EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 2574-2585
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A New Outlook on Omega Closed Functions in Bitopological Spaces and Related Aspects

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Abstract. Many studies have employed a variety of techniques to further investigate topological space, particularly the notion of bitopological spaces, due to the significance of topological space in data processing as well as certain implementations. Numerous extended topological structures have been laid out subsequently. Of those abstractions, functions in topology was one of which was most noteworthy. In order to assist in this trend, we focused our research on the idea of open and closed sets, which is one of the strongest techniques available to present scientists for the study of computer graphics and digital topology.New functions, pairwise $\omega-$closed functions, which are strictly weaker than pairwise closed functions, will be introduced in this study. By applying the $\ddot{\text{P}}-$space definition, whose is a $P-$space modification. Additionally, we establish different projection and product theories pertaining to pairwise Lindelöf and pairwise paracompact spaces utilizing $\ddot{\text{P}}-$spaces. We analyze images and inverse images that have been chosen topological attributes for every one of these functions. In the final analysis, we explore several counterexamples that correspond to the offered definitions and theorems.

2020 Mathematics Subject Classifications: 54E55,54B10,54D30

Key Words and Phrases: Bitopological spaces, $\ddot{\text{P}}-$space, pair$-$Lindelöf, pair$-\omega-$closed functions, perfect function, pair$-\omega-$continuous functions, pair- paracompact

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*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5347

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1. Introduction

Several broad topological configurations have been explored subsequently. In light of the topological space's significance in analysis and various other uses, see [2, 3, 5]. One of the most fundamental topological space improvements is represented by the closed functions. General topology informs us that closed sets are crucial for the creation of new set forms and have vital topological traits. To expand on multiple features of closed functions, $\omega-$closed functions are primarily included in the topology. Compactness and Lindelö̈f are the fundamental elements in standardized topology. Additionally, topology and closed function theories have a frequent application in mathematical evaluation and logical arithmetic correspondingly. Both of these notions are also very useful in real-world applications. A novel kind of mappings known as $\omega-$closed mappings, which are precisely weaker than closed mappings, was created by [11] in 1982. He then goes over a few more situations that have relevance to the definitions and theorems which are presented, as he proposed the subsequent definitions of $\omega-$open and $\omega-$closed sets. If J has all of its condensation points, then it is commonly referred to as being $\omega-$closed. $\omega-$open is the complement of an $\omega-$closed set. likewise the intersection of all $\omega-$closed sets that contain J will be indicated by $cl^{\omega}\ J$. The idea of the existence of a bitopological space was initially introduced by [13] in 1963. Since then, other single topological qualities, including Lindelöfness, mapping types, separation axioms, compactness and metacompactness, have also been stretched to bitopological spaces.We are going to utilize pairwise Lindelöf as pair-Lindelöf during the course of this investigation, and pair- signifies pairwise. The fundamental definitions employed in this investigation are presented in Section 2. We demonstrate some properties of pair$-\omega-$closed functions in Section 3. The association between particular weakened versions of pairwise closed functions and pair$-\omega-$closed functions is illustrated with several instances in. Subsequently, the more complex characteristics of the pair$-\omega-$closed functions, notably their product and projection, are covered in Section 4. In the end, in Section 5, we go through a variety of counterexamples that are pertinent to the definitions and theorems offered in the earlier sections.

2. Basic definitions and preliminary remarks

Some key ideas and details that were employed in the research are presented in this part.

Definition 1. *[4] A function $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ is referred to as pair$-$continuous,*

whether $\Upsilon_1 : (D, \kappa_1) \to (G, \upsilon_1)$ and $\Upsilon_2 : (D, \kappa_2) \to (G, \upsilon_2)$ are continuous functions.

Definition 2. *[8] A function $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ is referred to as pair$-$closed,*

if $\Upsilon_1 : (D, \kappa_1) \to (G, \upsilon_1)$ and $\Upsilon_2 : (D, \kappa_2) \to (G, \upsilon_2)$ are closed functions.

That is cruel $H_1$is closed in κ_1,then $\Upsilon(H_1)$ is closed in υ_1, and if H_2 is closed in κ_2, then $\Upsilon(H_2)$ is closed in υ_2.

