu)\Big\}\,d_2^2 \\ =\;& \Upsilon_2(y)(m_2+n_2)+\Upsilon_3(y)(m_1^2+n_1^2).\end{aligned} \tag{21}$$

Putting the value of $m_1^2+n_1^2$ from (20) in (21), we get

$$d_2^2 = \frac{(\Upsilon_2(y))^3(m_2+n_2)}{2\left\{\left(\Upsilon_2\left(y\right)\right)^2[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu) - Q\left(\Upsilon\left(y\right),\gamma,q,\mu\right)\right\}}, \tag{22}$$

where

$$\begin{aligned}&Q\left(\Upsilon\left(y\right),\gamma,q,\mu\right)\\ =\;& [2]_q^{2k}\,\Psi^2(s)(q-\gamma+[2]_q\,\mu)\left\{\left(\Upsilon_2\left(y\right)\right)^2(1+\gamma)\right.\\ &\left.+\,\Upsilon_3(y)(q-\gamma+[2]_q\right\}\mu),\end{aligned}$$

which yields (7) on using (14). Using (19) in the subtraction of (18) from (16), we obtain

$$d_3 = d_2^2 + \frac{\Upsilon_2(y)(m_2-n_1)}{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)}. \tag{23}$$

Then in view of (20), and (23), we get

$$d_3 = \frac{(\Upsilon_2(y))^2(m_1^2+n_1^2)}{2\,[2]_q^{2k}\,\Psi^2(s)(q-\gamma+[2]_q\,\mu)^2} + \frac{\Upsilon_2(y)(m_2-n_2)}{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)},$$

which yields (8) on using (14). From (22) and (23), for $\delta\in\mathbb{R}$, we get

$$\begin{aligned}\left|d_3-\delta d_2^2\right| \;=\;& |\Upsilon_2(y)|\left|\left(T(\delta,q,y)+\frac{1}{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)}\right)m_2 + \right.\\ &\left.\left(T(\delta,q,y)-\frac{1}{2\,[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)}\right)n_2\right|,\end{aligned}$$

where

$$T(\delta,q,y)=\frac{(1-\delta)}{2\left\{[2]_q\,[3]_q^k\,\Psi(s)(q-\gamma+[3]_q\,\mu)-(\Upsilon_2\,(y))^2\,Q\,(\Upsilon\,(y)\,,\gamma,q,\mu)\right\}}.$$

In view of (5), we conclude that

$$\left|d_3-\delta d_2^2\right|\leq\begin{cases}\frac{|\Upsilon_2(y)|}{[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)}; & 0\leq|T(\delta,q,y)|\leq\frac{1}{2[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)},\\ 2\left|\Upsilon_2(y)\right|\left|T(\delta,q,y)\right|; & |T(\delta,q,y)|\geq\frac{1}{2[2]_q[3]_q^k\Psi(s)(q-\gamma+[3]_q\mu)},\end{cases}$$

which yields (9). Evidently, this concludes Theorem 1.

Remark 7. *For $\mu=0$, $\gamma=0$, $k=0$ and $\Psi(s)=1$, in Theorem 1 we obtain Corollary* 1 *and Corollary* 3 *proved in [26].*

2.1.1. Coefficient estimates and Fekete-Szegö problem for the class $\mathcal{L}_\Sigma(y,\gamma,\mu,k,q,\Psi(s))$

Theorem 2. *Let $g(z)$ of the form (1) belong to $\mathcal{L}_\Sigma(y,\gamma,\mu,k,q,\Psi(s))$. Then*

$$|d_2|\leq\frac{|b(y)|\sqrt{|b(y)|}}{\sqrt{\left|(\Upsilon_2\,(y))^2\,[3]_q^{k+1}\,\Psi(s)(1-\gamma+[2]_q\,\mu)-R_1\,(\Upsilon,y,\gamma,\mu)\right|}}\tag{24}$$

and

$$|d_3|\leq\frac{\left|b^2y^2\right|}{[2]_q^{2k+2}\,\Psi^2(s)(1-\gamma+\mu)^2}+\frac{|b(y)|}{2\,[3]_q^{k+1}\,\Psi(s)(1-\gamma+[2]_q\,\mu)},\tag{25}$$

where

$$R_1\,(\Upsilon,y,\gamma,\mu)=[2]_q^{2k+2}\,\Psi^2(s)(1-\gamma+\mu)\left\{(\Upsilon_2\,(y))^2\,\gamma+\Upsilon_3(y)(1-\gamma+\mu)\right\}.$$

