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Comparative Analysis of Modified Admoian Decomposition Method and Homotopy Perturbation Mohand Transform Method for Solving Burger's Equations

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Abstract. This article examines the analytical solution for Burger's equations utilizing MADM and AHPMTM. We compare both analytical methods for convergence. The MADM uses a novel integral transform (the Mohand transform) with the Adomian decomposition method. The MADM solves the proposed problem using series form solutions that quickly converge to the exact solutions. The homotopy process with Mohand transformed and accelerated He's polynomials underlie the novel AHPMTM approach to accelerate the convergence of the homotopy perturbation Mohand transform method (HPMTM). We compare solutions for MADM and AHPMTM to an exact solution. The methodology can be applied to different models in applied sciences and technology.

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Key Words and Phrases: Burger's equation, Modified Adomian decomposition method, Homotopy perturbation Mohand transform method, Analytical solution

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1. Introduction

In mathematical language, differential equations (DEs) are major sources for modeling physical phenomena that arise in applied sciences and technology. These equations have different parameters that describe the present state of most physical phenomena. The DEs have numerous applications in real-world problems, such as in neural networks [17], the dynamics of the system are largely determined by the time delay, which is a fundamental

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component of the system [6], delays have been demonstrated to cause oscillations in coupled systems with synchronization, but they can also improve synchronization between coupled elements [16], time delay laser dynamics model [27], traffic dynamics that include a delay to account for drivers' finite reaction times [8], a time-delayed compensation applied to both the local and remote sites of the teleportation system can be used to stabilize the control for teleporting mobile robots [30], time delay for controlling unstable motion [15], the non-instantaneous consequences of relaxation stresses in some materials' delay system of viscoelastic effects [28].
The solutions such as type DEs are investigated by different researchers. The Boubaker used polynomials to solve DEs [14]. Sedaghat et al. proposed a new numerical technique based on the transferred Chebyshev polynomials. Also they used the variational iteration technique for solving the pantograph equation with delay, Cevik implemented the perturbation method with an iteration algorithm for the selected DEs [4]. The solution of high-order DEs is approximated by exponential polynomials given in [9]. The pantograph generalized equation solution with variable coefficient and delay is found by using Bessel polynomials [19]. In [26], the author used the homotopy perturbation method (HPM) for the numerical solution of DEs. In [18], the authors presented a combination of two semi-analytical methods called the "singular perturbed homotopy analysis method" and applied it to give a numerical s'imulation for the combustion of spray fuel droplets. Tohidi et al. and Akyuz and Sezer checked the validity and applicability of the Bernoulli collocation technique for the solution of pantograph generalized problems [10]. Some other methods are also implemented for the solution of DEs such as the Jacobi rational Gauss collocation method [13], Runge-Kutta methods [12], Hermite polynomials [21], Chebyshev wavelet method [11, 29] and others ([1, 2]).

Besides these methods, we have implemented two newly developed approaches for the analytical solution of these proposed models. The first one is the MADM, which is based on the Adomain decomposition procedure with the Mohand transform. The MADM provides a series-form solution that converges to the exact-form solution. The second method is the AHPMTM, which employs the homotopy perturbation procedure with the Mohand transform and utilizes accelerated He's polynomials for the nonlinearity terms. Solving nonlinear Burger's DEs verifies the applicability and validity of these methods. We compare the obtained results using tables and plotting.

We prearrange the rest of the article as follows: in Section 2, we introduce the studied problem; in Section 3, we define the fundamental concepts for completing this research work; in Section 4, we elaborate on the MADM general procedure; in Section 5, we explain the new AHPMTM for the solution of nonlinear PDEs; in Section 6, we consider testing problems to ensure the validity and applicability of the proposed methods; in Section 7, we present the results and discussions; and in Section 8, we conclude the comparison analysis.

2. The studied problems

In this article, the Burger's equations are considered as follows:

$$\vartheta_t(x,t) = \vartheta_{xx}(x,t)\,\vartheta_x(x,t) - \vartheta_x(x,t) - \vartheta(x,t), \qquad t > 0,\ x \in R,$$

$$\vartheta_t(x,t) = \vartheta_x(x,t)\,\vartheta(x,t) + \vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t), \qquad t > 0,\ x \in R.$$

Burger's equations can be used to model dynamics and communicate with acoustic waves, reaction devices, convection effects, heat conduction, diffusion transports, and more. Additional references ([24]-[31]) are included for more information. In [3], some well-known mathematical methods for coupled Burger's equations are compared. This comparison used the following methods: The optimal homotopy asymptotic method (OHAM), OHAM with Daftardar-Jafari polynomials, Laplace transform Adomian decomposition, and homotopy perturbation methods. Burger's model has been examined and researched by numerous scientists for various fluid dynamics and physical flow issues. Burger's equation, due to the diffusion term with viscosity coefficient and the non-linear convection term, resembles Navier-Stokes equations (NSEs) in structure. Thus, one may think of this equation as a simplified version of the NSEs.