Definition 3. *[15] A cover* T *of the bitopological space* (D, κ_1, κ_2) *has been referred to* $\kappa_1\kappa_2-$ *open if* $T \subset \kappa_1 \cup \kappa_2$*. Additionally,* T *contains at least one−nonempty member of* κ_2*,it is regarded as pair−open.*

Definition 4. *[12] If any pair−open cover of a bitopological space has a countable subcover, the space is commonly referred to as a pair−Lindelöf.*

Definition 5. *[13] If any* $\kappa_1\kappa_2-$*open cover of a bitopological space has a countable subcover, the space is commonly referred to as a s−Lindelöf.*

Definition 6. *[17] whether* $\underset{\sim}{T}$, $\underset{\sim}{P}$ *are pair−open covers,we say that* $\underset{\sim}{P}$ *is a parallel refinement of* $\underset{\sim}{T}$*, solely in the event that any* $P_1 \in \underset{\sim}{P}$*, in a way that* $P \in \kappa_1$ *is included in* $T_1 \in \underset{\sim}{T}$ *and* $T_1 \in \kappa_1$*, and* $P_2 \in \underset{\sim}{V}$ *, such that* $P \in \kappa_2$ *is included in* $T_2 \in \underset{\sim}{T}$ *and* $T_2 \in \kappa_2$*.*

Definition 7. *[4] A pair−open cover* $\underset{\sim}{P}$ *is known as locally finite particularly in the event that* $\forall d \in D$ *, There's an open set* $T_1 \in \kappa_1$*, to the extent that* T_1 *intersects numerous individuals of* $P \bigcap \kappa_1$*, or to the extent that an open set* $T_2 \in \kappa_2$*, to the extent that* T_1 *intersects numerous individuals of* $P \bigcap \kappa_2$*.*

Definition 8. *[13] A space* (D, κ_1, κ_2) *is defined as pair−paracompact, whether and only whether any pair−open cover has a pair−open locally finite parallel refinement.*

Definition 9. *[7] A point* d *of a space* D *is known as a condensation point of the set* $M \subset D$*, if an arbitrary neigborhood (briefly, nbd) of the point* d *contains an uncountable subset of this set.*

Definition 10. *[6] The intersection of countably several open sets is an open set when it is the case unless space* D *is referred to by the term pair−space.*

Definition 11. *[9] Whenever each countably pair−open cover of a bitopological space* (D, κ_1, κ_2) *has a finite subcover, therefore the space is referred to be pair−countably compact.*

Definition 12. *[9] When there is a finite subcover for each countably* $\kappa_1\kappa_2-$*open cover of a bitopological space*(D, κ_1, κ_2)*, subsequently the space has been referred to as s−countably compact.*

Definition 13. *[18] Whenever a function* $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ *is referred to as pair−weakly continuous, it means that* $\Upsilon^{-1}(T)$ *is pair−*ω*−open for each pair−open set* $T \subset G$*.*

Definition 14. *[14] Assuming a bitopological space* (D, κ_1, κ_2)*, we declare that* κ_1 *is locally Lindelöf with respect to* κ_2*.When there is a* κ_1 *nbd* T_d *of* d*. In a way that* $\overline{T_d}^{\kappa_2}$ *is pair−Lindelöf all of them* $d \in (D, \kappa_1, \kappa_2)$*.*

3. A Novel Categorization Of Closed Functions

The notion of $\omega-$closed functions in bitopological spaces is introduced and their relation to other spaces is illustrated in this section.

Definition 15. *A pair–$\omega-$closed function can be expressed as*

$$\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$$

when it mappings pair–closed sets onto pair–$\omega-$closed sets.

Definition 16. *A pair– semi–$\omega-$closed function can be expressed as*

$$\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$$

when it mappings pair– semi closed sets onto pair–semi–$\omega-$closed sets.

Definition 17. *Whenever $\Upsilon^{-1}(L)$ is pair– Lindelöf every individual pair– Lindelöf closed subset L of $(G, \upsilon_1, \upsilon_2)$, subsequently*

$$\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$$

is a pair– Lindelöf function.

Definition 18. *Whenever $\Upsilon^{-1}(L)$ is pair–semi Lindelöf every individual pair– semi Lindelöf closed subset L of $(G, \upsilon_1, \upsilon_2)$, subsequently*

$$\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$$

is a pair–semi Lindelöf function.

Definition 19. *If the intersection of a countably many open sets is an $\omega-$open set, therefore space D is known to as a $\H{P}-$space.*

Definition 20. *When there is a pair–open subset T_d including d that means $T_d - J$ is a countable set, therefore a subset J of a bitopological space (D, κ_1, κ_2) is pair– $\omega-$open. Pair– $\omega-$closed sets have been defined to be the complement of pair– $\omega-$open sets.*

Definition 21. *Pair–$\omega - BO(J)$ as well as pair– $\omega - BC(J)$ is the family of all pair– $\omega-$open as well as pair– $\omega-$closed subsets of a space (D, κ_1, κ_2). Moreover, pair–$\omega - BO(D; d)$ represents the family of all pair–$\omega-$open sets of (D, κ_1, κ_2) including d.*

Definition 22. *Whether there's a $\kappa_1\kappa_2-$open subset T_d comprising d that implies $T_d - J$ is a countable set. Consequently a subset J of a bitopological space (D, κ_1, κ_2) is pair–semi–$\omega-$open. Pair–semi-–$\omega-$closed sets deemed to be the complement of pair–semi– $\omega-$open sets.*

