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On the Category of (i, j)-Baire Bilocales

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Abstract. We define and characterize the notion of (i, j)-Baireness for bilocales. We also give internal properties of (i, j)-Baire bilocales which are not translated from properties of (i, j)-Baireness in bispaces. It turns out (i, j)-Baire bilocales are conservative in bilocales, in the sense that a bitopological space is almost (i, j)-Baire if and only if the bilocale it induces is (i, j)-Baire. Furthermore, in the class of Noetherian bilocales, (i, j)-Baireness of a bilocale coincides with (i, j)-Baireness of its ideal bilocale. We also consider relative versions of (i, j)-Baire where we show that a bilocale is (i, j)-Baire only if the subbilocale induced by the Booleanization is (i, j)-Baire. We use the characterization of (i, j)-Baire bilocales to introduce and characterize (τ_i, τ_j)-Baireness in the category of topobilocales.

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1. Introduction

In classical topology, a space is called *Baire* if the intersection of every sequence of dense open sets is dense. Baire spaces play an important role in different areas of mathematics such as analysis and mathematical logic. The concept of Baire spaces has also appeared in fuzzy set theory as well as soft set theory, see [22] and [2]. Fuzzy sets were introduced by Zadeh [23] and soft sets were initially introduced by Molodtsov [13]. Both of these sets were developed to solve the problem of modeling vagueness in real-life problems. Fuzzy sets have been applied in medical diagnosis [1] while fuzzy soft sets have been used to classify wood materials to prevent fire-related injuries and deaths [9].

In bispaces (spaces endowed with two topologies), an *almost* (i, j)*-Baire* bispace refers to a bispace (X, τ_1, τ_2) in which the intersection of any sequence of τ_i-dense τ_j-open subsets is τ_i-open. A study of almost (i, j)-Baire bispaces is documented in [8]. These bispaces also appear in a number of articles such as [7] and [6]. In locale theory, a *Baire* locale was introduced by Isbell [10] as one in which every non-void open sublocale is of second category. To our knowledge, (i, j)-Baireness has not yet appeared in the category of

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bilocales. In this paper, we introduce and study (i, j)-Baire bilocales. Our definition is rather an extension of almost (i, j)-Baire bispaces instead of Baire locales, with the prefix "almost" being dropped. Since the definition of almost (i, j)-Baire bispace is purely in terms of open subsets, we extend it to bilocales almost verbatim. We aim to extend some known bispaces results and also give some natural properties of (i, j)-Baire bilocales. Some of the natural results include (i, j)-Baireness of both the ideal bilocale and the subbilocale induced by the smallest dense sublocale. Extending results from spaces or bispaces to bilocales/biframes is not outrageous. For instance, Schauerte in [21] extended the notion of a normal space to a normal biframe. This paper contributes to the theory of bilocales.

This paper is organized as follows. Section two consists of the necessary background. In section three we introduce and characterize (i, j)-Baire bilocales. We also show that the class of (i, j)-Baire bilocales includes the following classes: (i) compact i-prefit bilocales, (ii) bilocales (L, L_1, L_2) where there is an i-prefit compactification $h : (M, M_1, M_2) \to (L, L_1, L_2)$ with which $h_*[L]$ is i-G_δ-dense in M, and (iii) i-pseudocomplete bilocales. In the class of Noetherian bilocales, a bilocale is (i, j)-Baire if and only if the induced ideal bilocale is (i, j)-Baire. In section four, we investigate relative versions of (i, j)-Baireness. We show that a bilocale is (i, j)-Baire only if the subbilocale induced by the smallest dense sublocale is (i, j)-Baire. We also introduce and characterize relatively (i, j)-Baire subbilocales. It turns out that in a class of dense subbilocales, (i, j)-Baire coincides with relatively (i, j)-Baire. In section five, we define and characterize (τ_i, τ_j)-Baire topobilocales.

2. Preliminaries

The book [17] is our main reference for notions of locales and sublocales. See [4, 15, 18] for the theory of bilocales.

2.1. Locales

A *locale* L is a complete lattice in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all $a \in L$, $B \subseteq L$. 1_L and 0_L, with subscripts dropped if there is no possibility of confusion, respectively denote the top element and the bottom element of a locale L. By a *point* of a locale L we mean an element a of L such that $a \neq 1$ and $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$ for all $b, c \in L$. We denote by a^* the *pseudocomplement* of an element $a \in L$. An element $a \in L$ is said to be *dense* and *complemented* in case $a^* = 0$ and $a \vee a^* = 1$, respectively. An element $x \in L$ is *compact* if $x \leq \bigvee A$ for $A \subseteq L$ implies $x \leq \bigvee F$ for some finite $F \subseteq A$. By a *compact* locale L we mean a locale in which the top element is compact. A *regular* locale is a locale L in which

$$a = \bigvee \{x \in L : x \prec a\}$$

for every $a \in L$, where $x \prec a$ means that $x^* \vee a = 1$.

By a *subframe* of a locale L, we mean a subset which is closed under joins and finite meets.

We denote by $\mathfrak{O}X$ the locale of open subsets of a topological space X.

A *localic map* is an infima-preserving function $f : L \to M$ between locales such that the corresponding left adjoint f^*, called the *frame homomorphism*, preserves binary meets. A frame homomorphism $h : M \to L$ is *dense* if $h(x) = 0$ implies $x = 0$ for all $x \in M$.

A *sublocale* of a locale L is a subset S closed under all meets and $x \twoheadrightarrow s \in S$ for every $x \in L$ and $s \in S$, where $\twoheadrightarrow$ is a *Heyting operation* on L satisfying that

$$a \leq b \twoheadrightarrow c \text{ if and only if } a \wedge b \leq c$$

for all $a, b, c \in L$. We denote by O the smallest sublocale of a locale L. We use $\mathcal{S}(L)$ to represent the coframe of sublocales of a locale L. For each $S \in \mathcal{S}(L)$, we define

$$L \smallsetminus S := \bigvee \{T \in \mathcal{S}(L) : T \cap S = \mathsf{O}\}.$$

The sublocales

$$\mathfrak{c}(a) = \{x \in L : a \leq x\} \quad \text{and} \quad \mathfrak{o}(a) = \{a \twoheadrightarrow x : x \in L\},$$

of a locale L are respectively the *closed* and *open* sublocales induced by an element a of L. They are complements of each other. The smallest closed sublocale of L that contains a sublocale S is called the *closure* of S and denoted by $\overline{S}$ or $\mathrm{cl}_L(S)$ with subscript L dropped when the locale is clear from the context.

For every Λ,

$$\mathfrak{c}\left(\bigvee_{\alpha \in \Lambda} x_\alpha\right) = \bigwedge_{\alpha \in \Lambda} \mathfrak{c}(x_\alpha).$$

A sublocale S of a locale L is *dense* and *nowhere dense* if $\overline{S} = L$ and $S \cap \mathfrak{B}L = \mathsf{O}$, respectively, where $\mathfrak{B}(L) = \{x \twoheadrightarrow 0 : x \in L\}$ is the *smallest dense* sublocale of L. We refer a reader to [14] for a comprehensive study of nowhere dense sublocales. By a *G_δ-sublocale* of a locale L, we mean a sublocale of the form $S = \bigwedge_{n \in \mathbb{N}} \mathfrak{o}(x_n)$.

For each sublocale $S \subseteq L$ there is an onto frame homomorphism $\nu_S : L \to S$ defined by $\nu_S(a) = \bigwedge \{s \in S : a \leq s\}$. Open sublocales and closed sublocales of a sublocale S of L are given by

$$\mathfrak{o}_S(\nu_S(a)) = S \cap \mathfrak{o}(a) \quad \text{and} \quad \mathfrak{c}_S(\nu_S(a)) = S \cap \mathfrak{c}(a),$$

respectively, for $a \in L$.

Each localic map $f : L \to M$ induces the functions $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$ given by the set-theoretic image of each sublocale of L under f, and $f_{-1}[-] : \mathcal{S}(M) \to \mathcal{S}(L)$ given by

$$f_{-1}[T] = \bigvee \{A \in \mathcal{S}(L) : A \subseteq f^{-1}(T)\}.$$

For a localic map $f : L \to M$ and $x \in M$,

$$f_{-1}[\mathfrak{c}_M(x)] = \mathfrak{c}_L(h(x)) \quad \text{and} \quad f_{-1}[\mathfrak{o}_M(x)] = \mathfrak{o}_L(h(x)).$$

We denote by $\widetilde{A}$ the sublocale of $\mathfrak{O}X$ *induced* by a subset A of a topological space X.