3. Basic definitions

The Mohand transform has some useful properties, including linearity, convolution, differentiation, and inversion, which make it a powerful tool in signal processing and other areas. It also has some connections with other well-known transforms, such as the Laplace transform and the Mellin transform. In this section, we present some key definitions and introductory ideas for the Mohand transform [20] and accelerated He's polynomials.

3.1. Mohand transform

Mahgoub and Mohand explained the Mohand transform for the first time in 2017 for the function $f(t)$ for $t \geq 0$. For a function $f(t)$, the Mohand transformation indicated by $\mathbb{M}$ is defined as:

$$\mathbb{M}\{f(t)\} = F(s) = s^2 \int_0^\infty f(t)e^{-st}dt, \qquad k_1 \leq s \leq k_2.$$

If the Mohand transform of a function $f(t)$ is $F(s)$ then $f(t)$ is known as the inverse of $F(s)$ which can be described by ([23], [25]):

$$\mathbb{M}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi\, i}\int_{x-i\infty}^{x+i\infty} e^{st}\frac{F(s)}{s^2}ds, \qquad \mathbb{M}^{-1} \text{ is the inverse Mohand operator.}$$

Some properties of the Mohand transform [20]:

- Linearity property for $\mathbb{M}\{.\}$: For arbitrary constants a_1, a_2, we have

$$\mathbb{M}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1\mathbb{M}\{f_1(t)\} + a_2\mathbb{M}\{f_2(t)\}.$$

- Change of scale property: If $\mathbb{M}\{f(t)\} = F(s)$, then

$$\mathbb{M}\{f(at)\} = aF\left(\frac{s}{a}\right).$$

- Shifting property for $\mathbb{M}\{.\}$: If $\mathbb{M}\{f(t)\} = F(s)$, then

$$\mathbb{M}\{e^{at}f(t)\} = \frac{s^2}{(s-a)^2}F(s-a).$$

- Convolution theorem for $\mathbb{M}\{.\}$: If $\mathbb{M}\{f_1(t)\} = F_1(s)$ and $\mathbb{M}\{f_2(t)\} = F_2(s)$, then

$$\mathbb{M}\{f_1(t) * f_2(t)\} = \frac{1}{s^2}F_1(s)\,F_2(s).$$

- Mohand transforms of the derivatives of the function $f(t)$: If $\mathbb{M}\{f(t)\} = F(s)$ then we have the following three properties of the derivatives:

$$\begin{aligned}
&\mathbb{M}\left\{f'(t)\right\} = sF(s) - s^2 f(0),\\
&\mathbb{M}\left\{f''(t)\right\} = s^2F(s) - s^3 f(0) - s^2 f'(0),\\
&\mathbb{M}\left\{f^{(n)}(t)\right\} = s^n\,F(s) - s^{n+1}f(0) - s^n f'(0) - \ldots - s^2\,f^{(n-1)}(0).
\end{aligned} \tag{1}$$

Mohand transforms for some famous functions:

$f(t)$	$\mathbb{M}\{f(t)\} = F(s)$
1	s
t	1
t^2	$\frac{2!}{s}$
$t^n, n \in N$	$\frac{n!}{s^{n-1}}$
$t^n, n > -1$	$\frac{\Gamma(n+1)}{s^{n-1}}$
e^{at}	$\frac{s^2}{s-a}$
sin(at)	$\frac{as^2}{s^2+a^2}$
cos(at)	$\frac{s^3}{s^2+a^2}$