Definition 23. *Pair–semi–$\omega - BO(J)$ as well as pair–semi– $\omega - BC(J)$ is the family of all pair–semi– $\omega-$open. Additionally pair–semi– $\omega-$closed subsets of a space (D, κ_1, κ_2). Furthermore, pair–semi–$\omega - BO(D; d)$ represents the family of all pair–semi–$\omega-$open sets of (D, κ_1, κ_2) encompassing d.*

Theorem 1. *In a space (D, κ_1, κ_2), any pair− Lindelöf, pair−ω−open subset J has the form $L \backslash N$, where L is a pair−open and N is a countable set; specifically, J is a G_δ−set.*

Proof. For any value of d in J, there's is a pair−open subset T_d which includes d and is countable set $T_d - J$. A countable set is created when there is a pair−open subset $T_d - J$ containing d for every d that is in J.

Claim $T_1, T_2, ... \in \kappa_1$, $T_1^*, T_2^*, ... \in \kappa_2$, thus $J \subset \bigcup_{i=1}^{\infty} T_i \cup \bigcup_{j=1}^{\infty} T_j^*$, during which $T_i \cap (J - D)$ is κ_1 countable,$T_j^* \cap (J - D)$ is κ_2−countable.

Presently, $T_i \cap (J - D) = \bigcup_{n=1}^{\infty} D_{i,n}, i = 1, 2, ..., T_j^* \cap (J - D) = \bigcup_{m=1}^{\infty} D_{h,m}, h = 1, 2...$

Right now, $J = \cup(T_i \backslash \bigcup_{n=1}^{\infty} D_{i,n}) \cup (T_h^* \backslash \bigcup_{m=1}^{\infty} D_{h,m}) = \cup \bigcup_{i=1}^{\infty} (T_i \backslash L_1) \cup \bigcup_{j=1}^{\infty} (T_h^* \backslash L_2)$.

Enable $L = L_1 \cup L_2$, and $L \subset \bigcup_{n=1}^{\infty} D_{i,n} \cup \bigcup_{m=1}^{\infty} D_{h,m}$.

Corollary 1. *Let's consider the hereditary Lindelöf space (D, κ_1, κ_2). Following this, a G_δ−set is any pair−ω−open subset of a space (D, κ_1, κ_2).*

Theorem 2. *In a space (D, κ_1, κ_2), any s− Lindelöf, s−ω−open subset J has the form $L \backslash N$, where L is a $\kappa_1\kappa_2$−open and N is a countable set; specifically, J is a G_δ−set.*

Proof. The proof use the same methodology as theorem 1.

Corollary 2. *Let's consider the hereditary Lindelöf space (D, κ_1, κ_2). Following this, a G_δ−set is any s−ω−open subset of a space (D, κ_1, κ_2).*

It is incorrect to assert that theorem 3.1 is contradictory. Considering a specific illustration:

Example 1. *Consider two topologies κ_1, κ_2 on R by the basis*

$$H_1 = \{(-\infty, j) : j > 0\} \cup \{\{d\} : d > 0\}, H_2 = \{(d, \infty) : d < 0\} \cup \{\{d\} : d < 0\},$$

then (R, κ_1, κ_2) is pair−Lindelöf .

The ensuing theorem extends the widely recognized theorem, which states that closed continuous functionings with pair−Lindelöf counter images maintain the pair−Lindelöf property under taking counter images.

Theorem 3. *Letting Υ represent a pair−continuous pair−ω−closed functioning of a space onto $(G, \upsilon_1, \upsilon_2)$ from (D, κ_1, κ_2). which means that for every $g \in (G, \upsilon_1, \upsilon_2)$, $\Upsilon^{-1}(g)$ is pair−Lindelöf . While $(G, \upsilon_1, \upsilon_2)$ is such a case, therefore (D, κ_1, κ_2) is pair−Lindelöf.*

Proof. Let us know $\underset{\sim}{T} = \{T_\delta : \delta \in \Xi\}$ is a pair−open cover of (D, κ_1, κ_2). In the meantime late $\forall g \in (G, \upsilon_1, \upsilon_2)$, $\Upsilon^{-1}(g)$ is pair−Lindelöf, the situation exists a countable subsets Ξ_y, Ξ_y^* of Ξ , which means $\Upsilon^{-1}(g) \subseteq \bigcup_{\delta\in\Xi_g} \{N_\delta : \delta \in \Xi_g\} \bigcup \bigcup_{\alpha\in\ \Lambda_y^*} \{M_\delta : \delta \in \Xi_g^*\}$, at which $\{N_\delta : \delta \in \Xi_g\}$ is κ_1-open ,$\{M_\delta : \delta \in \Xi_g^*\}$ is κ_2-open.