2.2. Bilocales

A *bilocale* is a triple (L, L_1, L_2) where L_1, L_2 are subframes of a locale L and for all $a \in L$,

$$a = \bigvee\{a_1 \wedge a_2 : a_1 \in L_1, a_2 \in L_2 \text{ and } a_1 \wedge a_2 \leq a\}.$$

We call L the *total part* of (L, L_1, L_2), and L_1 and L_2 the first and second parts, respectively. We use the notations L_i, L_j to denote the first or second parts of (L, L_1, L_2), always assuming that $i, j = 1, 2$, $i \neq j$.

For every bispace (X, τ_1, τ_2), there is a corresponding bilocale $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$. For example, let (X, τ_1, τ_2) be a bispace where $X = \{a, b\}$, $\tau_1 = \{\emptyset, X\}$ and $\tau_2 = \{\emptyset, X, \{a\}\}$. Then $(\tau_1 \vee \tau_2 = \tau_2, \tau_1, \tau_2)$ is a bilocale.

The *bilocale pseudocomplement* of $c \in L_i$ is given by

$$c^\bullet = \bigvee\{x \in L_j : x \wedge c = 0\}.$$

For all $a \in L_j, b \in L_i$, $a \wedge b = 0$ if and only if $a \leq b^\bullet$.

A bilocale (L, L_1, L_2) is *compact* if its total part is compact, and *regular* provided that

$$x = \bigvee\{a \in L_i : a \prec_i x\}$$

for every $x \in L_i$, where $a \prec_i x$ means that there is $c \in L_j$ such that $a \wedge c = 0$ and $c \vee x = 1$.

A *biframe homomorphism* (or *biframe map*) $h : (M, M_1, M_2) \to (L, L_1, L_2)$ is a frame homomorphism $h : M \to L$ for which $h(M_i) \subseteq L_i \quad (i = 1, 2)$. The map $h : M \to L$ is called the *total part* of $h : (M, M_1, M_2) \to (L, L_1, L_2)$. By a biframe map we mean a function $h : (M, M_1, M_2) \to (L, L_1, L_2)$ with a dense total part $h : M \to L$. It is *onto* if $h[M_i] = L_i$ for $i = 1, 2$.

A *subbilocale* of a bilocale (L, L_1, L_2) is a triple (S, S_1, S_2) where S is a sublocale of L and

$$S_i = \nu_S[L_i] \quad \text{for} \quad i = 1, 2.$$

We shall say that (S, S_1, S_2) is a P-subbilocale in case S has property P.

Recall that for a bilocale (L, L_1, L_2) and a sublocale S of L: [15]

$$\text{int}_i(S) = \bigvee\{\mathfrak{o}(a) : a \in L_i, \mathfrak{o}(a) \subseteq S\} \quad (i = 1, 2).$$

and [18]

$$\text{cl}_i(S) = \bigwedge\{\mathfrak{c}(a) : a \in L_i, S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}\left(\bigvee\{a \in L_i : S \subseteq \mathfrak{c}(a)\}\right) \quad (i = 1, 2).$$

For each $a \in L_i$, $\mathfrak{c}(a^\bullet) = \text{cl}_j(\mathfrak{o}(a))$.

A sublocale A of a bilocale (L, L_1, L_2) is *i-dense* if $\text{cl}_i(A) = L$. This is equivalent to saying that S is i-dense if and only if for each non-zero $x \in L_i$, $\mathfrak{o}(x) \cap S \neq \mathsf{O}$. Every sublocale containing an i-dense sublocale is i-dense.

Given a bilocale (L, L_1, L_2), a sublocale S of L is *(i, j)-nowhere dense* if $\text{int}_j(\text{cl}_i(S)) = \mathsf{O}$ $(i \neq j \in \{1, 2\})$. As a result, a sublocale S of a bilocale (L, L_1, L_2) is (i, j)-nowhere dense if and only if $L \smallsetminus \mathrm{cl}_i(S)$ is j-dense if and only if $\overline{S}$ is (i,j)-nowhere dense. Furthermore, an element $a \in L_i$ is j-dense if and only if $\mathfrak{c}(a)$ is (i,j)-nowhere dense.

When dealing with subbilocales, say (S, S_1, S_2), we write i_S-dense, i_S-open and (i_S, j_S)-nowhere dense instead of i-dense, i-open and (i,j)-nowhere dense.

By an *i-G_δ-sublocale* of a bilocale (L, L_1, L_2), we mean a sublocale of the form $S = \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ where each $x_n \in L_i$. A sublocale of a bilocale (L, L_1, L_2) is *i-G_δ-dense* if it meets every nonvoid i-G_δ-sublocales.

For the bilocale $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ induced by a bispace (X, τ_1, τ_2), we shall write U is τ_i-open if $U \in \tau_i$ and U is τ_i-dense if U is dense with respect to the topological space (X, τ_i).

3. (i,j)-Baire Bilocales

Recall that a bispace (X, τ_1, τ_2) is *almost (i,j)-Baire* [8] if any collection $\{U_n : n \in \mathbb{N}\}$ of τ_i-dense τ_j-open subsets of X satisfies the condition $\bigcap_{n\in\mathbb{N}} U_n$ is τ_i-dense. In this section, we extend the definition of almost (i,j)-Baire bispaces to bilocales where the prefix "almost" shall be dropped. We aim to define (i,j)-Baire bilocales in such a way that a bispace (X, τ_1, τ_2) is almost (i,j)-Baire if and only if the bilocale $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is (i,j)-Baire.

We shall call an open (resp. closed) sublocale *i-open* (resp. i-closed) in case the inducing element is an element of L_i.

Definition 1. *A bilocale (L, L_1, L_2) is said to be (i,j)-Baire if the intersection of countably many i-dense j-open sublocales is i-dense.*

Example 1. *Since, in locales, the intersection of dense sublocales is dense, every symmetric bilocale (bilocale of the form (L, L, L)) is (i,j)-Baire.*

For a bispace (X, τ_1, τ_2) and $A \subseteq X$, define

$$\widetilde{A} = \{\mathrm{int}_{\tau_1\vee\tau_2}((X \smallsetminus A) \cup G) : G \in \tau_1 \vee \tau_2\}.$$

It is clear that $\widetilde{A}$ is a sublocale of $\tau_1 \vee \tau_2$. For each $x \in X$, $\widetilde{x} = X \smallsetminus \mathrm{cl}_{\tau_1\vee\tau_2}\{x\}$ is a point of $\tau_1 \vee \tau_2$. Just like in the case of locales, $\mathfrak{o}(U) = \widetilde{U}$ for every $U \in \tau_1 \vee \tau_2$.

Recall from [11] that given a topological property P, a bispace (X, τ_1, τ_2) is sup-P if $(X, \tau_1 \vee \tau_2)$ has property P. In [16], we proved the following result.

Lemma 1. *Let (X, τ_1, τ_2) be a sup-T_D-bispace. Then $A \subseteq X$ is τ_i-dense in (X, τ_1, τ_2) iff $\widetilde{A}$ is i-dense in $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$.*

In [19], the authors show that if X is a topological space and $Y \subseteq X$, then

$$\widetilde{Y} = \bigvee\{\{X \smallsetminus \overline{\{y\}}, 1_{\mathfrak{O}X}\} : y \in Y\}.$$

Proposition 1. *Let (X, τ_1, τ_2) be a sup-T_D-bispace in which G_δ-sublocales of $(\tau_1\vee\tau_2, \tau_1, \tau_2)$ are complemented. Then (X, τ_1, τ_2) is almost (i,j)-Baire iff $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is (i,j)-Baire.*

Proof. ($\Longrightarrow$): Let $\{\mathfrak{o}(U_n) : n \in \mathbb{N}\}$ be a collection of i-dense j-open sublocales. It follows that $\{U_n : n \in \mathbb{N}\}$ is a collection of τ_i-dense τ_j-open subsets of X. It follows that $\bigcap_{n\in\mathbb{N}} U_n$ is τ_i-dense.

Claim: $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n)$ is i-dense.