For $\delta\in\mathbb{R}$

$$\left|d_3-\delta d_2^2\right|\leq\begin{cases}\frac{|by|}{[3]_q^{k+1}\Psi(s)(1-\gamma+[2]_q\mu)}, & |1-\delta|\leq M,\\ \frac{|by|^3|1-\delta|}{\left|2(by)^2[3]_q^{k+1}\Psi(s)(1-\gamma+[2]_q\mu)-R_2(\Upsilon,y,\gamma,\mu)\right|}, & |1-\delta|\geq M,\end{cases}\tag{26}$$

where

$$R_2\,(\Upsilon,y,\gamma,\mu)=[2]_q^{2k+2}\,\Psi^2(s)(1-\gamma+\mu)\left\{(by)^2\,\gamma+\left(pby^2+ra\right)(1-\gamma+\mu)\right\}$$

and

$$\begin{aligned}M\quad=\quad&\frac{1}{[3]_q^{k+1}\,\Psi(s)(1-\gamma+[2]_q\,\mu)}\left|[3]_q^{k+1}\,\Psi(s)(1-\gamma+[2]_q\,\mu)-\right.\\&\left.[2]_q^{2k+2}\,\Psi^2(s)(1-\gamma+\mu)\left\{\gamma+\left(\frac{pby^2+ra}{(by)^2}\right)(1-\gamma+\mu)\right\}\right|.\end{aligned}\tag{27}$$

Proof. Let $g(z) \in \mathfrak{L}_\Sigma(y, \gamma, \mu, k, q, \Psi(s))$. Then, for two analytic functions $m(z)$ and $n(z)$ such that

$$m(0) = n(0) = 0$$

and

$$|m(z)| < 1 \text{ and } |n(\omega)| < 1, \quad z, \omega \in \mathfrak{D}.$$

Using Definition 7, we can write

$$\frac{z\partial_q(D^k g_\Psi(z)) + \mu z^2 \partial_q^2(D_q^k g_\Psi(z))}{(1-\gamma)\, z + \gamma z \partial_q(D_q^k g_\Psi(z))} = F(y, m\,(z)) + 1 - \alpha \tag{28}$$

and

$$\frac{\omega\partial_q(D_q^k f_\Psi(\omega)) + \mu \omega^2 \partial_q^2(D_q^k f_\Psi(\omega))}{(1-\gamma)\, \omega + \gamma \omega \partial_q(D_q^k f_\Psi(\omega))} = F(y, n\,(\omega)) + 1 - \alpha. \tag{29}$$

Following (10), (11), (12), and (13) in the proof of Theorem 1, one gets the following in view of (28) and (29):

$$[2]_q^{k+1}\, \Psi(s)\, (1 - \gamma + \mu)\, d_2 = \Upsilon_2(y) m_1, \tag{30}$$

$$\begin{aligned} &\Big\{[3]_q^{k+1}\, \Psi(s)(1 - \gamma + \mu\, [2]_q) d_3 \\ &- [2]_q^{2k+2}\, \Psi^2(s)\, (1 - \gamma + \mu)\, \gamma d_2^2\Big\} \\ = \;& \Upsilon_2(y) m_2 + \Upsilon_3(y) m_1^2, \end{aligned} \tag{31}$$

$$-\, [2]_q^{k+1}\, \Psi(s)\, (1 - \gamma + \mu)\, d_2 = \Upsilon_2(y) n_1 \tag{32}$$

and

$$\begin{aligned} &-[3]_q^{k+1}\, \Psi(s)(1 - \gamma + \mu\, [2]_q) d_3 + \Big\{2\, [3]_q^{k+1}\, \Psi(s)(1 - \gamma + \mu\, [2]_q) \\ &- [2]_q^{2k+2}\, \Psi^2(s)\, (1 - \gamma + \mu)\, \gamma d_2\Big\} \\ = \;& \Upsilon_2(y) n_2 + \Upsilon_3(y) n_1^2. \end{aligned} \tag{33}$$

The results (24)-(26) of this theorem now follow from (30)-(33) by applying the procedure as in Theorem 1 with respect to (15)-(18).

Remark 8. *The results obtained in Theorem 2 coincide with Theorem* 2.1 *of [42] for* $k = 0$, $q \to 1-$ *and* $\Psi(s) = 1$.