Assume $H_g = (G, \upsilon_1, \upsilon_2) - \Upsilon((D, \kappa_1, \kappa_2) - \bigcup_{\delta\in\Xi_g} N_\delta)$ is a ρ_1-open set comprising g,

and $H_g^* = (G, \upsilon_1, \upsilon_2) - \Upsilon((D, \kappa_1, \kappa_2) - \bigcup_{\delta\in\Xi_g^*} M_\delta)$ is a ρ_2-open set comprising g, where

$\Upsilon^{-1}(H_g) \subseteq \bigcup_{\delta\in\Xi_g} V_\alpha,\ \Upsilon^{-1}\ (H_g^*\ ) \subseteq \bigcup_{\delta\in\Xi_g^*} M_\delta$.

Assume $\underset{\sim}{H}_g = \{H_g : g \in\ (G, \upsilon_1, \upsilon_2)\} \bigcup \{H_g^* : g \in\ (G, \upsilon_1, \upsilon_2)\}$ is a pair$-$open cover of $(G, \upsilon_1, \upsilon_2)$. Considering Υ is pair$-\omega-$closed, $\underset{\sim}{H}_g$ is pair$-\omega-$open for each $g \in (G, \upsilon_1, \upsilon_2)$.

Thus, there is an open pair$-$nbd $H_g^\setminus$ .In a manner that $H_g^\setminus\ \cap\ ((D, \kappa_1, \kappa_2) - H_g^\setminus\ )$ is countable. Now $H_g^\setminus\ = (\underset{\sim}{H}_g \cap H_g) \cup\ H_g^\setminus\ \cap\ ((D, \kappa_1, \kappa_2) - H_g\ )$.Consequently, $\Upsilon^{-1}\ (H_g^\setminus)$ is enclosed in a union of countably large number of members of $\underset{\sim}{T}$.

Because of this $\left\{H_g^\setminus,\ g \in (G, \upsilon_1, \upsilon_2)\right\}$ is a pair$-$open cover of $(G, \upsilon_1, \upsilon_2)$ and it is pair$-$Lindelöf, $\left\{H_g^\setminus\ , g \in (G, \upsilon_1, \upsilon_2)\right\}$ has a countable subcover. Therefore (D, κ_1, κ_2) is the union of countably large number of members of $\left\{\Upsilon^{-1}\ (H_g^\setminus)\ ,\ g \in (G, \upsilon_1, \upsilon_2)\right\}$,since each $\Upsilon^{-1}\ (H_g^\setminus)$ is contained in the union of countably large number of members of $\underset{\sim}{T}$ .Consequently, (D, κ_1, κ_2) is the union of countably large number of members of $\underset{\sim}{T}$. Hence, (D, κ_1, κ_2) is pair$-$Lindelöf .

Corollary 3. *(i) A pair$-$ $\omega-$subset of a pair$-$ Lindelöf space is pair$-$Lindelöf (ii) If $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ is pair$-$continuous function from (D, κ_1, κ_2) to $(G, \upsilon_1, \upsilon_2)$, so the subsequent ones are comparable :*

(a) Υ is pair$-\omega-$closed ; (b) for each $g \in (G, \upsilon_1, \upsilon_2)$ and any pair$-$open set T, that is to say $\Upsilon^{-1}(g) \subset T$, it actually exists a pair$-\omega-$open set Q_g such that $g \in Q_g$ and $\Upsilon^{-1}(Q_g) \subset U$.

Corollary 4. *(i) A s$-$ $\omega-$subset of a s$-$ Lindelöf space is s$-$Lindelöf (ii) If $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ is s$-$continuous function from (D, κ_1, κ_2) to $(G, \upsilon_1, \upsilon_2)$, so the subsequent ones are comparable :(a) Υ is s$-\omega-$closed ; (b) for each $g \in (G, \upsilon_1, \upsilon_2)$ and any s$-$open set T, that is to say $\Upsilon^{-1}(g) \subset T$, it actually exists a s$-\omega-$open set Q_g such that $g \in Q_g$ and $\Upsilon^{-1}(Q_g) \subset U$.*

Theorem 4. *Assume Υ be pair$-$continuous s-$\omega-$closed functioning of a space (D, κ_1, κ_2)*

onto $(G, \upsilon_1, \upsilon_2)$,so that $\Upsilon^{-1}(g)$ is s-Lindelöf, for every $g \in (G, \upsilon_1, \upsilon_2)$, subsequently (D, κ_1, κ_2) is s-Lindelöf, whether $(G, \upsilon_1, \upsilon_2)$ is indeed.

Proof. Using the identical method as the theorem previously mentioned, the proof is produced.

Theorem 5. *Suppose that Υ is a $\omega-$closed pair$-$continuous function of a* regular space (D,κ_1,κ_2) onto $(G,\upsilon_1,\upsilon_2)$.When $(G,\upsilon_1,\upsilon_2)$ is pair−paracompact and $\Upsilon^{-1}(g)$ is pair−paracompact relative to (D,κ_1,κ_2). For every g in $(G,\upsilon_1,\upsilon_2)$, then (D,κ_1,κ_2) is pair−paracompact .

