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Relationship Between the Second Largest Adjacency and Signless Laplacian Eigenvalues of Graphs and Properties of Planar Graphs

Machasri Manickam[1], Kalyani Desikan[1,*]

[1] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai, Tamilnadu, India*

Abstract. A graph's second largest eigenvalue is a significant algebraic characteristic that provides details on the graph's expansion, connectivity, and randomness. Bounds for the second largest eigenvalue of a graph, denoted as λ_2 were previously established in the literature in relation to graph parameters like edge connectivity and vertex connectivity, matching number, independence number, and edge expansion constant, among others. A graph is planar if it can be drawn in a plane without graph edges crossing. Determining the planarity of a graph helps in optimizing, simplifying, and understanding complex systems across various fields. Graph skewness, graph thickness, and graph crossing number are a few metrics that describe how much a graph deviates from planarity. In this work, we ascertain the relationship between the graph's properties, including graph skewness, thickness, and crossing number, and the graph's second largest eigenvalues of the adjacency matrix $A(G)$ and the signless Laplacian matrix $Q(G)$. Based on the skewness, thickness, and crossing number, we establish a lower bound for the graph's second largest adjacency and signless Laplacian eigenvalues. We also determine a lower bound for these graph properties in terms of the second largest adjacency and signless Laplacian eigenvalues of regular graphs.

2020 Mathematics Subject Classifications: 05C07, 05C10, 05C50

Key Words and Phrases: Second largest eigenvalue, Planar graph, Crossing number, Graph Thickness, Graph Skewness

1. Introduction

The goal of spectral graph theory, a branch of algebraic graph theory, is to apply ideas from linear algebra and spectral theory to the study of graph properties. Graphs are represented as matrices in spectral graph theory, including adjacency, Laplacian and signless Laplacian matrices. Several structural characteristics and graph properties are analysed using the eigenvalues and eigenvectors of these matrices. While both the spectral radius

*Corresponding author.
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Email addresses: machasri.m2019@vitstudent.ac.in (M. Machasri),
kalyanidesikan@vit.ac.in (K. Desikan)

and the second largest eigenvalue of a graph are crucial ideas in spectral graph theory, they signify distinct spectral features of the graph. In network analysis, machine learning, optimization, and computational science, the graph's second largest eigenvalue is widely used and advances our knowledge of graph structures and their functional characteristics.

A graph G is made up of two sets (V, E), where E is the set of unordered pairs of distinct vertices, known as edges, and V is a finite non-empty set of elements. The vertex set is denoted as $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set is denoted by $E = \{e_1, e_2, \ldots, e_m\}$. Basic notations and terminology are based on DB West's book, *Introduction to Graph Theory* [26]. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$ if v_i and v_j are adjacent, otherwise it is 0. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A known as the spectrum of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal degree matrix. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \geq \mu_{(n-1)} \geq \mu_n$ be the eigenvalues of the Laplacian matrix. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. Its eigenvalues are denoted as $q_1 \geq q_2 \geq \cdots \geq q_n$.

The graph planarization problem is to determine a minimum subset of edges to remove from a non-planar graph to make it planar. The problem has applications in computer science with regard to printed circuit board layout and Very-Large Scale Integration (VLSI) circuit routing.

Beyond the standard graph parameters, we intend to work on the parameters that are most challenging to compute. So far, no results have been found connecting the second largest eigenvalue to planarity-related characteristics. Parameters such as graph skewness, thickness, and crossing number are used to determine how much a graph deviates from planarity. Computation of these properties are often challenging. As a result, it is helpful to find bounds for them. In this work, we prove the relation of these graph parameters with the second largest eigenvalue of the graph. We employ quotient matrix and eigenvalue interlacing technique to arrive at these bounds.

Determining the skewness of a non planar graph is the graph-theoretic variant of the problem. The problem is known to be $NP-$complete [18]. The concept of skewness is crucial in optimizing graph layouts, designing efficient algorithms, ensuring network reliability, and advancing theoretical research in graph theory. In Very Large Scale Integration (VLSI) circuit design, skewness helps in minimizing the complexity of circuit layouts. By analyzing the skewness of interconnections in a circuit graph, designers can strategically remove certain connections to create a more planar layout, which is crucial for reducing interference and improving signal integrity. This application is vital in ensuring that circuits function efficiently without excessive heat generation or signal degradation.