3.2. Accelerated He's polynomials

The nonlinear term $N(\vartheta)$ in the differential equations under study can be expressed as a linear combination of the accelerated He's polynomials in the following series from:

$$N(\vartheta) = \sum_{n=0}^{\infty} \bar{H}_n, \tag{2}$$

where the accelerated He's polynomials $\bar{H}_n$ can be defined and constructed with the help of the standard He's polynomials as follows [22]:

$$\bar{H}_n = N(\vartheta_n) - \sum_{j=0}^{n-1} \hat{H}_j, \tag{3}$$

here, $\hat{H}$ is the general He's polynomials.
Consider a nonlinear term $N(\vartheta) = \vartheta_{xx}(x,t)\,\vartheta(x,t)$, by using the definition given in equations (2) and (3), we can get the accelerated He's polynomials as:

$$\bar{H}_0 = (\vartheta_0)_{xx}(x,t)\,\vartheta_0(x,t)\,,$$

$$\bar{H}_1 = (\vartheta_0)_{xx}(x,t)\,\vartheta_1(x,t) + (\vartheta_1)_{xx}(x,t)\,\vartheta_0(x,t) + (\vartheta_1)_{xx}(x,t)\,\vartheta_1(x,t)\,,$$

$$\bar{H}_2 = (\vartheta_0)_{xx}(x,t)\,\vartheta_2(x,t) + (\vartheta_1)_{xx}(x,t)\,\vartheta_2(x,t) + (\vartheta_2)_{xx}(x,t)\,\vartheta_2(x,t) + (\vartheta_2)_{xx}(x,t)\,\vartheta_0(x,t) + (\vartheta_2)_{xx}(x,t)\,\vartheta_1(x,t)\,,$$

$$\bar{H}_3 = (\vartheta_0)_{xx}(x,t)\,\vartheta_3(x,t) + (\vartheta_1)_{xx}(x,t)\,\vartheta_3(x,t) + (\vartheta_2)_{xx}(x,t)\,\vartheta_3(x,t) + (\vartheta_3)_{xx}(x,t)\,\vartheta_3(x,t) + (\vartheta_3)_{xx}(x,t)\,\vartheta_2(x,t) + (\vartheta_3)_{xx}(x,t)\,\vartheta_1(x,t)\,.$$

In the current section, we briefly explained the procedure of a newly adopted modified technique. We have considered the general Burger's equation which is defined as:

$$\vartheta_t(x,t) = \pounds\left(\vartheta(x,t)\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\,, \qquad t>0,\ x\in R, \tag{4}$$

with the initial condition:

$$\vartheta(x,0) = w_1(x),$$

where, $\pounds, N$ are the linear and nonlinear terms, respectively, and $\delta(x,t)$ is the source function.
Applying the Mohand transform with Burger's equation (4) gives us:

$$\mathbb{M}\left\{\vartheta_t(x,t)\right\} = \mathbb{M}\left\{\pounds\left(\vartheta(x,t)\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\},$$

by applying the transformation property (1) we can get:

$$s\{R(x,s) - s\,\vartheta(x,0)\} = \mathbb{M}\left\{\pounds\left(\vartheta(x,t)\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\}, \tag{5}$$

after some calculation, the equation (5) was simplified as:

$$R(x,s) = s\,\vartheta(x,0) + \frac{1}{s}\mathbb{M}\left\{\pounds\left(\vartheta(x,t)\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\},$$

by using the inverse Mohand transformation we get:

$$\vartheta(x,t) = \vartheta(x,0) + \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\pounds\left(\vartheta(x,t)\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\}\right\}. \tag{6}$$

4. The implementation of the modified Adomian decomposition method

Thus, the first initial component for the approximate solution of the given problem (4) will be obtained as follows:

$$\vartheta_0(x,t) = \vartheta(x,0) + \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\delta(x,t)\right\}\right\},$$

then, the final iterative scheme for the other terms becomes as:

$$\vartheta_{m+1}(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\pounds\left(\vartheta_m(x,t)\right) + A_m\right\}\right\}, \qquad m \geq 0. \tag{7}$$

The nonlinear term N is decomposed by using the Adomian's polynomials defined as:

$$N(\vartheta) = \sum_{m=0}^{\infty} A_m, \tag{8}$$

where,

$$A_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\vartheta_i\right)\right]\right]_{\lambda=0}, \qquad m = 0, 1, \cdots \tag{9}$$

For more details and applications of the modified decomposition method see [7].

5. Basic concepts of AHPMTM

In this section, a semi-analytical method known as the AHPMTM is used for the solution of Burger's equation. Consider the same general nonlinear Burger's equation, which takes the form (4).