Proof. Present alongside $\underset{\sim}{T} = \{T_\delta{:}\delta \in \Xi\}$ is a pair−open cover of (D,κ_1,κ_2). Meanwhile, early $\forall g \in (G,\upsilon_1,\upsilon_2)$, $\Upsilon^{-1}(g)$ is pair−paracompact, $\underset{\sim}{T}$ has a pair−open locally finite refinement in(D,κ_1,κ_2) which at first cover $\Upsilon^{-1}(g)$,It is a real issue a countable subsets Ξ_g, Ξ_g^* of Ξ , this implies $\Upsilon^{-1}(g) \subseteq \bigcup_{\delta\in\Xi_g} \{N_\delta : \delta \in \Xi_g\} \bigcup \bigcup_{\alpha\in\ \Lambda_y^*} \{M_\delta : \delta \in \Xi_g^*\}$, at which $\{N_\delta : \delta \in \Xi_g\}$ is κ_1−open ,$\{M_\delta : \delta \in \Xi_g^*\}$ is κ_2−open.

Consider $H_g = (G,\upsilon_1,\upsilon_2) - \Upsilon((D,\kappa_1,\kappa_2) - \bigcup_{\delta\in\Xi_g} N_\delta)$ is a ρ_1-open set comprising g,

and $H_g^* = (G,\upsilon_1,\upsilon_2) - \Upsilon((D,\kappa_1,\kappa_2) - \bigcup_{\delta\in\Xi_g^*} M_\delta)$ is a ρ_2-open set comprising g, where $\Upsilon^{-1}(\ H_g) \subseteq \bigcup_{\delta\in\Xi_g} V_\alpha$, $\Upsilon^{-1}\ (H_g^*\ ) \subseteq \bigcup_{\delta\in\Xi_g^*} M_\delta$. Assume $\underset{\sim}{H}_g = \{H_g : g \in\ (G,\upsilon_1,\upsilon_2)\}$

$\bigcup\ \{H_g^* : g \in\ (G,\upsilon_1,\upsilon_2)\}$ is a pair−open cover of $(G,\upsilon_1,\upsilon_2)$. Taking into account Υ is pair−ω−closed, $\underset{\sim}{H}_g$ is pair−ω−open for each $g \in (G,\upsilon_1,\upsilon_2)$. Thus, there is an open pair−neibourhood $H_g^{\setminus}$ .In away that $H_g^{\setminus} \cap ((D,\kappa_1,\kappa_2) - H_g^{\setminus})$ is countable. Considering $(G,\upsilon_1,\upsilon_2)$ is pair−paracompact $\underset{\sim}{H}$ has pair−open locally finite parallel refinement declare that:$\underset{\sim}{Q} = \{Q_D : D \in \Omega_1\ \} \bigcup \{Q_D^* : D \in \Omega_2\ \}$, where $\{Q_D : D \in \Omega_1\ \}$ is υ_1-locally finite paracompact of H_g,and $\{Q_D^* : D \in \Omega_2\}$ is υ_2-locally finite paracompact of $H_g^{\setminus}$, $\Omega = \Omega_1 \bigcup\ \Omega_2$.Let $L_1 = \{\Upsilon^{-1}(Q_D) \bigcap_{\delta_i} N_\delta\ , i\ = 1,2,...,n, D \in \Omega_1, \delta \in \Xi_g\}$ is κ_1− open locally finite parallel refinement of $\{N_\delta : \delta \in \Xi_g\}$,and let $L_2 = \{\Upsilon^{-1}(Q_D^*) \bigcap M_{\delta_i}\ , i\ = 1,2,...,n, D \in \Omega_2,\ \delta \in \Xi_g^*\}$ is κ_2− open locally finite parallel refinement of $\{M_\delta : \delta \in \Xi_g^*\}$.

Let $\underset{\sim}{L} = \{L_1 \bigcup L_2\}$, then $\underset{\sim}{L}$ is pair−open locally finite parallel refinement of $\underset{\sim}{T}$, so (D,κ_1,κ_2) is pair−paracompact space.

Theorem 6. *Allow $\Upsilon : (D,\kappa_1,\kappa_2) \to (G,\upsilon_1,\upsilon_2)$ is pair−continuous function from (D,κ_1,κ_2)onto $(G,\upsilon_1,\upsilon_2)$, where $(G,\upsilon_1,\upsilon_2)$is pair−locally Lindelöf pair−Hausdorff $\H{P}$−space. Therefore the subsequent statements are comparable:*

(a) Υ is a pair−ω−closed function and for each $g \in (G,\upsilon_1,\upsilon_2)$, $\Upsilon^{-1}(g)$ is pair−Lindelöf .

(b) Υ is a pair− Lindelöf function.