Proof: Let $\mathfrak{o}(V)$ be an i-open sublocale such that $\mathfrak{o}(V) \cap \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n) = \mathsf{O}$. Then

$$\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(V \cap U_n) = \mathsf{O}.$$

We must have that $\bigcap_{n\in\mathbb{N}}(V \cap U_n) = \emptyset$. Otherwise, there is $x \in V \cap U_n$ for each $n \in \mathbb{N}$. Since each $V \cap U_n$ is $\tau_1 \vee \tau_2$,

$$\widetilde{V \cap U_n} = \mathfrak{o}(V \cap U_n) \ni \widetilde{x}$$

for each $n \in \mathbb{N}$. Therefore $\widetilde{x} \in \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(V \cap U_n)$ which is impossible. Therefore $V = \emptyset$ so that $\mathfrak{o}(V) = \mathsf{O}$. Thus $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n)$ is i-dense.

($\Longleftarrow$): Let $\{U_n : n \in \mathbb{N}\}$ be a collection of τ_i-dense τ_j-open subsets of X. Then $\{\mathfrak{o}(U_n) : n \in \mathbb{N}\}$ is a collection of i-dense j-open sublocales of $\tau_1 \vee \tau_2$. It follows that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n)$ is τ_i-dense. To show that $\bigcap_{n\in\mathbb{N}} U_n$ is τ_i-dense, let V be a nonempty τ_i-open subset of X such that $V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right) = \emptyset$. Then

$$\bigcup_{n\in\mathbb{N}} (X \smallsetminus (V \cap U_n)) = X.$$

Observe that $\bigvee_{n\in\mathbb{N}} \mathfrak{c}(V \cap U_n) = \mathfrak{O}X$: Let $p \in X$. Then $p \in X \smallsetminus (V \cap U_n)$ for some $n \in \mathbb{N}$. Therefore $\overline{\{p\}} \subseteq X \smallsetminus (V \cap U_n)$ for some $n \in \mathbb{N}$ so that $V \cap U_n \subseteq X \smallsetminus \overline{\{p\}}$. This implies that $X \smallsetminus \overline{\{p\}} \in \mathfrak{c}(V \cap U_n)$. As a result,

$$\{X \smallsetminus \overline{\{p\}}, 1_{\tau_1 \vee \tau_2}\} \subseteq \mathfrak{c}(V \cap U_n).$$

Therefore

$$\begin{aligned} \tau_1 \vee \tau_2 &= \bigvee \{\{X \smallsetminus \overline{\{p\}}, 1_{\tau_1 \vee \tau_2}\} : p \in X\} \\ &\subseteq \bigvee \{\mathfrak{c}(V \cap U_n) : n \in \mathbb{N}\} \\ &\subseteq \tau_1 \vee \tau_2. \end{aligned}$$

Since $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n)$ is i-dense and $\mathfrak{o}(V)$ is non-void i-open,

$$\mathfrak{o}(V) \cap \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(U_n) \neq \mathsf{O},$$

so that

$$\mathsf{O} \neq \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(V \cap U_n).$$

Because $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(V\cap U_n)$ is a complemented G_δ-sublocale, we have that

$$\bigvee_{n\in\mathbb{N}} \mathfrak{c}(V\cap U_n)\neq \tau_1\vee\tau_2,$$

which is impossible.

Definition 2. *Let (L, L_1, L_2) be a bilocale. A sublocale S of L is said to be of (i, j)-first category if there are countably many (i, j)-nowhere dense sublocales S_n, $n\in\mathbb{N}$, such that $S\subseteq\bigvee_{n\in\mathbb{N}} S_n$. It is of (i, j)-second category if it is not of (i, j)-first category.*

Theorem 1. *Let (L, L_1, L_2) be a bilocale whose j-G_δ-sublocales are complemented in L. The following statements are equivalent:*

(i) (L, L_1, L_2) *is* (i, j)*-Baire.*

(ii) Each non-void i-open sublocale is of (j, i)-second category.

(iii) Every sublocale of (j, i)-first category has void i-interior.

(iv) The supplement of every sublocale of (j, i)-first category is i-dense.

Proof. $(i)\implies(ii)$: Let U be a non-void i-open sublocale of L and assume that $U\subseteq\bigvee_{n\in\mathbb{N}} S_n$ for some collection $\{S_n : n\in\mathbb{N}\}$ of (j, i)-nowhere dense sublocales. Then $U\subseteq\bigvee_{n\in\mathbb{N}} \overline{S_n}$ where members of the collection $\{\overline{S_n} : n\in\mathbb{N}\}$ are (j, i)-nowhere dense. It follows that members of the collection $\{L\smallsetminus \mathrm{cl}_j(\overline{S_n}) : n\in\mathbb{N}\}$ are i-dense j-open sublocales. By hypothesis, $\bigwedge_{n\in\mathbb{N}}(L\smallsetminus \mathrm{cl}_j(\overline{S_n})$ is i-dense so that

$$U\cap\left(\bigwedge_{n\in\mathbb{N}}(L\smallsetminus \mathrm{cl}_j(\overline{S_n}))\right)\neq \mathsf{O}.$$

Therefore

$$\begin{aligned}\mathsf{O}\neq&\left(\bigvee_{k\in\mathbb{N}}\overline{S_k}\right)\cap\left(\bigwedge_{n\in\mathbb{N}}(L\smallsetminus \mathrm{cl}_j(\overline{S_n}))\right)\\ &=\bigvee_{k\in\mathbb{N}}\left(\overline{S_k}\cap\left(\bigwedge_{n\in\mathbb{N}}(L\smallsetminus\overline{S_n})\right)\right)\\ &\subseteq\bigvee_{k\in\mathbb{N}}\left(\overline{S_k}\cap(L\smallsetminus\overline{S_k})\right)\\ &=\mathsf{O}\end{aligned}$$

which is impossible.

$(ii)\implies(iii)$: Let S be a sublocale of (j, i)-first category with a non-void i-interior. We then get that $\mathrm{int}_i(S)$ is a non-void i-open sublocale which must be of (j, i)-second category by (ii). This is a contradiction.

$(iii) \implies (iv)$: Let S be a sublocale of L which is of (j,i)-first category and choose $x \in L_i$ with $\mathfrak{o}(x) \cap (L \smallsetminus S) = \mathsf{O}$. Then $\mathfrak{o}(x) \subseteq S$. Since S satisfies the conditions hypothesized in (iii), $\mathrm{int}_i(S) \neq \mathsf{O}$, so that the i-open sublocale $\mathfrak{o}(x)$ is void.

$(iv) \implies (i)$: Let $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ be a collection of i-dense j-open sublocales and assume that there is an i-open sublocale $\mathfrak{o}(y)$ with

$$\mathfrak{o}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n) \right) = \mathsf{O}$$

Then $\mathfrak{o}(y) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{c}(x_n)$, making $\mathfrak{o}(y)$ a sublocale of (j,i)-first category. By (iv), $L \smallsetminus \mathfrak{o}(y) = \mathfrak{c}(y)$ is i-dense, i.e., $\mathrm{cl}_i(\mathfrak{c}(y)) = \mathfrak{c}(y) = 1$. Therefore $y = 0$ so that $\mathfrak{o}(y) = \mathsf{O}$. Hence $\left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right)$ is i-dense.

We shall say that a collection $\mathcal{C}$ of sublocales of L has the *Finite Intersection Property (FIP)* if the intersection of every finite subcollection of $\mathcal{C}$ has a non-void intersection.

Proposition 2. *If a bilocale (L, L_1, L_2) is compact, then every collection of closed sublocales with the FIP has a non-void intersection.*

Proof. Let $\{\mathfrak{c}(x_\alpha) : \alpha \in \Lambda\}$ be a collection with the FIP and assume that $\bigwedge_{\alpha\in\Lambda} \mathfrak{c}(x_\alpha) = \mathsf{O}$. Then

$$L = L \smallsetminus \bigwedge_{\alpha\in\Lambda} \mathfrak{c}(x_\alpha) = \bigvee_{\alpha\in\Lambda} \mathfrak{o}(x_\alpha),$$

making $\mathfrak{o}\left(\bigvee_{\alpha\in\Lambda} x_\alpha\right) = L$. Therefore $\bigvee_{\alpha\in\Lambda} x_\alpha = 1$. Since (L, L_1, L_2) is compact, there is a finite set $F \subseteq \Lambda$ such that $\bigvee_{\alpha\in F} x_\alpha = 1$. We get that

$$\mathsf{O} = \mathfrak{c}\left(\bigvee_{\alpha\in F} x_\alpha\right) = \bigwedge_{\alpha\in F} \mathfrak{c}(x_\alpha),$$

which contradicts that $\{\mathfrak{c}(x_\alpha) : \alpha \in \Lambda\}$ has the FIP.