By using the homotopy perturbation technique to its corresponding equation (6), we get:

$$\begin{aligned}(1-\rho)\left(\vartheta(x,t) - \vartheta(x,0)\right) + \rho\left(\vartheta(x,t) - \vartheta(x,0)\right)\right) - \rho\Big(\mathbb{M}^{-1}\Big\{\frac{1}{s}\mathbb{M}\Big\{\pounds\left(\vartheta(x,t)\right) \\ + N\left(\vartheta(x,t)\right) + \delta(x,t)\Big\}\Big\}\Big) = 0,\end{aligned} \tag{10}$$

here, $\vartheta(x,t)$ takes the following form:

$$\vartheta(x,t) = \sum_{n=0}^{\infty} \vartheta_n(x,t)\rho^n, \tag{11}$$

and the nonlinear term takes the following form:

$$N\left(\vartheta(x,t)\right) = \sum_{n=0}^{\infty} H_n\rho^n, \tag{12}$$

where H_n represent the accelerated He's polynomials and are defined as:

$$H_n(\vartheta_0, \vartheta_1, ..., \vartheta_n) = N(\vartheta_n) - \sum_{j=o}^{n-1} \hat{H}_j,$$

where $\hat{H}_j$ are the general He's polynomials.
Now, by substituting the equations (11) and (12) in equation (10), we get the final recursive scheme:

$$\sum_{n=0}^{\infty} \vartheta_n\left(x,t\right)\rho^n = \rho\left(\vartheta\left(x,0\right) + \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\sum_{n=0}^{\infty} \pounds\left(\vartheta_n(x,t)\right)\rho^n + \sum_{n=0}^{\infty} H_n\rho^n + \delta\left(x,t\right)\right\}\right\}\right), \tag{13}$$

and the approximated terms will be obtained by comparing the coefficient of the like powers of ρ.

6. Numerical applications

In this section, we have tested the applicability and validity of the MADM and AHPMTM in the solution for nonlinear Burger's equations.

6.1. Implementation of the MADM on Problem 1

Consider the nonlinear Burger's equation defined as [5]:

$$\vartheta_t\left(x,t\right) = \vartheta_{xx}\left(x,t\right)\vartheta_x\left(x,t\right) - \vartheta_x\left(x,t\right) - \vartheta\left(x,t\right) + \delta(x,t), \qquad t > 0,\ x \in R, \tag{14}$$

with the initial condition $\vartheta(x,0) = 0$. Here, $\delta(x,t) = e^{-x} + t^2\, e^{-2x}$, and the nonlinear operator $N\left(\vartheta(x,t)\right) = \vartheta_{xx}\left(x,t\right)\vartheta_x\left(x,t\right)$.

Using Mohand transformation to equation (14) as:

$$\mathbb{M}\left\{\vartheta_t\left(x,t\right)\right\} = \mathbb{M}\left\{-\vartheta_x\left(x,t\right) - \vartheta\left(x,t\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\},$$

by using the transformation property defined by (1), we get:

$$s\{R(x,s) - s\,\vartheta(x,0)\} = \mathbb{M}\left\{-\vartheta_x\left(x,t\right) - \vartheta\left(x,t\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\}, \tag{15}$$

after the simplification of the equation (15), the simplified form takes the following form:

$$R(x,s) = s\,\vartheta(x,0) + \frac{1}{s}\mathbb{M}\left\{-\vartheta_x\left(x,t\right) - \vartheta\left(x,t\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\}, \tag{16}$$

implementing the inverse of Mohand transformation equation (16), the scheme becomes as:

$$\vartheta\left(x,t\right) = \vartheta(x,0) + \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{-\vartheta_x\left(x,t\right) - \vartheta\left(x,t\right) + N\left(\vartheta(x,t)\right) + \delta(x,t)\right\}\right\}. \tag{17}$$

Thus, the first term becomes as:

$$\vartheta_0\left(x,t\right)=\vartheta(x,0)+\mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\delta(x,t)\right\}\right\},$$

then, the final iterative scheme for other terms becomes as:

$$\vartheta_{m+1}\left(x,t\right)=\mathbb{M}^{-1}\left\{\tfrac{1}{s}\mathbb{M}\left\{-(\vartheta_m)_x\left(x,t\right)-\vartheta_m\left(x,t\right)+A_m\right\}\right\},\qquad m\geq 0, \tag{18}$$

with using the initial condition and the nonlinear term $N\left(\vartheta_m(x,t)\right.$ which is decomposed by using the formula (8), the approximated term $\vartheta_0\left(x,t\right)=t\,e^{-x}+\frac{1}{3}t^3\,e^{-2x}$, and the others can be obtained.
Thus, the solution is obtained by summation of approximated terms:

$$\vartheta(x,t)=\vartheta_0(x,t)+\vartheta_1(x,t)+\vartheta_2(x,t)+\vartheta_3(x,t)+\cdots=t\,e^{-x}+\frac{1}{3}t^3\,e^{-2x}+\cdots.$$

This series form solution converges to the exact solution $\vartheta(x,t)=t\,e^{-x}$ of the given problem (14) as m approaches to infinity.