Proof. $(a) \to (b)$ originates using the identical method as in Theorem 3.

$(b) \to (a)$: Let $\Upsilon : (D,\kappa_1,\kappa_2) \to (G,\upsilon_1,\upsilon_2)$ be pair−continuous function pair−Lindelöf, where $(G,\upsilon_1,\upsilon_2)$ is pair−locally Lindelöf pair−Hausdorff P̋−space. Demonstrating that Υ is pair−ω−closed function is adequate. Let $S_1$is closed in κ_1.Assume $\Upsilon(S_1)$ is not ω− closed in υ_1, therefor a point is present $g_0 \in (G,\upsilon_1,\upsilon_2) -\ \Upsilon(S_1)$. In this way in order for every neighborhood N of $g_0, N \cap \Upsilon(S_1)$ is uncountable. Because of $(G,\upsilon_1,\upsilon_2)$ is pair−locally Lindelöf, there exists υ_1−neighborhood M of g_0, so that $\overline{M}^{\upsilon_2}$ is pair−Lindelöf . Check now $\Upsilon(S_1) \cap \overline{M}^{\upsilon_2}$ is not pair− Lindelöf. While such is the case, it is evident that it is pair−ω−closed, therefore there is a υ_1− neighborhood K of g_0. In a way that $\Upsilon(S_1) \cap K$ is countable, and these is not feasible. Presently $\overline{M}^{\upsilon_2}$ is pair− Lindelöf, so $\Upsilon^{-1}(\overline{M}^{\upsilon_2})$ is pair−Lindelöf and $S_1 \cap \Upsilon^{-1}(\overline{M}^{\upsilon_2})$ is pair−Lindelöf subset of (D, κ_1, κ_2).Consequently $\Upsilon(S_1 \cap \Upsilon^{-1}(\overline{M}^{\upsilon_2})) = \Upsilon(S_1) \cap \overline{M}^{\upsilon_2}$ is pair−Lindelöf, it is paradoxical.Therefore $\Upsilon(S_1)$ is not $\omega-$ closed in υ_1. Comparative to S_2 is closed in κ_2, $\Upsilon(S_2)$ is not $\omega-$ closed in υ_2. Hence $\Upsilon(S)$ is pair− $\omega-$ closed .

4. A novel applications of projection and product theorems

Here, we derive various applications of projection and product theorems for pair−Lindelöf, pair−paracompact spaces using the findings from the preceding sections.

Theorem 7. *Assume (D, κ_1, κ_2) is a pair−Lindelöf space and $(G, \upsilon_1, \upsilon_2)$ be a $\H{P}-$space, subsequent the projection $p : (D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2) \to (G, \upsilon_1, \upsilon_2)$ is pair− $\omega-$ closed functions.*

Proof. Let $g \in (G, \upsilon_1, \upsilon_2)$ and $\H{N}= \{k_\delta : \delta \in \Xi\} \times \{l_\delta : \delta \in \Xi\}$ be a $(\kappa_1 \times \upsilon_1), (\kappa_2 \times \upsilon_2)$ open cover of $D \times G$,where $\{k_\delta : \delta \in \Xi\}$ is pair−open cover of (D, κ_1, κ_2) and $\{l_\delta : \delta \in \Xi\}$ is pair−open cover of $(G, \upsilon_1, \upsilon_2)$. In a way that $p^{-1}(g) = D_g = D \times \{g\} \subset N$. For every $(d, g) \in D \times \{g\}$. Let J_d and $J_g(D)$ be a pair−open neighborhood of (D, κ_1, κ_2) and $(G, \upsilon_1, \upsilon_2)$, such that $(d, g) \in J_d \times J_g(D) \subset U$. Now $\{J_d : d \in D\}$ is pair−open cover of (D, κ_1, κ_2). Consequently it has a countable subcover $\{J_{d_i}\}_{i=1}^{\infty}$ .Thus, $D \times \{g\} \subset \bigcup_{i=1}^{\infty} J_{d_i} \times J_g(D_i) \subset \H{N}$. Let $W_g = \bigcap_{i=1}^{\infty} J_g(D_i)$ and $W = \{W_g : g \in G\}$, then $D \times \{g\} \subset \bigcup_{i=1}^{\infty} J_{d_i} \times G_y \subset \H{N}$ and W_g is pair−$\omega-$ open set, as of late $(G, \upsilon_1, \upsilon_2)$ is a $\H{P}-$space. Consequently, for every $g \in (G, \upsilon_1, \upsilon_2)$, there is pair−$\omega-$ open set W_g such that $g \in W_g$ and $p^{-1}(g) \subset \H{N}$. Thus, according to Theorem 1, the projection p is a pair− $\omega-$ closed functions.