Remark 1. *The converse of the preceding result holds. We are however interested in the forward direction, that is why we only proved it.*

Recall from [20] that a locale L is *prefit* if for each nonzero $x \in L$ there is a nonzero $y \in L$ such that $y^\star \vee x = 1$. A bispace (X, τ_1, τ_2) is *almost regular* if for each nonempty $U \in \tau_i$, there is nonempty $V \in \tau_i$ such that $\mathrm{cl}_{\tau_j}(V) \subseteq U$. Since prefitness is a localic version of almost regularity in spaces (spaces in which every nonempty open set contains some closure of a nonempty open subset), we define a *prefit bilocale* (L, L_1, L_2) using the notion of almost regular bispace as one in which for each $x \in L_i$, $i = 1, 2$, there is $y \in L_i$ such that $y^\bullet \vee x = 1$. Related to prefit bilocales, we give the following definition.

Definition 3. *Call a bilocale (L, L_1, L_2) i-prefit in case for every nonzero $x \in L$, there is a nonzero $y \in L_i$ such that $y^\bullet \vee x = 1$.*

We consider some examples.

Example 2. *(i) The bilocale of reals is an example of a prefit bilocale which is not i-prefit.*

(ii) For any almost regular bispace (X, τ_1, τ_2) with $\tau_1 \subseteq \tau_2$, the bilocale $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is 2-prefit. In particular, if L is prefit, then (L, L, L) is i-prefit ($i = 1, 2$).

(iii) By [21], a bilocale (L, L_1, L_2) is Boolean if for each $x \in L_i$, $i = 1, 2$, there is $c \in L_j$ ($i \neq j$) such that $x \wedge c = 0$ and $x \vee c = 1$. Boolean and i-prefit are incomparable: Consider the set $X = \{a, b, c, d\}$ endowed with topologies $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$. It is clear that $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is Boolean. This bilocale is not i-prefit ($i = 1, 2$) since for the set $\{a\} \in \tau_1 \vee \tau_2$, there is no nonempty $U \in \tau_i$ satisfying that $U^\bullet \vee \{a\} = X$.

For any non-Boolean prefit locale L, (L, L, L) is an example of a non-Boolean i-prefit ($i = 1, 2$) bilocale.

In the following result, we show that the class of (i, j)-Baire bilocales contains compact i-prefit bilocales.

Proposition 3. *Every compact i-prefit bilocale is (i, j)-Baire.*

Proof. Let (L, L_1, L_2) be a compact i-prefit bilocale and choose a collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}, x_n \in L_i\}$ of i-dense j-open sublocales and a non-void j-open sublocale $\mathfrak{o}(y)$. Then

$$\mathfrak{o}(y) \cap \mathfrak{o}(x_n) \neq \mathsf{O}$$

for each $n \in \mathbb{N}$. This makes $y \wedge x_n \neq 0$. Since (L, L_1, L_2) is i-prefit, there is nozero $b_1 \in L_i$ such that $b_1^\bullet \vee (y \wedge x_1) = 1$. Because $\mathfrak{o}(x_2)$ is i-dense, we have that $\mathfrak{o}(x_2) \cap \mathfrak{o}(b_1) \neq \mathsf{O}$ so that $x_2 \wedge b_1$ is a nonzero element of L. By i-prefitness again, there is nonzero $b_2 \in L_i$ such that $b_2^\bullet \vee (x_2 \wedge b_1) = 1$. Continuing like this for $n = 3, 4, ..,$ we find $b_n \in L_i$ such that $b_n^\bullet \vee (x_n \wedge b_{n-1}) = 1$. Therefore $\mathfrak{c}(b_n^\bullet) \subseteq \mathfrak{o}(x_n) \cap \mathfrak{o}(b_{n-1})$. Since each $b_n \in L_i$, we have that $\mathfrak{c}(b_n^\bullet) = \mathrm{cl}_j(\mathfrak{o}(b_n))$. Therefore

$$... \subseteq \mathfrak{c}(b_3^\bullet) = \mathrm{cl}_j(\mathfrak{o}(b_3)) \subseteq \mathfrak{c}(b_2^\bullet) = \mathrm{cl}_j(\mathfrak{o}(b_2)) \subseteq \mathfrak{c}(b_1^\bullet) = \mathrm{cl}_j(\mathfrak{o}(b_1)) \subseteq \mathfrak{o}(y) \cap \mathfrak{o}(x_1).$$

We now have the decreasing sequence $\mathfrak{c}(b_1^\bullet), \mathfrak{c}(b_2^\bullet), \mathfrak{c}(b_3^\bullet), ...$ of closed sublocales, so that the collection $\{\mathfrak{c}(b_n^\bullet) : n \in \mathbb{N}\}$ has the FIP. By compactness of (L, L_1, L_2), $\bigwedge_{n\in\mathbb{N}} \mathfrak{c}(b_n^\bullet) \neq \mathsf{O}$. Because

$$\bigwedge_{n\in\mathbb{N}} \mathfrak{c}(b_n^\bullet) \subseteq \mathfrak{o}(y) \cap \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n),$$

we then have that

$$\mathfrak{o}(y) \cap \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n) \neq \mathsf{O},$$

making $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ an i-dense sublocale.

The converse of Proposition 3 is not always true, as shown below.

Example 3. *For the bispace $(\mathbb{N}, \tau_D, \tau_{cf})$, $(\tau_D \vee \tau_{cf} = \tau_D, \tau_D, \tau_{cf})$ is (τ_D, τ_{cf})-Baire but not compact. Its (τ_D, τ_{cf})-Baireness follows since $\mathbb{N}$ is the only τ_D-dense member of τ_{cf}.*

Recall that for any onto frame homomorphism $h : M \to L$, $h_* : L \to h_*[L]$ is a frame isomorphism. If h is further dense, then $h_*(0) = 0$.

Lemma 2. *Let $h : (M, M_1, M_2) \to (L, L_1, L_2)$ be a dense and onto biframe map. If $x \in L_i$ is j-dense, then $h_*(x) \wedge a \neq 0$ whenever $a \in M_i$ is nonzero.*

Proof. Let $a \in M_i$ be nonzero such that $a \wedge h_*(x) = 0$. Then

$$0 = h(a) \wedge h(h_*(x)) = h(a) \wedge x.$$

Since $h[M_i] \subseteq L_i$, $h(a) \in L_i$ so that $h(a) = 0$. Therefore $a \leq h_*(h(a)) = h_*(0) = 0$.

Proposition 4. *Let (L, L_1, L_2) be a bilocale. If there is a dense onto biframe map $h : (M, M_1, M_2) \to (L, L_1, L_2)$ from an (i, j)-Baire bilocale (M, M_1, M_2) with which $h_*[L]$ is i-G_δ-dense in M, then (L, L_1, L_2) is (i, j)-Baire.*

Proof. Suppose that the hypothesized statement is true and let $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ be a collection of i-dense j-open sublocales of L. Now, if $\mathfrak{o}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right) = \mathsf{O}$ for some i-open sublocale $\mathfrak{o}(y)$ of L, then

$$\mathsf{O} = h_*[\mathfrak{o}(y)] \cap h_* \left[\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right] = h_*[\mathfrak{o}(y)] \cap \left(\bigwedge_{n\in\mathbb{N}} h_*[\mathfrak{o}(x_n)]\right)$$

where the first equality follows since $h_*[\mathsf{O}] = \mathsf{O}$ and h_* is injective, and the second equality follows since the total part $h_* : L \to M$ is a right adjoint. By virtue of $h_* : L \to h_*[L]$ being a frame isomorphism and hence open, we get that

$$\mathfrak{o}_{h_*[L]}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{h_*[L]}(h_*(x_n))\right) = \mathsf{O}.$$

For each $n \in \mathbb{N}$, $h_*(x_n) = h_*(h(a_n))$ for some $a_n \in M_i$. Therefore

$$\begin{aligned} \mathsf{O} &= \mathfrak{o}_{h_*[L]}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{h_*[L]}(h_*(h(a_n)))\right) \\ &= h_*[L] \cap \mathfrak{o}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} (h_*[L] \cap \mathfrak{o}(h_*(h(a_n))))\right) \\ &= h_*[L] \cap \mathfrak{o}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(h_*(h(a_n)))\right) \\ &\supseteq h_*[L] \cap \mathfrak{o}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(a_n)\right). \end{aligned}$$

Since i-G$_\delta$-dense sublocales are dense, $h_*[L]$ is a dense sublocale of M. Each of the a_n's is i-dense: Pick $b \in M_i$ such that $b \wedge a_n = 0$. Then $h(b) \wedge h(a_n) = 0$ so that

$$h_*(h(b)) \wedge h_*(h(a_n)) = h_*(0) = 0,$$

where the latter equality follows since $h_* : L \to M$ is a dense localic map. Therefore $b \wedge h_*(x_n) = 0$. By Lemma 2, $b = 0$ and hence each a_n is i-dense.