6.2. Implementation of the AHPMTM on Problem 1

Consider the same nonlinear Burger's equation defined by equation (14), with the same initial condition.
Now, from equation (17) and applying the procedure of AHPMTM, we get:

$$\sum_{n=0}^{\infty}\vartheta_n\left(x,t\right)\rho^n=\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(\delta(x,t)\right)\right\}+\rho\,\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(-\sum_{n=0}^{\infty}\left((\vartheta_n)_x+\vartheta_n\right)\rho^n+\sum_{n=0}^{\infty}H_n\,\rho^n\right)\right\},$$

by using He's polynomials and comparing different powers of ρ, we get the approximated terms:

$$\begin{aligned}
\rho^0:&\qquad \vartheta_0\left(x,t\right)=\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(\delta(x,t)\right)\right\}=t\,e^{-x}+\frac{1}{3}t^3\,e^{-2x},\\
\rho^1:&\qquad \vartheta_1\left(x,t\right)=\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(-(\vartheta_0)_x-\vartheta_0+H_0\right)\right\},\\
\rho^2:&\qquad \vartheta_2\left(x,t\right)=\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(-(\vartheta_1)_x-\vartheta_1+H_1\right)\right\},\\
&\qquad\vdots
\end{aligned}$$

The AHPMTM solution in series form (at $\rho=1$) becomes as:

$$\vartheta\left(x,t\right)=\sum_{n=0}^{\infty}\vartheta_n\left(x,t\right), \tag{19}$$

inserting the related iterations, we get $\vartheta\left(x,t\right)=t\,e^{-x}$, which is the exact solution for equation (14).

6.3. Implementation of the MADM on Problem 2

Consider the nonlinear Burger's equation defined as:

$$\vartheta_t(x,t) = \vartheta_x(x,t)\,\vartheta(x,t) + \vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t) + \delta(x,t), \qquad t>0, \quad x\in R, \tag{20}$$

with the initial condition $\vartheta(x,0)=0$, where,

$$\pounds = \vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t), \qquad N(\vartheta(x,t)) = \vartheta_x(x,t)\,\vartheta(x,t),$$

are the linear and nonlinear terms, respectively. Here, $\delta(x,t) = e^x - t^2\,e^{2x} - \frac{3}{2}t\,e^x$.

Using the Mohand transformation to equation (20) as:

$$\mathbb{M}\{\vartheta_t(x,t)\} = \mathbb{M}\left\{\vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t) + N(\vartheta(x,t)) + \delta(x,t)\right\},$$

utilizing the transform property (1), we get:

$$s\left(R(x,s) - s\,\vartheta(x,0)\right) = \mathbb{M}\left\{\vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t) + N(\vartheta(x,t)) + \delta(x,t)\right\}, \tag{21}$$

after the simplification of the equation (21), the simplified form becomes as:

$$R(x,s) = s\,\vartheta(x,0) + \frac{1}{s}\mathbb{M}\left\{\vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t) + N(\vartheta(x,t)) + \delta(x,t)\right\}, \tag{22}$$

implementing the inverse of the Mohand transformation equation (22), the scheme becomes as:

$$\vartheta(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\{\delta(x,t)\}\right\} + \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{\vartheta_{xx}(x,t) + \frac{1}{2}\vartheta(x,t) + N(\vartheta(x,t))\right\}\right\}. \tag{23}$$

Thus, the first term takes the form:

$$\vartheta_0(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\{\delta(x,t)\}\right\} = t\,e^x - \frac{3}{4}t^2e^x - \frac{1}{3}t^3e^{2x},$$

then, the final iterative scheme for the other terms becomes as:

$$\vartheta_{m+1}(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\mathbb{M}\left\{(\vartheta_m)_{xx}(x,t) + \frac{1}{2}\vartheta_m(x,t) + A_m\right\}\right\}, \qquad m=0,1,2..., \tag{24}$$

with using the initial condition and the nonlinear term $N(\vartheta(x,t))$ which is decomposed by using definition (8), the approximated term $\vartheta_0(x,t) = t\,e^x - \frac{3}{4}t^2e^x - \frac{1}{3}t^3e^{2x}$, and the others can be obtained.