Theorem 8. *Assume (D, κ_1, κ_2) is a s−Lindelöf space and $(G, \upsilon_1, \upsilon_2)$ be a $\H{P}-$space, subsequent the projection $p : (D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2) \to (G, \upsilon_1, \upsilon_2)$ is s− $\omega-$ closed functions.*

Proof. The proof use the same methodology as Theorem 7.

Theorem 9. *Let $(D, \kappa_1, \kappa_2), (G, \upsilon_1, \upsilon_2)$ be any bitopological spaces,(D, κ_1, κ_2) be a pair−Lindelöf space and $(G, \upsilon_1, \upsilon_2)$ be a $\H{P}-$space then the projection function*

$$\pi : (D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2) \to (G, \upsilon_1, \upsilon_2)$$

is pair− $\omega-$closed.

Proof. If (D, κ_1, κ_2) is pair−Lindelöf, then (D, κ_1) is Lindelöf and (D, κ_2) is Lindelöf. Consequently, the projection functions: $\pi_1 : (D \times G, \kappa_1 \times \upsilon_1) \to (G, \upsilon_1)$, $\pi_2 : (D \times G, \kappa_2 \times \upsilon_2) \to (G, \upsilon_2)$ are $\omega-$ closed. Thus π is pair−$\omega-$closed.

Theorem 10. *Let $(D,\kappa_1,\kappa_2),(G,\upsilon_1,\upsilon_2)$ be any bitopological spaces, (D,κ_1,κ_2) be a s−Lindelӧf space and $(G,\upsilon_1,\upsilon_2)$ be a $\acute{\acute{P}}-$space then the projection function*

$$\pi:(D\ \times G,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)\to(G,\upsilon_1,\upsilon_2)$$

is s− ω−closed.

Proof. If $(G,\upsilon_1,\upsilon_2)$ is s−Lindelӧf, then (D,κ_1) is Lindelӧf and (D,κ_2) is Lindelӧf. Consequently, the projection functions: $\pi_1:(D\ \times G,\kappa_1\times\upsilon_1)\to(G,\upsilon_1)$, $\pi_2:(D\ \times G,\kappa_2\times\upsilon_2)\to(G,\upsilon_2)$ are $\omega-$ closed. Thus π is s−ω−closed.

Theorem 11. *Assume $(G,\upsilon_1,\upsilon_2)$ be a topological space in which a $F_\sigma-$set which is not pair−ω−closed, and (D,κ_1,κ_2) be any bitopological space. If the projection*

$$(D\ \times G,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)\to(G,\upsilon_1,\upsilon_2)$$

is pair− ω−closed, then (D,κ_1,κ_2) is pair−countably compact.

Proof. Assume $\bigcup_{i=1}^{\infty} J_i$ is a pair−F−subset of $(G,\upsilon_1,\upsilon_2)$ which is not pair− ω−closed, and (D,κ_1,κ_2) is not pair−countably compact. Subsequently, there a decreasing pairwise sequence $\{K_i\}_{i=1}^{\infty}$of pair−closed subsets of (D,κ_1,κ_2), in a manner that $\bigcap_{i=1}^{\infty} K_i=\phi$. Let $F=\bigcup_{i=1}^{\infty}(J_i\times K_i,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)$, afterwards it is evident to us that F is a pair−closed subset of $(J\times K,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)$. Likewise for each of the points $(d,g)\in(J\times K,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)$, $p(d,g)=g$. Next $p(F)=\bigcup_{i=1}^{\infty} J_i$ is not pair−ω−closed, consequently the projection is not pair− ω−closed.Thus, the outcome.

Corollary 5. *Assume $(G,\upsilon_1,\upsilon_2)$ be a topological space in which a $F_\sigma-$set which is not s−ω−closed, and (D,κ_1,κ_2) be any bitopological space. If the projection*

$$(D\ \times G,\ \kappa_1\times\upsilon_1,\ \kappa_2\times\upsilon_2)\to(G,\ \upsilon_1,\ \upsilon_2)$$

is s− ω−closed, then (D,κ_1,κ_2) is s−countably compact.

Theorem 12. *A space $(G,\upsilon_1,\upsilon_2)$ is $\acute{\acute{P}}-$space if and only if for pair−Lindelӧf space (D,κ_1,κ_2), then the projection $(D\ \times G,\ \kappa_1\times\upsilon_1,\ \kappa_2\times\upsilon_2)\to(G,\upsilon_1,\upsilon_2)$ is pair−ω−closed.*

Proof. The requirement portion is derived from theorem 7, as the condition must be sufficient. Presume $(G,\upsilon_1,\upsilon_2)$ is not $\acute{\acute{P}}-$space. For any pair−Lindelӧf space (D,κ_1,κ_2) then the projection $(D\ \times G,\kappa_1\times\upsilon_1,\kappa_2\times\upsilon_2)\to(G,\upsilon_1,\upsilon_2)$ is pair− ω−closed. Let $D=R$ is the set of real numbers with usual topology (R,κ_u,κ_u). Hence by the earlier theorem, (D,κ_1,κ_2) is pair−coutably compact, it is paradoxical.