Therefore the collection $\{\mathfrak{o}(a_n) : n \in \mathbb{N}\}$ consists of i-dense j-open sublocales of M. We then get that $\mathfrak{o}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} (\mathfrak{o}(a_n))\right)$ is an i-G$_\delta$-sublocale. Because $h_*[L]$ is i-G$_\delta$-dense, so

$$\mathfrak{o}(h_*(y)) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(a_n)\right) = \mathsf{O}.$$

Since (M, M_1, M_2) is (i, j)-Baire, it follows that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(a_n)$ is i-dense, so that $\mathfrak{o}(h_*(y)) = \mathsf{O}$. Therefore $h_*(y) = 0$, so that

$$0 = h(h_*(y)) = y.$$

This means that $\mathfrak{o}(y) = \mathsf{O}$. Thus $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ is i-dense, and hence (L, L_1, L_2) is (i, j)-Baire.

A *compactification* of a bilocale (L, L_1, L_2) is a dense and onto biframe map $h : (M, M_1, M_2) \to (L, L_1, L_2)$ from a compact regular bilocale (M, M_1, M_2). So, for a bilocalic property P, we shall say that (L, L_1, L_2) has a P-compactification in case (M, M_1, M_2) has property P.

Corollary 1. *Let (L, L_1, L_2) be a bilocale. If there is a i-prefit compactification $h : (M, M_1, M_2) \to (L, L_1, L_2)$ with which $h_*[L]$ is i-G$_\delta$-dense in M, then (L, L_1, L_2) is (i, j)-Baire.*

Definition 4. *Let (L, L_1, L_2) be a bilocale. An i-π-base for (L, L_1, L_2) is a collection $\mathcal{C}$ of non-void i-open sublocales such that each non-void i-open sublocale of L contains a member of $\mathcal{C}$. A bilocale is said to be i-pseudocomplete if it is i-prefit and it has a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of i-π-bases such that whenever $\mathfrak{o}(x_n) \in \mathcal{C}_n$ and $\mathrm{cl}_j(\mathfrak{o}(x_{n+1})) \subseteq \mathfrak{o}(x_n)$ for each n, then $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n) \neq \mathsf{O}$.*

Proposition 5. *Every i-pseudocomplete bilocale is (i, j)-Baire.*

Proof. Let (L, L_1, L_2) be a pseudocomplete bilocale and pick a collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ of i-dense j-open sublocales. Since (L, L_1, L_2) is pseudocomplete, there is a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of i-π-bases with the corresponding pseudocompleteness property. For each non-void i-open sublocale $\mathfrak{o}(y)$, we have that each $\mathfrak{o}(y) \cap \mathfrak{o}(x_n)$ is a non-void open sublocale, so that $y \wedge x_n \neq 0$. Since (L, L_1, L_2) is i-prefit, for $n = 1$, there is nonzero $a_1 \in L_i$ such that $a_1^\bullet \vee (y \wedge x_1) = 1$. This makes

$$\mathsf{O} \neq \mathfrak{o}(a_1) \subseteq \mathrm{cl}_j(\mathfrak{o}(a_1)) \subseteq \mathfrak{o}(y) \cap \mathfrak{o}(x_1).$$

For the i-π-base $\mathcal{C}_1$, there is $\mathfrak{o}(c_1) \in \mathcal{C}_1$ with

$$\mathfrak{o}(c_1) \subseteq \mathrm{cl}_j(\mathfrak{o}(c_1)) \subseteq \mathfrak{o}(a_1) \subseteq \mathfrak{o}(y) \cap \mathfrak{o}(x_1).$$

Using the fact that $\mathfrak{o}(x_2)$ is i-dense and $\mathfrak{o}(c_1)$ is non-void i-open, we get that $\mathfrak{o}(c_1) \cap \mathfrak{o}(x_2)$ is a non-void open sublocale. Since (L, L_1, L_2) is i-prefit, there is nonzero $a_2 \in L_i$ such that

$$\mathsf{O} \neq \mathfrak{o}(a_2) \subseteq \mathrm{cl}_j(\mathfrak{o}(a_2)) \subseteq \mathfrak{o}(c_1) \cap \mathfrak{o}(x_2).$$

An application of pseudocompleteness to the i-π-base $\mathcal{C}_2$ yields an existence of $\mathfrak{o}(c_2) \in \mathcal{C}_2$ such that

$$\mathfrak{o}(c_2) \subseteq \mathrm{cl}_j(\mathfrak{o}(c_2)) \subseteq \mathfrak{o}(a_2) \subseteq \mathfrak{o}(c_1) \cap \mathfrak{o}(x_2).$$

Since $\mathfrak{o}(x_3)$ is i-dense and $\mathfrak{o}(c_2)$ is a non-void i-open sublocale, it follows that $\mathfrak{o}(c_2) \cap \mathfrak{o}(x_3) \neq \mathsf{O}$. Applying that (L, L_1, L_2) is i-prefit again implies that there is a nonzero $a_3 \in L_i$ such that

$$\mathsf{O} \neq \mathfrak{o}(a_3) \subseteq \mathrm{cl}_i(\mathfrak{o}(a_3)) \subseteq \mathfrak{o}(c_2) \cap \mathfrak{o}(x_3).$$

Therefore, for the i-π-base $\mathcal{C}_3$, there is $\mathfrak{o}(c_3) \in \mathcal{C}_3$ such that

$$\mathfrak{o}(c_3) \subseteq \mathrm{cl}_j(\mathfrak{o}(c_3)) \subseteq \mathfrak{o}(a_3) \subseteq \mathfrak{o}(c_2) \cap \mathfrak{o}(x_3).$$

Continuing like this for $n = 4, 5, ...$, we get that for each i-π-base $\mathcal{C}_n$, there is $\mathfrak{o}(c_n) \in \mathcal{C}_n$ such that

$$\mathfrak{o}(c_n) \subseteq \mathrm{cl}_j(\mathfrak{o}(c_n)) \subseteq \mathfrak{o}(c_{n-1}) \cap \mathfrak{o}(x_n).$$

Since (L, L_1, L_2) is i-pseudocomplete, $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(c_n) \neq \mathsf{O}$. Because

$$\mathfrak{o}(c_n) \subseteq \mathfrak{o}(y) \cap \mathfrak{o}(x_n)$$

for each $n \in \mathbb{N}$, we have

$$\mathsf{O} \neq \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(c_n) \subseteq \mathfrak{o}(y) \cap \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n),$$

making $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ an i-dense sublocale. Thus (L, L_1, L_2) is (i, j)-Baire.

Recall from [5] that the triple $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2)$, where $\mathfrak{J}L$ is the locale of all ideals of L and $(\mathfrak{J}L)_i$ $(i = 1, 2)$ is the subframe of $\mathfrak{J}L$ consisting of all ideals $J \subseteq L$ generated by $J \cap L_i$, is a bilocale called the *ideal bilocale*.

Call a bilocale (L, L_1, L_2) *Noetherian* in case its total part L is Noetherian, i.e., all of its elements are compact. In a Noetherian locale, all ideals are principal [3]. This suggests that in a Noetherian locale, the locale $\mathfrak{J}L$ of ideals of a locale L is isomorphic to L.

For use below, we recall from [15, Proposition 6.9.] that in a bilocale (L, L_1, L_2), if $x \in L_i$ is j-dense, then $\downarrow x \in \mathfrak{J}L_i$ is $\mathfrak{J}L_j$-dense. Furthermore, for a Noetherian bilocale (L, L_1, L_2), $\bigvee J \in L_i$ is j-dense whenever $J \in \mathfrak{J}L_i$ is $\mathfrak{J}L_j$-dense.

Proposition 6. *Let (L, L_1, L_2) be a bilocale. Then (L, L_1, L_2) is (i, j)-Baire only if $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2)$ is (i, j)-Baire. Moreover, if (L, L_1, L_2) is Noetherian, then (L, L_1, L_2) is (i, j)-Baire iff $(\mathfrak{J}L, (\mathfrak{J}L)_1, (\mathfrak{J}L)_2)$ is (i, j)-Baire.*

Proof.