2.2. Coefficient estimates and Fekete-Szegö problem for the class $B_{\Sigma}(y, \xi, \tau, k, q, \Psi(s))$

We derive the estimates for the coefficients $|d_2$ and $|d_3|$ and Fekete-Szegö problem in the following result.

Theorem 3. *Let $g(z)$ of the form 1 is in $\mathfrak{B}_{\Sigma}(y, \xi, \tau, k, \Psi(s))$. Then*

$$|d_2| \leq \frac{|b(y)| \sqrt{|b(y)|}}{\sqrt{[3]_q^{k+1} \Psi(s) \left(\xi\tau \left([2]_q + 1\right) - 1\right) (by)^2 - H(s, \xi, \tau, b, y)}}, \tag{34}$$

$$|d_3| \leq \frac{(by)^2}{(2\xi\tau - 1)^2 [2]_q^{2k+2} \Psi^2(s)} + \frac{|b(y)|}{[3]_q^{k+1} \Psi(s) \left(\xi\tau \left([2]_q + 1\right) - 1\right)}, \tag{35}$$

where

$$H(s, \xi, \tau, b, y) = \Big\{[2]_q^{k+1} \Psi^2(s)(2\xi\tau(\tau - 1) - 2\xi\tau + 1)(by)^2 - (2\xi\tau - 1)[2]_q^{2k+2} \Psi^2(s)\left(pby^2 + ra\right)\Big\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[3]_q^{k+1} \Psi(s)\left(\xi\tau\left([2]_q + 1\right) - 1\right)}, & |1 - \delta| \leq \Omega, \\ \frac{|by|^3 |1-\delta|}{\left|[3]_q^{k+1} \Psi(s)\left(\xi\tau\left([2]_q + 1\right) - 1\right)(by)^2 - H(s, \xi, \tau, b, y)\right|}, & |1 - \delta| \geq \Omega, \end{cases} \tag{36}$$

where

$$\Omega = \frac{\left|[3]_q^{k+1} \Psi(s) \left(\xi\tau \left([2]_q + 1\right) - 1\right) (b^2 y^2) - H(s, \xi, \tau, b, y)\right|}{4 [3]_q^{k+1} \Psi(s) \left(\xi\tau \left([2]_q + 1\right) - 1\right) (b^2 y^2)}.$$

Proof. Let $g(z) \in \mathfrak{B}_{\Sigma}(y, \xi, \tau, k, q, \Psi(s))$, we have

$$\frac{(1 - \xi) + \xi \left[\partial_q (z \partial_q \left(D_q^k g_{\Psi}(z)\right))\right]^{\tau}}{\partial_q (D_q^k g_{\Psi}(z))} = F(y, m(z)) + 1 - \alpha, \ z \in \mathfrak{D} \tag{37}$$

and

$$\frac{(1 - \xi) + \xi \left[\partial_q (\omega \partial_q \left(D_q^k f_{\Psi}(\omega)\right))\right]^{\tau}}{\partial_q (D_q^k f_{\Psi}(\omega))} = F(y, n(\omega)) + 1 - \alpha, \ \omega \in \mathfrak{D}. \tag{38}$$

Solving the both side of (37) and (38), we get following equations:

$$(2\xi\tau - 1)[2]_q^{k+1} \Psi(s) d_2 = \Upsilon_2(y) m_1, \tag{39}$$

$$\Big\{[3]_q^{k+1} \Psi(s) \left(\xi\tau \left([2]_q + 1\right) - 1\right) d_3 -$$

$$\begin{aligned}&[2]_q^{k+1}\Psi^2(s)(2\xi\tau(\tau-1)-2\xi\tau+1)d_2^2\Big\}\\ =\ & \Upsilon_2(y)m_2+\Upsilon_3(y)m_1^2, \end{aligned}\tag{40}$$

$$-(2\xi\tau-1)[2]_q^{k+1}\Psi(s)d_2=\Upsilon_2(y)n_1 \tag{41}$$

and

$$\begin{aligned}&-[3]_q^{k+1}\Psi(s)\left(\xi\tau\left([2]_q+1\right)-1\right)d_3+\Big(2[3]_q^{k+1}\Psi(s)\left(\xi\tau\left([2]_q+1\right)-1\right)\\ &-[2]_q^{k+1}\Psi^2(s)(2\xi\tau(\tau-1)-2\xi\tau+1)\Big)d_2^2\\ =\ & \Upsilon_2(y)n_2+\Upsilon_3(y)n_1^2. \end{aligned}\tag{42}$$

By using the same procedure of Theorem 3, we have the required result.