Thus, the solution is obtained by summation of iteration terms:

$$\vartheta(x,t) = \vartheta_0(x,t) + \vartheta_1(x,t) + \vartheta_2(x,t) + \vartheta_3(x,t) + \cdots = t\,e^x - \frac{3}{4}t^2e^x - \frac{1}{3}t^3e^{2x} + \cdots.$$

This series form solution converges to the exact solution $\vartheta(x,t) = t\,e^x$ of the given problem (20) as m approaches to infinity.

6.4. Implementation of the AHPMTM on Problem 2

Consider the same nonlinear Burger's equation defined by equation (20), with the same initial condition.
Now, from the equation (22), and using the procedure of AHPMTM, we get:

$$\sum_{n=0}^{\infty} \vartheta_n\,(x,t)\,\rho^n = \mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\,(\delta(x,t))\right\} + \rho\,\mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\left(\sum_{n=0}^{\infty}((\vartheta_n)_{xx}+0.5\vartheta_n)\,\rho^n + \sum_{n=0}^{\infty} H_n\rho^n\right)\right\},$$

by using He's polynomials and comparing different powers of ρ, we get the approximation terms:

$$\begin{aligned}
\rho^0: &\quad \vartheta_0\,(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\,(\delta(x,t))\right\} = t\,e^x - \frac{3}{4}t^2e^x - \frac{1}{3}t^3e^{2x},\\
\rho^1: &\quad \vartheta_1\,(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\,((\vartheta_0)_{xx} + 0.5\vartheta_0 + H_0)\right\},\\
\rho^2: &\quad \vartheta_2\,(x,t) = \mathbb{M}^{-1}\left\{\frac{1}{s}\,\mathbb{M}\,((\vartheta_1)_{xx} + 0.5\vartheta_1 + H_1)\right\},\ldots.
\end{aligned}$$

The AHPMTM solution in series form (at $\rho = 1$) becomes as in (19), and by inserting the related iterations, we get:

$$\vartheta(x,t) = \vartheta_0(x,t) + \vartheta_1(x,t) + \vartheta_2(x,t) + \vartheta_3(x,t) + \cdots = t\,e^x - \frac{3}{4}t^2e^x - \frac{1}{3}t^3e^{2x} + \cdots.$$

This series form solution converges to the given exact solution $\vartheta\,(x,t) = x\,e^t$ of the given problem (20) as m approaches to infinity.

7. Results and discussions

Here, we behold the two models (14) and (20) with different values of m and the corresponding initial conditions for each one of them in $(x,t) \in [0,1] \times [0,1]$. We give a numerical simulation for Burger's equation by implementing the indicated scheme during in Figures 1-2.

(i) Figure 1, recognizes a comparison between the exact $(\vartheta_{ex}(x,t))$ and approximate solutions by MADM $(\vartheta_{MADM}(x,t))$, and AHPMTM $(\vartheta_{AHPMTM}(x,t))$ for the Problem 1 (14). Where the order of approximation $m = 6$ (the number of the considered terms of the series solution).

(ii) Figure 2, gives a comparison between the exact $(\vartheta_{ex}(x,t))$ and approximate solutions by MADM $(\vartheta_{MADM}(x,t))$, and AHPMTM $(\vartheta_{AHPMTM}(x,t))$ for the Problem 2 (20). Where the order of approximation $m = 6$ (the number of the considered terms of the series solution).