Choose a collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ of i-dense j-open sublocales. Then $\{\mathfrak{o}_{\mathfrak{J}L}(\downarrow x_n) : n \in \mathbb{N}\}$ is a collection of $\mathfrak{J}L_i$-dense $\mathfrak{J}L_j$-open sublocales. By hypothesis, $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{\mathfrak{J}L}(\downarrow x_n)$ is $\mathfrak{J}L_i$-dense. To show that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ is i-dense, let $\mathfrak{o}(y)$ be an i-open sublocale such that $\mathfrak{o}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right) = \mathsf{O}$.

Claim: $\mathfrak{o}_{\mathfrak{J}L}(\downarrow y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{\mathfrak{J}L}(\bigvee \downarrow x_n)\right) = \mathsf{O}$.

Proof: Otherwise, $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(\downarrow y \cap \downarrow x_n) \neq \mathsf{O}$ which implies that

$$\mathsf{O} \neq \downarrow y \cap \downarrow x_n = \downarrow (y \wedge x_n)$$

for each $n \in \mathbb{N}$. Therefore $y \wedge x_n \neq 0$ for each $n \in \mathbb{N}$ so that

$$\mathsf{O} \neq \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(y \wedge x_n) = \mathfrak{o}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right)$$

which is a contradiction.

Thus $\mathfrak{o}_{\mathfrak{J}L}(\downarrow y) = \mathsf{O}$ implying that $\downarrow y = \mathsf{O}$. Therefore $\mathfrak{o}(y) = \mathsf{O}$ and hence $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ is i-dense.

The particular case follows since L is isomorphic to $\mathfrak{J}L$.

4. Concerning relative versions of (i, j)-Baire bilocales

In this section, we consider (i, j)-Baireness of subbilocales.

We recall the following lemma from [16].

Lemma 3. *Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2). An element y of L_i is j-dense iff $\nu_S(y)$ is j_S-dense.*

Corollary 2. *Let (S, S_1, S_2) be a dense subbilocale of a bilocale (L, L_1, L_2). An element y of L_i is j-dense iff $\mathfrak{o}_S(\nu_S(y)) = S \cap \mathfrak{o}(y)$ is j_S-dense i_S-open.*

We also have the following result.

Lemma 4. *Let (L, L_1, L_2) be a bilocale with (S, S_1, S_2) as its dense subbilocale. A sublocale A of S is i_S-dense iff it is i-dense.*

Proof. ($\Longrightarrow$): Choose a non-void i-open sublocale $\mathfrak{o}(x)$ of L. Then

$$\mathsf{O} \neq S \cap \mathfrak{o}(x) = \mathfrak{o}(\nu_S(x))$$

where $\nu_S(x) \in S_i$. This makes $\mathfrak{o}_S(\nu_S(x))$ an non-void i_S-open sublocale of S. Since A is i_S-open,

$$\mathsf{O} \neq A \cap \mathfrak{o}_S(\nu_S(x)) = A \cap \mathfrak{o}(x).$$

Thus A is i-dense.

($\Longleftarrow$): Let $\mathfrak{o}_S(x)$ be a non-void i_S-open sublocale of S. Then $x = \nu_S(y)$ for some $y \in L_i$. It follows from Lemma 3 that y is i-dense. Therefore

$$\mathsf{O} \neq A \cap \mathfrak{o}(y) = A \cap \mathfrak{o}_S(x).$$

Thus A is i_S-dense.

Proposition 7. *A bilocale (L, L_1, L_2) is (i,j)-Baire only if it contains some dense (i,j)-Baire subbilocale.*

Proof.

Let (S, S_1, S_2) be a dense and (i,j)-Baire subbilocale of (L, L_1, L_2) and pick a collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ of i-dense j-open sublocales. Since the subbilocale (S, S_1, S_2) is dense, it follows from Corollary 2 that $\{S \cap \mathfrak{o}(x_n) : n \in \mathbb{N}\}$ is a collection of i_S-dense j_S-open sublocales. By hypothesis, $\bigwedge_{n\in\mathbb{N}}(S \cap \mathfrak{o}(x_n))$ is i_S-dense, so that it is i-dense by Lemma 4. Since

$$\bigwedge_{n\in\mathbb{N}} (S \cap \mathfrak{o}(x_n)) \subseteq \bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n),$$

it follows that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ is i-dense.

Corollary 3. *A bilocale (L, L_1, L_2) is (i,j)-Baire only if $(\mathfrak{B}L, \nu_{\mathfrak{B}}[L_1], \nu_{\mathfrak{B}}[L_2])$ is (i,j)-Baire as a bilocale.*

Call a bilocale (L, L_1, L_2) (i,j)-*submaximal* if every i-dense sublocale of L is j-open

Proposition 8. *Let (L, L_1, L_2) be an (i,j)-submaximal bilocale. Then (L, L_1, L_2) is (i,j)-Baire iff $(\mathfrak{B}L, \nu_{\mathfrak{B}}[L_1], \nu_{\mathfrak{B}}[L_2])$ is (i,j)-Baire as a bilocale.*

Proof. We only prove the forward implication:

Let $\{\mathfrak{o}_{\mathfrak{B}L}(x_n) : n \in \mathbb{N}\}$ be a collection of $i_{\mathfrak{B}L}$-dense $j_{\mathfrak{B}L}$-open sublocales. It follows that $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ is a collection of i-dense j-open sublocales. Since (L, L_1, L_2) is (i,j)-Baire, $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)$ is i-dense. We must have that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{\mathfrak{B}L}(x_n)$ is $i_{\mathfrak{B}L}$-dense, otherwise there is a non-void $\nu_{\mathfrak{B}L}[L_i]$-open sublocale $\mathfrak{o}_{\mathfrak{B}L}(y)$ such that

$$\mathfrak{o}_{\mathfrak{B}L}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_{\mathfrak{B}L}(x_n) \right) = \mathsf{O}.$$

Therefore

$$\mathfrak{o}_{\mathfrak{B}L}(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n) \right) = \mathsf{O}.$$

Since every dense sublocale is i-dense and (L, L_1, L_2) is (i,j)-submaximal, we have that $\mathfrak{B}L$ is j-open so that $\mathfrak{o}_{\mathfrak{B}L}(y) = \mathfrak{B}L \cap \mathfrak{o}(y)$ is a j-open sublocale. Therefore $\mathfrak{o}_{\mathfrak{B}L}(y) = 0$ which is impossible.

Proposition 9. *Every i-open subbilocale of an (i,j)-Baire bilocale is (i,j)-Baire.*

Proof. Let (S,S_1,S_2) be an i-open subbilocale of an (i,j)-Baire bilocale (L,L_1,L_2). Choose a collection $\{\mathfrak{o}_S(x_n) : n \in \mathbb{N}\}$ of i_S-dense j_S-open sublocales. We show that $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)$ is i_S-dense. Pick an i_S-open sublocale $\mathfrak{o}_S(y)$ such that

$$\left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)\right) \cap \mathfrak{o}_S(y) = \mathsf{O}.$$

Since $\mathfrak{o}_S(y) \subseteq \overline{S}$, $\mathfrak{o}_S(y) \cap (L \smallsetminus \overline{S}) = \mathsf{O}$. Therefore

$$\begin{aligned}\mathfrak{o}_S(y) \cap \left(\left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)\right) \vee (L \smallsetminus \overline{S})\right) &= \left(\left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)\right) \cap \mathfrak{o}_S(y)\right) \vee \big(\mathfrak{o}_S(y) \cap (L \smallsetminus \overline{S})\big)\\ &= \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)\right) \cap \mathfrak{o}_S(y)\\ &= \mathsf{O}.\end{aligned}$$

Because $\mathfrak{o}_S(x_n) \vee (L \smallsetminus \overline{S})$ is i-dense, it follows that

$$\bigwedge_{n\in\mathbb{N}} \big(\mathfrak{o}_S(x_n) \vee (L \smallsetminus \overline{S})\big) = (L \smallsetminus \overline{S}) \vee \bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)$$

is i-dense. Therefore $\mathfrak{o}_S(y) = \mathsf{O}$. Thus $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)$ is i_S-dense.