Remark 9. *The results obtained in Theorem 3 coincide with Theorem* 2.2 *of [42], when* $k=0$ *and* $\Psi(s)=1$.

In the next section, we present some interesting consequences of our main results.

3. Corollaries and Consequences

Corollary 1. *Let* $g(z)$ *be in the family* $J_\Sigma(y,k,q,\Psi(s))$. *Then*

$$|d_2|\le\frac{|by|\sqrt{|by|}}{\sqrt{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)-B(s,q,y)\right\}\right|}}$$

and

$$|d_3|\le\frac{(by)^2}{[2]_q^{2k}\Psi^2(s)\left(q-\frac12+\frac12[2]_q\mu\right)^2}+\frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)},$$

where

$$B(s,q,y)=[2]_q^{2k}\Psi^2(s)\left(q-\frac12+\frac12[2]_q\right)\left\{\frac32(\Upsilon_2(y))^2+\Upsilon_3(y)\left(q-\frac12+\frac12[2]_q\right)\right\}.$$

For $\delta\in\mathbb{R}$

$$\left|d_3-\delta d_2^2\right|\le\begin{cases}\frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)}, & |1-\delta|\le J_1,\\ \frac{|by|^3|1-\delta|}{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)-B(s,q,y)\right\}\right|}, & |1-\delta|\ge J_1,\end{cases}$$

where

$$J_1=\frac{\left|\left\{[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)(\Upsilon_2(y))^2-B(s,q,y)\right\}\right|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac12+\frac12[3]_q\right)(\Upsilon_2(y))^2}.$$

Corollary 2. *Let $g(z) \in K_{\Sigma}(y,k,q,\Psi(s))$. Then*

$$|d_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\mu\right)-B_1(s,q,y)\right\}\right|}},$$

$$|d_3| \leq \frac{(by)^2}{[2]_q^{2k}\Psi^2(s)\left(q+\frac{1}{2}[2]_q\right)^2}+\frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\right)},$$

where

$$B_1(s,q,y) = [2]_q^{2k}\Psi^2(s)\left(q+\frac{1}{2}[2]_q\right)\left\{(\Upsilon_2(y))^2+\Upsilon_3(y)\left(q+\frac{1}{2}[2]_q\right)\right\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\right)}, & |1-\delta| \leq J_2 \\ \frac{|by|^3|1-\delta|}{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\right)-B_1(s,q,y)\right\}\right|}, & |1-\delta| \geq J_2, \end{cases}$$

where

$$J_2 = \frac{\left|\left\{[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\right)(\Upsilon_2(y))^2-B_1(s,q,y)\right\}\right|}{[2]_q[3]_q^k\Psi(s)\left(q+\frac{1}{2}[3]_q\mu\right)(\Upsilon_2(y))^2}.$$

Corollary 3. *Let $g(z) \in L_{\Sigma}(y,k,q,\Psi(s))$. Then*

$$|d_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)-B_2(s,q,y)\right\}\right|}},$$

$$|d_3| \leq \frac{(by)^2}{[2]_q^{2k}\Psi^2(s)\left(q-\frac{1}{2}+[2]_q\right)^2}+\frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)},$$

where

$$B_2(s,q,y) = [2]_q^{2k}\Psi^2(s)(q-\frac{1}{2}+[2]_q)\left\{\frac{3}{2}(\Upsilon_2(y))^2+\Upsilon_3(y)(q-\frac{1}{2}+[2]_q)\right\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)}, & |1-\delta| \leq J_3, \\ \frac{|by|^3|1-\delta|}{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)-B_2(s,q,y)\right\}\right|}, & |1-\delta| \geq J_3, \end{cases}$$

where

$$J_3 = \frac{\left|\left\{[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)(\Upsilon_2(y))^2-B_2(s,q,y)\right\}\right|}{[2]_q[3]_q^k\Psi(s)\left(q-\frac{1}{2}+[3]_q\right)(\Upsilon_2(y))^2}.$$