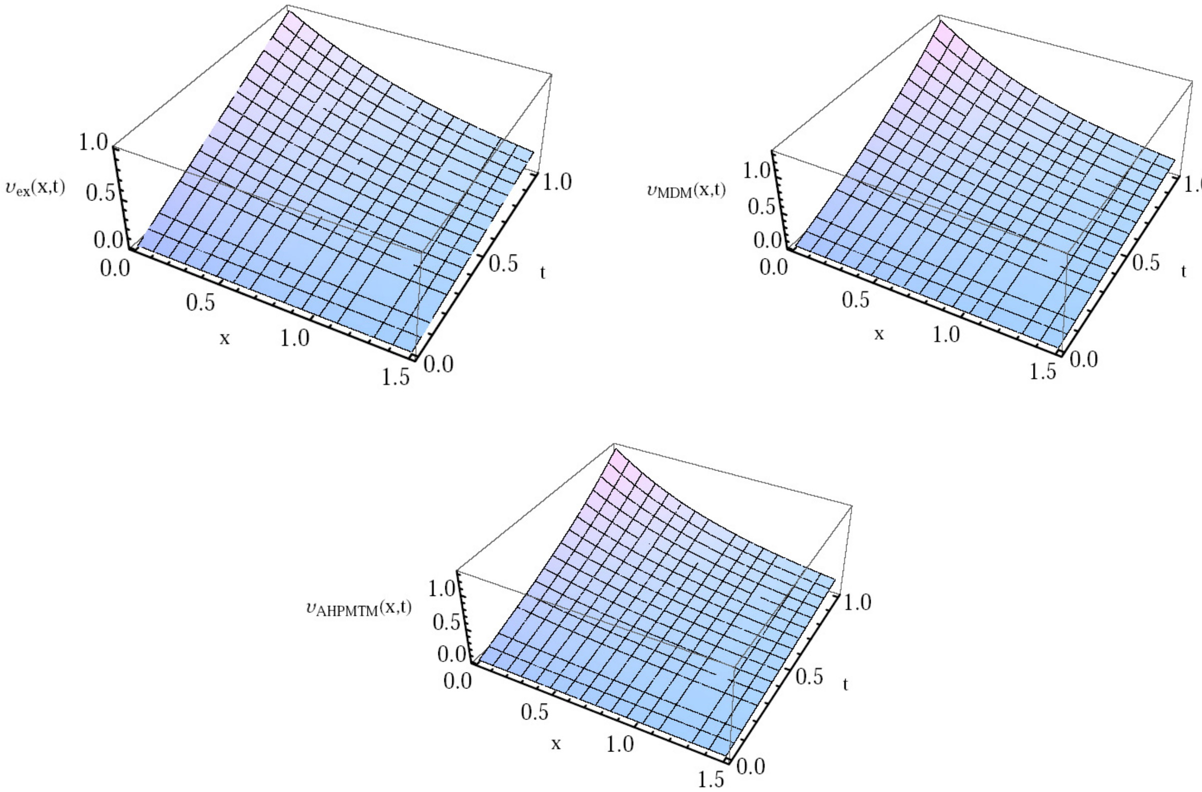

Fig. 1: Plot of Problem 1 for the exact and approximate solutions by MADM, and AHPMTM.

When looking at these two figures, we can emphasize that although the behavior of the approximate solution using the two methods used is similar to the behavior of the exact solution to the model under study, the approximate solution using the AHPMTM method is better than the solution using the MADM method.

In addition, to validate our numerical solutions by the two proposed approximation methods (MADM, and AHPMTM), we present a comparison in Tables 1 and 2 between the absolute error $|\vartheta_{exact}(x,t) - \vartheta_{approximation}(x,t)|$ of the testing Problems 1 and 2 respectively, with the order of approximation $m = 10$. This comparison shows that the AHPMTM method is more accurate than the MADM method, and this indicates that the AHPMTM is faster in convergence than the MADM.

8. Conclusions

In this case study, we compared the MADM and AHPMTM by solving the nonlinear Burger's differential equation. The MADM has a straight-forward decomposition procedure with the Mohand transform and requires less computational work. It provides a series of approximate solutions that converge with high accuracy to exact solutions for the given problems. The AHPMTM, on the other hand, used the homotopy perturbation pro-