Definition 5. *Let (L,L_1,L_2) be a bilocale. A subbilocale (S,S_1,S_2) of (L,L_1,L_2) is relatively (i,j)-Baire if for every collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ of i-dense j-open sublocales, $S \cap \big(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\big)$ is i_S-dense.*

Proposition 10. *In a class of dense subbilocales, (i,j)-Baire coincides with relatively (i,j)-Baire.*

Proof. Let (S,S_1,S_2) be an (i,j)-Baire subbilocale of a bilocale (L,L_1,L_2) and choose a collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ of i-dense j-open sublocales of L. If

$$\mathfrak{o}_S(y) \cap S \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right) = \mathsf{O},$$

then

$$\mathsf{O} = \mathfrak{o}_S(y) \cap \left(\bigwedge_{n\in\mathbb{N}} (S \cap \mathfrak{o}(x_n))\right) = \mathfrak{o}_S(y) \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(\nu_S(x_n))\right)$$

where each $\mathfrak{o}_S(\nu_S(x_n))$ is i_S-dense and j_S-open. Since (S,S_1,S_2) is (i,j)-Baire as a bilocale, $\bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(\nu_S(x_n))$ is i_S-dense so that $\mathfrak{o}_S(y) = \mathsf{O}$. Thus $S \cap \big(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\big)$ is i_S-dense.

On the other hand, let (S, S_1, S_2) be a relatively (i,j)-Baire subbilocale and pick a collection $\{\mathfrak{o}_S(x_n) : n \in \mathbb{N}\}$ of i_S-dense j_S-open sublocales of S. For each x_n, there is $a_n \in L_j$ such that $x_n = \nu_S(a_n)$. Now, members of the collection $\{\mathfrak{o}_S(a_n) : n \in \mathbb{N}\}$ are i-dense j-open in (L, L_1, L_2). Since (S, S_1, S_2) is relatively (i,j)-Baire,

$$S \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(a_n) \right) = \bigwedge_{n\in\mathbb{N}} \mathfrak{o}_S(x_n)$$

is i_S-dense. Thus (S, S_1, S_2) is (i,j)-Baire.

Here is an example of what is illustrated in Proposition 10.

Example 4. *Given a bilocale (L, L_1, L_2), the subbilocale $(\mathfrak{B}L, \nu_{\mathfrak{B}}[L_1], \nu_{\mathfrak{B}}[L_2])$ of (L, L_1, L_2) is (i,j)-Baire if and only if it is relatively (i,j)-Baire.*

We close this section with a characterization of relatively (i,j)-Baire subbilocales.

Proposition 11. *Let (S, S_1, S_2) be a dense and complemented subbilocale of a bilocale (L, L_1, L_2) whose j-G_δ-sublocales are complemented. The following statements are equivalent:*

(i) *(S, S_1, S_2) is relatively (i,j)-Baire.*

(ii) *For every non-void i-open sublocale U of L, $S \cap U$ is of (j,i)-second category in (S, S_1, S_2).*

(iii) *For every sublocale U of (j,i)-first category in (L, L_1, L_2), $\mathrm{int}_{i_S}(S \cap U) = \mathsf{O}$.*

(iv) *If V is a sublocale of (j,i)-first category in (L, L_1, L_2), then $S \cap (L \smallsetminus V)$ is i_S-dense.*

Proof. $(i) \implies (ii)$: Let $\mathfrak{o}(x)$ be non-void i-open and assume that

$$S \cap \mathfrak{o}(x) \subseteq \bigvee_{n\in\mathbb{N}}^{S} \mathfrak{c}_S(x_n)$$

for some collection $\{\mathfrak{c}_S(x_n) : n \in \mathbb{N}\}$ of (j_S, i_S)-nowhere dense sublocales of S. Then

$$S \cap \mathfrak{o}(x) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{c}(x_n)$$

where each $\mathfrak{c}(x_n)$ is (j,i)-nowhere dense because (S, S_1, S_2) is dense. It is clear that the collection $\{\mathfrak{o}(x_n) : n \in \mathbb{N}\}$ consists of i-dense j-open sublocales. It follows from (i) that $S \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n)\right)$ is i_S-dense. Since $S \cap \mathfrak{o}(x) \neq \mathsf{O}$ because of density of (S, S_1, S_2), we have that

$$S \cap \mathfrak{o}(x) \cap S \cap \left(\bigwedge_{n\in\mathbb{N}} \mathfrak{o}(x_n) \right) \neq \mathsf{O}.$$

Therefore

$$\left(\bigvee_{k\in\mathbb{N}}\mathfrak{c}(x_k)\right)\cap\left(\bigwedge_{n\in\mathbb{N}}\mathfrak{o}(x_n)\right)\neq\mathsf{O}.$$

Since $\bigwedge_{n\in\mathbb{N}}\mathfrak{o}(x_n)$ is a j-G_δ-sublocale of L, it follows that it is complemented. Therefore

$$\mathsf{O}\neq\bigvee_{k\in\mathbb{N}}\left(\mathfrak{c}(x_k)\cap\left(\bigwedge_{n\in\mathbb{N}}\mathfrak{o}(x_n)\right)\right)\subseteq\bigvee_{k\in\mathbb{N}}\left(\mathfrak{c}(x_k)\cap\mathfrak{o}(x_k)\right)=\bigvee_{n\in\mathbb{N}}(\mathsf{O})=\mathsf{O}$$

which is impossible. Thus $S\cap\mathfrak{o}(x)$ is (j,i)-second category.

$(ii)\implies(i)$: Let $\{\mathfrak{o}(x_n):n\in\mathbb{N}\}$ be a collection of i-dense j-open sublocales and assume that there is non-void i_S-open sublocale $\mathfrak{o}_S(y)$ of S such that

$$\mathfrak{o}_S(y)\cap S\cap\left(\bigwedge_{n\in\mathbb{N}}\mathfrak{o}(x_n)\right)=\mathsf{O}.$$

Then $\mathfrak{o}(y)$ is non-void i-open and

$$\mathfrak{o}_S(y)\cap\left(\bigwedge_{n\in\mathbb{N}}\mathfrak{o}(x_n)\right)=\mathsf{O}$$

which implies

$$\mathfrak{o}_S(y)\subseteq S\cap\left(\bigvee_{n\in\mathbb{N}}\mathfrak{o}(x_n)\right)=\bigvee_{n\in\mathbb{N}}\mathfrak{c}_S(\nu_S(x_n))$$

where the latter equality holds since S is complemented. Since each $\mathfrak{c}_S(\nu_S(x_n))$ is (j_S,i_S)-nowhere dense, $S\cap\mathfrak{o}(y)=\mathfrak{o}_S(y)$ is of (j,i)-first category in (S,S_1,S_2) which is a contradiction.

$(ii)\implies(iii)$: Let U be a sublocale of L which is of (j,i)-first category in (L,L_1,L_2) and assume that $\mathrm{int}_{i_S}(S\cap U)\neq\mathsf{O}$. Then

$$\mathrm{int}_{i_S}(S\cap U)=\mathfrak{o}(x)\cap S$$

for some $x\in L_i$. Such $\mathfrak{o}(x)$ is a non-void i-open sublocale of L, so

$$\mathrm{int}_{i_S}(S\cap U)=S\cap\mathfrak{o}(x)$$

must be of (j,i)-second category in (S,S_1,S_2) by (ii). But $U\subseteq\bigvee_{n\in\mathbb{N}}\mathfrak{c}(x_n)$ for some collection $\{\mathfrak{c}(x_n):n\in\mathbb{N}\}$ of (j,i)-nowhere dense sublocales of L, so

$$\mathrm{int}_{i_S}(S\cap U)=\mathfrak{o}(x)\cap S\subseteq U\cap S\subseteq S\cap\bigvee_{n\in\mathbb{N}}\mathfrak{c}(x_n)=\bigvee_{n\in\mathbb{N}}\mathfrak{c}_S(\nu_S(x_n))$$

where each $\mathfrak{c}_S(\nu_S(x_n))$ is (j_S,i_S)-nowhere dense in (S,S_1,S_2). This makes $\mathfrak{o}(x)\cap S$ a sublocale of (j,i)-first category in (S,S_1,S_2) which is impossible.

(iii) $\Longrightarrow$ *(iv)*: Let V be a sublocale of L which is of (j,i)-first category in (L, L_1, L_2) and choose an i_S-open sublocale $\mathfrak{o}_S(x)$ such that

$$\mathfrak{o}_S(x) \cap S \cap (L \smallsetminus V) = \mathsf{O}.$$

Then

$$\mathfrak{o}_S(x) \subseteq S \cap V.$$

By *(iii)*, $\mathrm{int}_{iS}(S \cap V) = \mathsf{O}$, making $\mathfrak{o}_S(x) = \mathsf{O}$.