Corollary 4. *Let $g(z)$ be in the family $M_\Sigma(y,\mu,k,\Psi(s))$. Then*

$$|d_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)-B_3(s,q,y)\right\}\right|}},$$

$$|d_3| \leq \frac{(by)^2}{[2]_q^{2k}\Psi^2(s)\left(q+[2]_q\mu\right)^2}+\frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)}$$

where

$$B_3(s,q,y) = [2]_q^{2k}\Psi^2(s)\left(q+[2]_q\mu\right)\left\{(\Upsilon_2(y))^2+\Upsilon_3(y)\left(q+[2]_q\mu\right)\right\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)}, & |1-\delta| \leq J_4, \\ \frac{|by|^3|1-\delta|}{\left|\left\{(\Upsilon_2(y))^2[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)-B_3(s,q,y)\right\}\right|}, & |1-\delta| \geq J_4, \end{cases}$$

where

$$J_4 = \frac{\left|\left\{[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)(\Upsilon_2(y))^2-B_3(s,q,y)\right\}\right|}{[2]_q[3]_q^k\Psi(s)\left(q+[3]_q\mu\right)}.$$

Corollary 5. *Let $g(z) \in N_\Sigma(y,\mu,k,q,\Psi(s))$. Then*

$$|d_2| \leq \frac{|b(y)|\sqrt{|b(y)|}}{\sqrt{\left|(\Upsilon_2(y))^2[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)-B_3(s,q,y,\mu)\right|}},$$

$$|d_3| \leq \frac{\left|b^2y^2\right|}{[2]_q^{2k+2}\Psi^2(s)(1+\mu)^2}+\frac{|b(y)|}{2[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)},$$

where

$$B_3(s,q,y,\mu) = [2]_q^{2k+2}\Psi^2(s)(1+\mu)\left\{\Upsilon_3(y)(1+\mu)\right\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)}, & |1-\delta| \leq M_1, \\ \frac{|by|^3|1-\delta|}{\left|2(by)^2[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)-B_3(s,q,y,\mu)\right|}, & |1-\delta| \geq M_1, \end{cases}$$

where

$$M_1 = \frac{1}{[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)(by)^2}\left|[3]_q^{k+1}\Psi(s)\left(1+[2]_q\mu\right)(by)^2-B_3(s,q,y,\mu)\right|.$$

Corollary 6. *Let $g(z) \in O_{\Sigma}(y, \mu, k, \Psi(s))$. Then*

$$|d_2| \leq \frac{|b(y)| \sqrt{|b(y)|}}{\sqrt{\left|(\Upsilon_2(y))^2 [3]_q^{k+1} \Psi(s) [2]_q \mu - [2]_q^{2k+2} \Psi^2(s)\mu \left\{(\Upsilon_2(y))^2 + \Upsilon_3(y)\mu\right\}\right|}},$$

$$|d_3| \leq \frac{\left|b^2 y^2\right|}{[2]_q^{2k+2} \Psi^2(s)\mu^2} + \frac{|b(y)|}{2 [3]_q^{k+1} \Psi(s) [2]_q \mu}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[3]_q^{k+1} \Psi(s) [2]_q \mu}, & |1-\delta| \leq M_2, \\ \frac{|by|^3 |1-\delta|}{\left|2(by)^2 [3]_q^{k+1} \Psi(s)[2]_q \mu - [2]_q^{2k+2} \Psi^2(s)\mu\left\{(by)^2\gamma + (pby^2 + ra)\mu\right\}\right|}, & |1-\delta| \geq M_2, \end{cases}$$

where

$$M_2 = \frac{1}{[3]_q^{k+1} \Psi(s) [2]_q \mu} \left|[3]_q^{k+1} \Psi(s) [2]_q \mu - [2]_q^{2k+2} \Psi^2(s)\mu \left\{1 + \left(\frac{pby^2 + ra}{(by)^2}\right)\mu\right\}\right|.$$

Corollary 7. *Let $g(z) \in P_{\Sigma}(y, \xi, k, \Psi(s))$. Then*

$$|d_2| \leq \frac{|b(y)| \sqrt{|b(y)|}}{\sqrt{[3]_q^{k+1} \Psi(s) \left(\xi \left([2]_q + 1\right) - 1\right) (by)^2 - B_4(r, s, q, \xi)}}$$

and

$$|d_3| \leq \frac{(by)^2}{(2\xi - 1)^2 [2]_q^{2k+2} \Psi^2(s)} + \frac{|b(y)|}{[3]_q^{k+1} \Psi(s) \left(\xi \left([2]_q + 1\right) - 1\right)},$$

where

$$B_4(r, s, q, \xi) = \Big\{[2]_q^{k+1} \Psi^2(s)(1 - 2\xi) (by)^2 - (2\xi - 1) [2]_q^{2k+2} \Psi^2(s) \left(pby^2 + ra\right)\Big\}.$$