The thickness of a graph is a valuable measure for simplifying and understanding complex graphs by breaking them into planar components. This concept finds applications in various fields, enhancing visualization, improving design efficiency, and aiding in the development of more effective algorithms and systems. Whether in circuit design, network optimization, biological research, or software engineering, the ability to work with planar subgraphs is a powerful tool for tackling complexity. Particularly in computational biology, graph thickness can aid in the analysis of biological networks, such as protein-protein interaction networks. Understanding the thickness of these graphs can help researchers identify modular structures within biological systems, facilitating insights into cellular functions and interactions. This application is essential for advancements in fields like genomics and systems biology, where complex interactions need to be understood and modeled accurately.

Crossing numbers play a significant role in various practical and theoretical aspects of graph theory and related disciplines. In network design, such as designing circuit layouts or communication networks, minimizing crossing numbers can lead to more efficient and less congested layouts. This optimization can improve the performance and reliability of the network. In map labeling, graphs are used to represent geographical features and their relationships. Minimizing crossing numbers in these graphs can lead to clearer and more informative maps. In transportation networks, such as railway or road systems, the crossing number can aid the design of routes to minimize intersections, which can improve traffic flow and safety. By analyzing the crossing number of a graph representing a transportation network, planners can make decisions that reduce congestion and enhance overall efficiency.

Bounds in graph theory serve as fundamental tools for estimation, algorithm optimization, structural analysis, and practical applications across various domains. They not only facilitate a deeper understanding of graph properties but also enhance computational efficiency in solving complex problems. In the literature, bounds for certain graph parameters have been determined in relation to other graph parameters. Peter Firby and Julie Haviland in [9] established lower bounds for the average distance in terms of the independence number of the graph. In [4] M.Aouchiche et al. established a sharp upper bound on the algebraic connectivity of a connected graph in terms of the domination number. Xiaofeng Gu and Muhuo Liu in [11], proved sharp lower bounds on the matching number of graphs in terms of the Laplacian eigenvalues. In [2] Nasir Ali et al. explored commutative rings such as the ring of Gaussian integers, the ring Z_n of integers modulo n, and quotient polynomial rings in order to establish general bounds for the multiset dimension in Zero Divisor graphs (ZD-graphs). Also, they analyzed the behavior of Mdim under algebraic operations and discussed bounds in terms of diameter and maximum degree. Some general bounds on the dominating metric dimension (Ddim) of the ZD-graph of R in terms of the maximum degree, girth, clique number, and diameter were determined by Nasir Ali et al. in [3].

The second largest eigenvalue of a regular graph G has impact on graph's diameter [6], covering number [8] and the convergence properties of random walks [10]. In 2006, Stanic [23] discovered for the first time the star complements for the graphs such as complete graphs and trees with second largest eigenvalue 1. Ramezani and Tayfeh-Rezaiea [20] found the maximal graphs and regular graphs which have $K_{r,s} + tK_1$ as a star complement with second largest eigenvalue 1.

The relationship between the matching number and $\partial_2(G)$, the second largest distance Laplacian eigenvalue of G, was examined by Tian and Wong [24], who also provided lower bounds for $\partial_2(G)$ in terms of $m(G)$. Additionally, all extremal graphs that achieved lower bounds were described. In 2019, Vladislav Kabanova et al. [14] studied the eigenfunctions of the Star graph S_n with $n \geq 3$, where S_n is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(12),(13),\dots,(1n)\}$ corresponding to $\lambda_2 = n-2$. A characterisation of eigenfunctions with the smallest cardinality of the support was found for $n \geq 8$ and $n = 3$. They also got the minimal cardinality of the support of an eigenfunction of S_n corresponding to the second largest eigenvalue. In terms of the order and matching number of G, Shuchao Li and Wanting Sun [15] set sharp lower bounds on $q_2(G)$. Among the $n-$vertex connected graphs with fixed connectivity, they found the one and only graph with the least $q_2(G)$.
The maximum number of vertices of a connected $k-$regular graph with the second largest eigenvalue at most λ is denoted as $v(k,\lambda)$. The Alon-Boppana Theorem implies that $v(k,\lambda)$ is finite when $k > \frac{\lambda^2+4}{4}$. Jae Young Yang and Jack H. Koolen [27] proved that for fixed $\lambda \geq 1$, there exists a constant $C(\lambda)$ such that $2k+2 \leq v(k,\lambda) \leq 2k + C(\lambda)$ when $k > \frac{\lambda^2+4}{4}$. The relationship between the local valency of an edge-regular graph and its λ_2 was discovered by Jongyook Park [19]. For some connected sets H, Siemons et al. [21] calculated the value of $\lambda_2(\Gamma)$ by looking at the second largest eigenvalue of the Cayley graph $\Gamma = Cay(G,H)$ over $G = S_n$ or A_n.