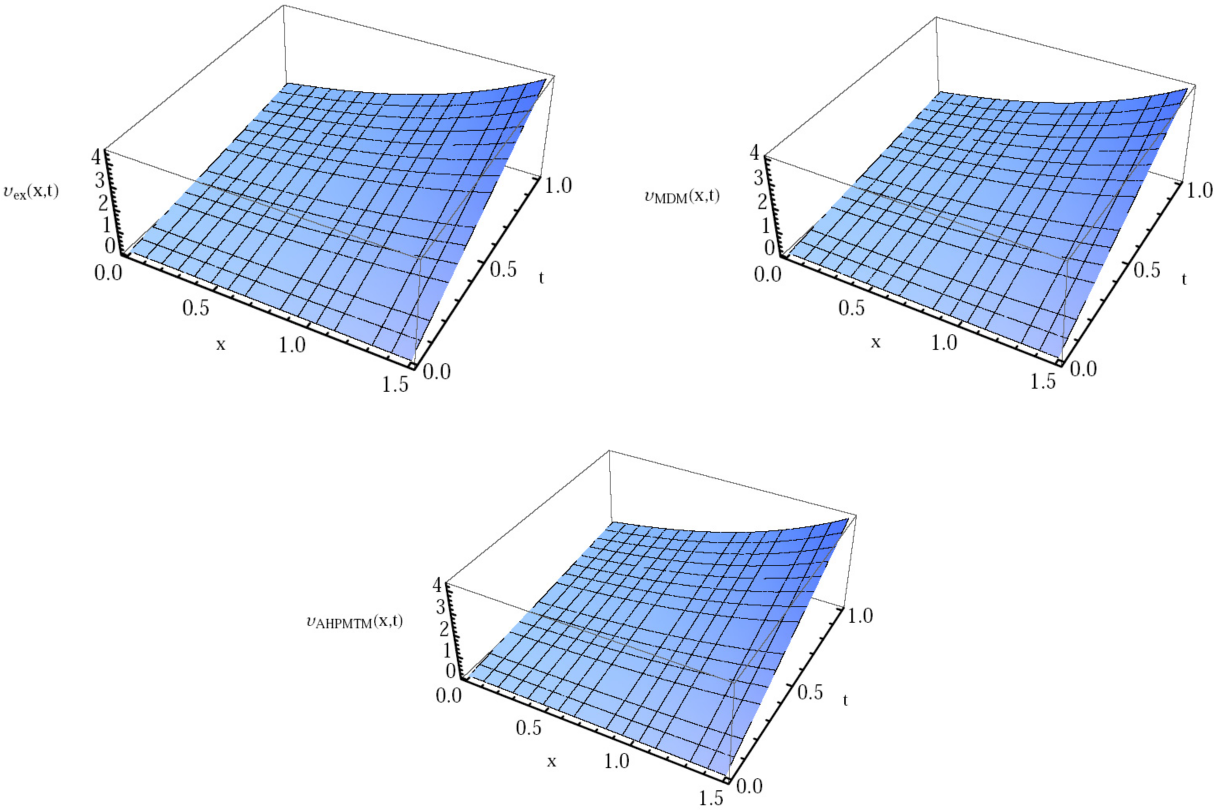

Fig. 2: Plot of Problem 2 for the exact and approximate solutions by MADM, and AHPMTM.

Table 1: The absolute error of testing Problem 1 by using MADM and AHPMTM

t	x	AR of MADM	AR of AHPMTM
0.0	0.0	0.0	0.0
0.1	0.1	0.0000085232	2.672×10^{-7}
0.2	0.2	0.0000123952	0.054×10^{-7}
0.3	0.3	0.0007418025	0.0000065142
0.4	0.4	0.0009021720	0.0000096325
0.5	0.5	0.0006500430	0.0000654120
0.6	0.6	0.0006548526	0.0000951019
0.7	0.7	0.0002540567	0.0000254075
0.8	0.8	0.0009871258	0.0000654987
0.9	0.9	0.0009602581	0.0000369025
1.0	1.0	0.0	0.0

cedure with the Mohand transform to accelerate He's polynomials. The accelerated He's polynomials improve the proposed method's convergence. The testing problems showed

Table 2: The absolute error of testing Problem 2 by using MADM and AHPMTM

t	x	AR of MADM	AR of AHPMTM
0.0	0.0	0.0	0.0
0.1	0.1	0.0000038259	3.19×10^{-8}
0.2	0.2	0.0000555400	4.889×10^{-7}
0.3	0.3	0.0002511656	0.0000023462
0.4	0.4	0.0006948186	0.0000068870
0.5	0.5	0.0014439690	0.0000151782
0.6	0.6	0.0024475200	0.0000272779
0.7	0.7	0.0034758120	0.0000410602
0.8	0.8	0.0040415190	0.0000505910
0.9	0.9	0.0033103111	0.0000438964
1.0	1.0	0.0	0.0

that the AHPMTM method provided the exact solution, while the MADM provided a series-form solution that converged to the exact solution. Both table values evaluated up to three terms of the proposed two methods. Overall, it is concluded that the AHPMTM has a large computational procedure but has a higher rate of convergence as compared to MADM. In future work, we will attempt to provide a theoretical study of the convergence and stability of the presented methods in some depth, as well as to apply this study to problems with industrial, biological, or other applications and to more complex systems.

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