For $\delta \in \mathbb{R}$

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[3]_q^{k+1} \Psi(s)\left(\xi\left([2]_q + 1\right) - 1\right)}, & |1-\delta| \leq \Omega_1, \\ \frac{|by|^3 |1-\delta|}{\left|[3]_q^{k+1} \Psi(s)\left(\xi\left([2]_q + 1\right) - 1\right)(by)^2 - B_4(r,s,q,\xi)\right|}, & |1-\delta| \geq \Omega_1, \end{cases}$$

where

$$\Omega_1 = \frac{\left|[3]_q^{k+1} \Psi(s) \left(\xi \left([2]_q + 1\right) - 1\right) b^2 y^2 - B_4(r, s, q, \xi)\right|}{4 [3]_q^{k+1} \Psi(s) \left(\xi \left([2]_q + 1\right) - 1\right) b^2 y^2}.$$

Corollary 8. *Let $g(z)$ be in the family $Q_\Sigma(y,\tau,k,Q,\Psi(s))$. Then*

$$|d_2| \leq \frac{|b(y)|\sqrt{|b(y)|}}{\sqrt{[3]_q^{k+1}\Psi(s)\left(\xi\left([2]_q+1\right)-1\right)(by)^2 - B_5(r,s,\tau,y)}}$$

and

$$|d_3| \leq \frac{(by)^2}{(2\tau-1)^2[2]_q^{2k+2}\Psi^2(s)} + \frac{|b(y)|}{[3]_q^{k+1}\Psi(s)\left(\tau\left([2]_q+1\right)-1\right)},$$

where

$$\begin{aligned} B_5(r,s,\tau,y) &= \left\{[2]_q^{k+1}\Psi^2(s)(2\tau^2-4\tau+1)(by)^2\right. \\ &\quad -\left\{(2\tau-1)[2]_q^{2k+2}\Psi^2(s)\left(pby^2+ra\right).\right. \end{aligned}$$

For $\delta\in\mathbb{R}$

$$\left|d_3-\delta d_2^2\right| \leq \begin{cases} \frac{|by|}{[3]_q^{k+1}\Psi(s)\left(\tau\left([2]_q+1\right)-1\right)}, & |1-\delta|\leq\Omega_2, \\ \frac{|by|^3|1-\delta|}{\left|[3]_q^{k+1}\Psi(s)\left(\tau\left([2]_q+1\right)-1\right)(by)^2-B_5(r,s,\tau,y)\right|}, & |1-\delta|\geq\Omega_2, \end{cases}$$

where

$$\Omega_2 = \frac{\left|[3]_q^{k+1}\Psi(s)\left(\xi\left([2]_q+1\right)-1\right)-B_6(r,s,\tau,y)\right|}{4[3]_q^{k+1}\Psi(s)\left(\tau\left([2]_q+1\right)-1\right)}$$

and

$$\begin{aligned} B_6(r,s,\tau,y) &= \left\{[2]_q^{k+1}\Psi^2(s)(2\tau^2-4\tau+1)-\right. \\ &\quad \left.(\xi\tau-1)[2]_q^{2k+2}\Psi^2(s)\left(\frac{pby^2+qa}{b^2y^2}\right)\right\}. \end{aligned}$$

4. Conclusions

This research aims to introduce new subfamilies of bi-univalent functions within the open unit disk, leveraging the connections between Horadam polynomials, modified Sigmoid functions, and the principles of subordination. By utilizing the power of q-calculus, quantum difference operators, and the modified Sigmoid function, we define and investigate three novel subclasses of bi-univalent functions linked to Horadam polynomials. Our study yields estimates for the Fekete-Szegö functional problems and the Taylor-Maclaurin coefficients $|d_2|$ and $|d_3|$ for each of these subclasses. Furthermore, by examining the variables in our main results, we uncover additional original findings. This methodology paves the way for the introduction of new subclasses of bi-univalent functions related to other generating functions, such as Fibonacci numbers and square-root functions. By applying the Faber polynomial technique, we can derive bounds for the n^{th} coefficients of these subclasses, specifically the first two initial coefficients and Fekete-Szegö functional problems.

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References

[1] C. Abirami, N. Magesh, and J. Yamini. Initial bounds for certain classes of bi-univalent functions defined by horadam polynomials. *Abstr. Appl. Anal.*, Art. ID 7391058:p 8, 2020.