This paper is organised as follows. We present some fundamental concepts related to the parameters of planar graphs in section 2. In section 3, we investigate the relation between the second largest eigenvalue of a graph with the parameters of planar graph such as skewness, thickness and crossing number. Section 3.1 provides the relation between the second largest eigenvalue of adjacency and signless Laplacian matrices with the skewness of the graph $sk(G)$; that is, lower bounds for λ_2 and q_2 with respect to the skewness of the graph. Section 3.2 gives the lower bounds for λ_2 and q_2 with respect to the thickness of the graph $\tau(G)$. Section 3.3 presents the lower bounds for λ_2 and q_2 with respect to the crossing number of the graph $cr(G)$. In all the sections lower bounds for the graph parameters $sk(G)$, $\tau(G)$, and $cr(G)$ in terms of the second largest adjacency and signless Laplacian eigenvalues for regular graphs are also presented.

2. Preliminaries

Planar graph is a graph that can be drawn in such a way that no edges cross each other [25]. Such a drawing is called a plane graph or planar embedding of the graph.

The skewness of a graph G is the minimum number of edges whose removal results in a planar graph. It is denoted by $sk(G)$.

From [7], we have

$$sk(G) \geq m - (3n - 6) \tag{1}$$

where m and n are the size and the order of G, respectively.

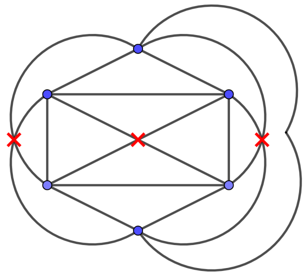

Figure 1: Complete graph K_6

In Figure 1, if we remove one edge from each red crossing, it will become planar. Skewness of the complete graph K_6 is given by $sk(K_6) = 3$.

The thickness $\tau(G)$ of a graph G is the minimum number of planar edge-induced subgraphs P_i of G needed such that the graph union $\bigcup P_i = G$ [22].

From the definition of thickness of a graph, the graph K_6 has 2 planar edge-induced subgraphs. Therefore, the thickness of K_6 is $\tau(K_6) = 2$.

A lower bound for the thickness of a graph is given by [22]

$$\tau(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil \tag{2}$$

where m is the number of edges, $n \geq 3$ is the number of vertices, and $\lceil x \rceil$ is the ceiling function. The thickness of the hypercube graph Q_n [13] is given by

$$\tau(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

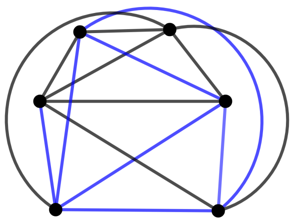

Figure 2: Thickness of K_6

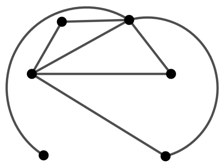

Figure 3: Planar edge-induced subgraph 1 of K_6

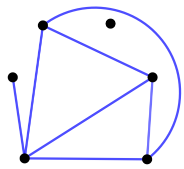

Figure 4: Planar edge-induced subgraph 2 of K_6

The crossing number $cr(G)$ of a graph G is the lowest number of edge crossings of a plane drawing of the graph G. A graph with crossing number 0 is known as a planar graph. For example, the complete graph K_6 illustrated in Figure 1 has crossing number 3. Ajtai et al. (1982) showed that there is an absolute constant $c > 0$ such that