(iv) $\Longrightarrow$ *(ii)*: Let $\mathfrak{o}(x)$ be a non-void i-open sublocale of L and assume that $S \cap \mathfrak{o}(x)$ is of (j,i)-first category. By *(iv)*,

$$S \cap (L \smallsetminus \mathfrak{o}(x)) = S \cap \mathfrak{c}(x) = \mathfrak{c}_S(\nu_S(x))$$

is i_S-dense which implies that $\nu_S(x) = 0$. Therefore $\mathfrak{o}(x) = \mathsf{O}$ which is a contradiction.

5. Baireness of topobilocales

The aim of this section is to introduce and characterize Baireness in the category of topobilocales.

A *topobilocale* [12] is a triple (L, τ_1, τ_2) where L is a locale, L_1 and L_2 are subframes of L all of whose elements are complemented in L. Each member of τ_i $(i = 1, 2)$ is called τ_i-open.

For each $a \in L$, the τ_i-*closure* $(i = 1, 2)$ of a in L is defined by

$$\mathrm{cl}_{(L,\tau_i)}(a) = \bigwedge\{b \in \tau_i' : a \leq b\}$$

and the τ_i-*interior* of a is defined by

$$\mathrm{int}_{(L,\tau_i)}(a) = \bigvee\{b \in \tau_i : b \leq a\}.$$

We have the following result. See [24] for the proofs of some of the statements. For the rest of the statements, the proofs resemble that of [16, Proposition 5.1.3.].

Proposition 12. *Let (L, τ_i, τ_j) be a topobilocale. Then*

(i) $\mathrm{cl}_{(L,\tau_i)}(0) = \mathrm{int}_{(L,\tau_i)}(0) = 0.$

(ii) $\mathrm{cl}_{(L,\tau_i)}(1) = \mathrm{int}_{(L,\tau_i)}(1) = 1.$

(iii) $a \leq \mathrm{cl}_{(L,\tau_i)}(a).$

(iv) *If $a \leq b$, then* $\mathrm{cl}_{(L,\tau_i)}(a) \leq \mathrm{cl}_{(L,\tau_i)}(b).$

(v) $\mathrm{int}_{(L,\tau_i)}(a) \leq a.$

(vi) *If $a \leq b$, then* $\mathrm{int}_{(L,\tau_i)}(a) \leq \mathrm{int}_{(L,\tau_i)}(b).$

(vii) *For each* $a \in L$, $(\mathrm{cl}_{(L,\tau_i)}(a))' = \mathrm{int}_{(L,\tau_i)}(a')$.

(viii) *For each* $a \in L$, $(\mathrm{int}_{(L,\tau_i)}(a))' = \mathrm{cl}_{(L,\tau_i)}(a')$.

Call an element $a \in L$ τ_i*-dense* if $\mathrm{cl}_{(L,\tau_i)}(a) = 1$. Clearly, $a \in L$ is τ_i-dense if and only if $a \wedge x \neq 0$ for all nonzero $x \in \tau_i$, see [15, Proposition 2.8.] for the proof.

Definition 6. *Call a topobilocale* (L, τ_1, τ_2) (τ_i, τ_j)*-Baire if any sequence* $(x_n)_{n\in\mathbb{N}}$ *of* τ_i*-dense elements of* τ_j *satisfies the condition* $\bigwedge_{n\in\mathbb{N}} x_n$ *is* τ_i*-dense.*

Call an element $a \in L$ (τ_i, τ_j)*-nowhere dense* if $\mathrm{int}_{(L,\tau_j)}(\mathrm{cl}_{(L,\tau_i)}(a)) = 0$.

An element $a \in L$ is of (τ_i, τ_j)*-first category* if $a \leq \bigvee_{n\in\mathbb{N}} x_n$ for some collection $\{x_n : n \in \mathbb{N}\}$ of (τ_i, τ_j)-nowhere dense elements of L. Otherwise it is of (τ_i, τ_j)*-second category.* It is clear that if a is of (τ_i, τ_j)*-first category* and $b \leq a$, then b is of (τ_i, τ_j)*-first category.*

For use below, we give the following result with a proof similar to that of [16, Proposition 2.1.4.]

Proposition 13. *Let* (L, τ_1, τ_2) *be a topobilocale. Then* $a \in L$ *is* (τ_i, τ_j)*-nowhere dense iff* $(\mathrm{cl}_{(L,\tau_i)}(a))'$ *is* τ_j*-dense.*

Proposition 14. *Let* (L, τ_1, τ_2) *be a topobilocale. The following statements are equivalent.*

(i) (L, τ_1, τ_2) *is* (τ_i, τ_j)*-Baire.*

(ii) *Each nonzero* τ_i *element is of* (τ_j, τ_i)*-second category.*

(iii) *Every element of* (τ_j, τ_i)*-first category has a zero* τ_i*-interior.*

(iv) *The complement an element of* (τ_j, τ_i)*-first category is* τ_i*-dense.*

Proof. $(i) \Longrightarrow (ii)$: Assume that there is a nonzero element $a \in \tau_i$ which is of (τ_j, τ_i)-first category. Then $a \leq \bigvee_{n\in\mathbb{N}} x_n$ for some collection $\{x_n : n \in \mathbb{N}\}$ of (τ_j, τ_i)-nowhere dense elements. It is clear members of the collection $\{(\mathrm{cl}_{(L,\tau_j)}(x_n))' : n \in \mathbb{N}\}$ are τ_i-dense. By (i), $\bigwedge_{n\in\mathbb{N}}(\mathrm{cl}_{(L,\tau_j)}(x_n))'$ is τ_i-dense. It follows that

$$a \wedge \left(\bigwedge_{n\in\mathbb{N}} (\mathrm{cl}_{(L,\tau_j)}(x_n))' \right) \neq 0.$$

Therefore

$$\begin{aligned}
0 &\neq \left(\bigvee_{k\in\mathbb{N}} x_k \right) \wedge \left(\bigwedge_{n\in\mathbb{N}} (\mathrm{cl}_{(L,\tau_j)}(x_n))' \right) \\
&= \bigvee_{k\in\mathbb{N}} \left(x_k \wedge \left(\bigwedge_{n\in\mathbb{N}} (\mathrm{cl}_{(L,\tau_j)}(x_n))' \right) \right) \text{ since } L \text{ is a locale} \\
&\leq \bigvee_{k\in\mathbb{N}} \left(x_k \wedge (\mathrm{cl}_{(L,\tau_j)}(x_k))' \right)
\end{aligned}$$

$$\leq \bigvee_{k\in\mathbb{N}} \Big(\mathrm{cl}_{(L,\tau_i)}(x_k) \wedge \mathrm{cl}_{(L,\tau_j)}(x_k)\Big)$$
$$= 0$$

which is a contradiction.

(ii) ⇒ *(iii)*: Let $a \in L$ be of (τ_j, τ_i)-first category and assume that $\mathrm{int}_{(L,\tau_i)}(a) \neq 0$. We now have $\mathrm{int}_{(L,\tau_i)}(a)$ as a nonzero τ_i element. It follows from *(ii)* that $\mathrm{int}_{(L,\tau_i)}(a)$ is of (τ_j, τ_i)-second category. This is not possible.

(iii) ⇒ *(iv)*: Let $a \in L$ be of (τ_j, τ_i)-first category and suppose that a' is not τ_i-dense. Then $\mathrm{cl}_{(L,\tau_i)}(a') \neq 1$. Because $\mathrm{cl}_{(L,\tau_i)}(a') = (\mathrm{int}_{(L,\tau_i)}(a))'$, we have that $(\mathrm{int}_{(L,\tau_i)}(a))' \neq 1$. Since a is of (τ_j, τ_i)-first category, it follows from *(iii)* that $\mathrm{int}_{(L,\tau_i)}(a) = 0$ so that $(\mathrm{int}_{(L,\tau_i)}(a))' = 1$, which is a contradiction.

(iv) ⇒ *(i)*: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of τ_i-dense elements of τ_j and assume that there is $y \in \tau_i$ such that $y \wedge \left(\bigwedge_{n\in\mathbb{N}} x_n\right) = 0$. Then

$$y \leq \left(\bigwedge_{n\in\mathbb{N}} x_n\right)' = \bigvee_{n\in\mathbb{N}} x_n'$$

since each x_n is complemented. This makes y to be of (τ_j, τ_i)-first category. By *(iv)*, y' is τ_i-dense so that $y = 0$. Thus $\bigwedge_{n\in\mathbb{N}} x_n$ is τ_i-dense. Hence (L, τ_1, τ_2) is (τ_i, τ_j)-Baire.

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