[2] C. Abiramim, N. Magesh, J. Yamini, and N. B. Gatti. Horadam polynomial coefficient estimates for the classes of λ -bi-pseudo-starlike and bi-bazilevic functions. *J. Anal*, pages 1–10, 2020.

[3] H. Airault and A. Bouali. Differential calculus on the faber polynomials. *Bull. Sci. Math.*, 130:179–222, 2006.

[4] A. G. Alamoush. Certain subclasses of bi-univalent functions involving the poisson distribution associated with horadam polynomials. *Malaya J. Mat.*, 7:618–624, 2019.

[5] A. G. Alamoush. On a subclass of bi-univalent functions associated to horadam polynomials. *Int. J. Open Problems Complex Anal.*, 12:58–65, 2020.

[6] H. Aldweby and M. Darus. Some subordination results on q*Abst. Appl. Anal*, 2014:ID 958563, 2014.

[7] M. K. Aouf, A. O. Mostafa, and R. E. El. Morsy. Coefficient bounds for general class of bi-univalent functions of complex order associated with q-salagean operator and chebyshev polynomials. *Electr. J. Math. Anal. Appl.*, 8:251–260, 2020.

[8] I. T. Awolere and A. T. Oladipo. Coefficients of bi-univalent functions involving pseudo-starlikeness associated with chebyshev polynomials. *Khayyam J. Math.*, 5:140–149, 2019.

[9] R. Bucur and D. Breaz L. Andre and. Coefficient bounds and fekete-szego problem for a class of analytic functions defined by using a new differential operator. *Appl. Math. Sci*, 9:1355–1368, 2015.

[10] P. L. Duren. *Univalent functions.* Grundlehren der Mathematischen Wissenschaften, Band 259. Springer-Verlag, New York, 1983.

[11] J. Dziok. A general solution of the fekete-szego problem. *Boundary Value Problems*, 98:1–13, 2013.

[12] O. A. Fadipe-Joseph, A. T. Oladipo, and U. A. Ezeafulukwe. Modified sigmoid function in univalent function theory. *Int. J. Math. Sci. Engr. Appl*, 7:313–317, 2013.

[13] O.A. Fadipe-Joseph, B.B. Kadirand S. E. Akinwumi, and E. O. Adeniran. Polynomial bounds for a class of univalent function involving sigmoid function. *Khayyam J. Math*, 4:88–101, 2018.

[14] M. Fekete and G. Szegő. Eine bemerkung über ungerade schlichte funktionen. *J. Lond. Math. Soc*, 8:85–89, 1933.

[15] P. Filipponi and A. F. Horadam. Derivative sequences of fibonacci and lucas polynomials. *in: G. E. Bergum, A. N. Philippou, A. F. Horadam (eds) Applications of Fibonacci Numbers*, 4:99–108, 1991.

[16] B. A. Frasin and M. K. Aouf. New subclasses of bi-univalent functions. *Appl. Math. Lett.*, 22:1569–1573, 2011.

[17] B. A. Frasin, Y. Sailaja, S. R. Swamy, and A. K. Wanas. Coefficients bounds for a family of bi-univalent functions defined by horadam polynomials. *Acta Comment. Univ. Tartu. Math.*, 6:25–32, 2022.

[18] B.A. Frasin, S.R. Swamy, and J. Nirmala. Some special families of holomorphic and al-oboudi type bi-univalent functions related to k-fibonacci numbers involving modified sigmoid activation function. *Afr. Mat.*, 32:631–643, 2021.

[19] M. Govindaraj and S. Sivasubramanian. On a class of analytic functions related to conic domains involving q-calculus. *Analysis Mathematica*, 43:1–13, 2017.

[20] A. F. Horadam. acobsthal representation polynomials. *Fibonacci Quart.*, 35:137–148., 1997.

[21] A. F. Horadam and J. M. Mahon. Pell and pell-lucas polynomials. *Fibonacci Quart*, 23:7–20, 1985.

[22] T. Hörçum and E.G. Koçer. On some properties of horadam polynomials. *Internat. Math. Forum.*, 4:1243–1252, 2009.

[23] M. E. H. Ismail, E. Merkes, and D. Styer. A generalization of starlike functions. *Com. Vari. Theo. Appl*, 14:77–84, 1990.