Theorem 13. *Assume $(D, \kappa_1, \kappa_2), (G, \upsilon_1, \upsilon_2)$ is any bitopological spaces with the property that every pair−Lindelöf subset is pair−ω−closed. when $\Upsilon : (D, \kappa_1, \kappa_2) \to (G, \upsilon_1, \upsilon_2)$ is pair−Lindelöf, then Υ is pair−weakly continuous .*

Proof. Let $p_1 : (D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2) \to (D, \kappa_1, \kappa_2)$, $p_2 : (D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2) \to (G, \upsilon_1, \upsilon_2)$ be the projections, then (D, κ_1, κ_2) and range Υ of a pair−Lindelöf set as images of pair−Lindelöf sets under $p_1$and p_2.Let $Z_1^* = p_1 \backslash \Upsilon$. Ensure that Z_1^* is pair−ω−closed. Actually, if T is pair−closed, then T is pair−Lindelöf, $Z_1^*(T)$ is pair−Lindelöf. Therefore, it is pair−ω−closed. Since Υ is a function defined on (D, κ_1, κ_2), Z_1^* is a bijection onto (D, κ_1, κ_2).This combined with reality that Z_1^* is pair−ω−closed, means that for each pair−open set $V, Z_1^*(V)$ is pair−ω−open in (D, κ_1, κ_2). Presently $\Upsilon = p_2 \circ Z_1^{*^{-1}}$.$\Upsilon$ thus possesses the necessary attribute.

Corollary 6. *Let (D, κ_1, κ_2) be a pair−Lindelöf space and $(G, \upsilon_1, \upsilon_2)$ be a $\H{P}$−space. Therefore the subsequent statement is true:*

(i) $(D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2)$ is pair−Lindelöf if and only if $(G, \upsilon_1, \upsilon_2)$ is indeed,
(ii) $(D \times G, \kappa_1 \times \upsilon_1, \kappa_2 \times \upsilon_2)$ is pair−paracompact if and only if $(G, \upsilon_1, \upsilon_2)$ is so.

5. Some Counter Examples

We go over a number of counterexamples in this section that are pertinent to the definitions and theorems in the preceding sections. We will begin with some instances pertaining to the pair−ω− closed functions.

Example 2. *Let Υbe functioning from a discrete countable space (D, κ_1, κ_2) onto the space of rationals $(G, \upsilon_1, \upsilon_2)$.Next, $\Upsilon$is a pair−continuous pair−ω−closed function. But still Υ is not pair−closed. Additionally, for every g in $(G, \upsilon_1, \upsilon_2), \Upsilon^{-1}(g)$ is pair−Lindelöf. Additionally (D, κ_1, κ_2), $(G, \upsilon_1, \upsilon_2)$ are a pair−Lindelöf spaces, therefore pair−paracompact spaces. Theorem 3 is therefore more generic than the one that presumes the function to be pair−closed.*

In connection theorem 13, the example that follows is examined.

Example 3. *Assume S be the sorgenfry line and the sorgenfry plane $S \times S$. It is aware of this (R, κ_s, κ_s) is pair−Lindelöf spaces, therefore pair−paracompact spaces. However $S \times S$ is not pair−normal so it is not pair−paracompact .*

Example 4. *We are going to concentrate on $\H{P}$−space. Pay attention to any space lacking a condensation point is a $\H{P}$−space, but not a pair−space. Considering the foregoing, any countable space is a $\H{P}$−space. As an illustration of uncountable $\H{P}$−space $N \cup R$, that is first countable, locally compact and 0−dimensional, however, it lacks condensation point. Consequently it is $\H{P}$−space though not a pair−space.*

6. Conclusions

This study has shown us that the pair $-\omega-$closed functions are an extension of pair$-$closed functions. They are specified on topological spaces and have an effective method of holding onto sequence bounds. This suggests that if a series has a sequence of points in the function's domain that converge to a point, then the image of the series under the function will also converge to the image of the point. It's a way to extend the notion of closest to more complicated situations, such weakening these functions, therefore we obtain and investigate their key characteristics in this study, to ensure the concepts of pairwise pair$-\omega-$closed are understood. We have examined the salient features of these concepts and shown how they apply to different situations. We determined their overall fundamental features and the prerequisites that need to be satisfied for similar linkages to be made between them. We discussed their key characteristics and gave examples of how they complement one another. The study provided multiple examples of various functions along with highlighting their properties.These functions will act as a basis for additional studies into the potential applications of each of these functions. Other versions of these duties including fuzzy, soft, and group, [1],[10],[12],[16],[18] and [19]. , might be the subject of future investigation.

Acknowledgements

We sincerely thank everyone who made a contribution to this research. Their cooperation, guidance, and support have been crucial to completing the work.

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