[24] F. H. Jackson. On q-functions and a certain difference operator. *Trans. Royal Soc. Edinburgh*, 46:253–281, 1908.

[25] F. H. Jackson. On q-definite integrals. *Quart. J. Pure Appl. Math*, 41:193–203, 1910.

[26] S. Kanas and D. Raducanu. Some class of analytic functions related to conic domains. *Math. Slovaca*, 64:1183–1196, 2014.

[27] T. Koshy. Fibonacci and lucas numbers with applications. *John Wiley and Sons: New York, NY, USA*, 2001.

[28] A. Lupas. A guide of fibonacci and lucas polynomials. *Octagon Math. Mag.*, 7:2–12, 1999.

[29] N. Magesh and S. Bulut. Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions. *Afr. Mat.*, 29:203–209, 2018.

[30] S. Selvaraj O. S. Babu and G.Murugusundaramoorthy. Subclasses of bi-univalent functions based on hohlov operator. *Int. J. Pure Appl. Math.*, 102:473–482, 2015.

[31] S. D. Purohit and R. K Raina. Certain subclasses of analytic functions associated with fractional q-calculus operators. *Math. Scand*, 109:55–70, 2011.

[32] V. Ravichandran R. M. Ali, S. K. Lee and S. Subramaniam. Coefficient estimates for bi-univalent ma-minda starlike and convex functions. *Appl. Math. Lett.*, 25:344–351, 2012.

[33] G. S. Salagean. Subclasses of univalent functions. *in: Complex Analysis, fifthRomanian–Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes inMathematics*, 1013:362–372, 1983.

[34] A. Shammaky, B.A. Frasin, and S.R. Swamy. Fekete-szegö inequality for bi-univalent functions subordinate to horadam polynomials. *Journal of Function Spaces*, 2022:7, 2022.

[35] H . M. Srivastava and J. Choi. Zeta and q-zeta functions and associated series and integrals. *Elsevier Science Publishers, Amsterdam, London and New York*, 2012.

[36] H. M. Srivastava. Univalent functions. *fractional calculus, and associated generalized hypergeometric functions, in univalent functions. Fractional Calculus; and Their Applications (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto*, pages 329–354, 1989.

[37] H. M. Srivastava, S. Altinkaya, and S. Yalcin. Certain subclasses of biunivalen functions associated with the horadam polynomials. *Iran. J. Sci. Technol. Trans. A Sci.*, 43:1873–1879, 2019.

[38] H. M. Srivastava, D. Breaz, S. Khan, and F. Tchier. Certain new applications of symmetric q-calculus for new subclasses of multivalent functions associated with the cardioid domain. *Axioms*, 13:366, 2024.

[39] H. M. Srivastava, S. Khan, S. N. Malik, F. Tchier, A. Saliu, and Q. Xin. Faber polynomial coefficient inequalities for bi-bazilevic functions associated with the fibonacci-number series and the square-root functions. *Journal of Inequalities and Applications*, 2024:doi.org/10.1186/s13660–024–03090–9, 2024.

[40] H. M. Srivastava, A. K. Mishra, and P. Gochhayat. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.*, 23:1188–1192, 2010.

[41] H. M. Srivastava, S. Altınkaya, and Ş Yalçin. Certain subclasses of bi-univalent functions associated with the horadam polynomials. *Iran. J. Sci. Technol. Trans. A Sci*, 43:1873–1879, 2019.

[42] S. R. Swamy and Y. Sailaja. Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions. *International Journal of Mathematics Trends and Technology*, 66:131–138, 2020.

[43] A. W. Wanas and A. A. Lupas. Applications of horadam polynomials on bazilevic bi-univalent function satisfying subordinate conditions. *IOP Conf. Series: Journal of Physics: Conf. Series*, 1294:032003, 2019.

[44] T. T. Wang and W. P. Zhang. Some identities involving fibonacci, lucas polynomials and their applications. *Bull. Math. Soc. Sci. Math. Roumanie (New Ser.)*, 55:95–103, 2012.

[45] Q. H. Xu, Y. C. Gui, and H. .M. Srivastava. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Appl. Math. Lett.*, 25:990–994, 2012.

[46] Q. H Xu, H.G. Xiao, and H. M. Srivastava. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Appl. Math. Comput*, 218:11461–11465, 2012.

[47] A. Zireh and S. Hajiparvaneh. Coefficient bounds for certain subclasses of analytic and bi-univalent functions. *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, 8:133–144, 2016.