$$cr(G) \geq \frac{cm^3}{n^2}.$$

This inequality is known as crossing number inequality or crossing lemma.
Due to Ackerman [1], the constant $c = \frac{1}{29}$ is the best known to date. Therefore,

$$cr(G) \geq \frac{m^3}{29n^2}. \tag{3}$$

Definition 1. *[5] Consider two sequences of real numbers:*
$\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_m$ with $m \leq n$. The second sequence is said to interlace the first one whenever $\xi_i \leq \eta_i \leq \xi_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is called tight if there exists an integer $k \in [0, m]$ such that $\xi_i = \eta_i$ for $1 \leq i \leq k$ and $\xi_{n-m+i} = \eta_i$ for $k+1 \leq i \leq m$.

Definition 2. *[16] Let $M = (M_{i,j})_{t\times t}$ be a real matrix of order n where M_{ij} are the blocks of M and $i, j = 1, 2, \ldots, t$. Then the Quotient matrix of M is a matrix $B(M) = (b_{ij})$ where b_{ij} is the sum of all entries in M_{ij} divided by the number of rows of M_{ij}.*

Lemma 1. *[12, 17] Let A_Q be the quotient matrix of a symmetric matrix A whose rows and columns are partitioned according to a partitioning*
$(X_1, X_2, \ldots, X_m)$. Then

i. The eigenvalues of A_Q interlace the eigenvalues of A.
ii. If the interlacing is tight then the partition is equitable.

3. Main Results

In this section, the relation between the second largest eigenvalue of graph with the parameters of planar graphs such as skewness, thickness and crossing number are discussed.

3.1. Skewness

The theorems relating the second largest adjacency and signless Laplacian eigenvalues of a graph and skewness of the graph are presented in this section. We establish lower bounds for λ_2 and q_2 in terms of $sk(G)$. Also, lower bounds for the skewness of regular graphs in terms of λ_2 and q_2 are obtained.

Theorem 1. *Let G be a connected graph with skewness $sk(G)$. Then*

$$\lambda_2 \geq \frac{1}{2}\left\{\left[\delta - \frac{3\Delta}{m - sk(G) + 3}\right] - \sqrt{\left[\delta + \frac{3\Delta}{m - sk(G) + 3}\right]^2 + \frac{12\Delta(\Delta - \delta)}{m - sk(G) + 3}}\right\}.$$

Proof. Let A_Q be the quotient matrix of the adjacency matrix A of G with respect to the partitions P_1 and P_2 where P_1 consists of one vertex of G with maximum degree Δ. Then, we have $|P_1| = n_1 = 1$ and $|P_2| = n_2 = (n-1)$. Then

$$A_Q = \begin{bmatrix} d_1' - t & t \\ \frac{t}{(n-1)} & d_2' - \frac{t}{(n-1)} \end{bmatrix}$$

where d_1', d_2' are the average degrees of the two partitions P_1, P_2 and t is the number of edges between P_1 and P_2. The characteristic equation of the above matrix is given by

$$\eta^2 - \left[d_1' + d_2' - t - \frac{t}{(n-1)}\right]\eta + \left(d_1' - t\right)\left(d_2' - \frac{t}{(n-1)}\right) - \frac{t^2}{(n-1)} = 0.$$

The roots of the above characteristic equation are

$$\eta_1 = \tfrac{1}{2}\left\{\left(d_1' + d_2' - \tfrac{tn}{(n-1)}\right) + \sqrt{r}\right\} \text{ and } \eta_2 = \tfrac{1}{2}\left\{\left(d_1' + d_2' - \tfrac{tn}{(n-1)}\right) - \sqrt{r}\right\}$$

where

$$\begin{aligned} r &= \left[\left(d_1' - t\right) - \left(d_2' - \frac{t}{(n-1)}\right)\right]^2 + \frac{4t^2}{(n-1)} \\ &= \left(d_1' - d_2' - t - \frac{t}{(n-1)}\right)^2 + \frac{4t(d_1' - d_2')}{(n-1)}. \end{aligned}$$

Therefore, we have

$$\eta_2 = \frac{1}{2}\left\{\left(d_1' + d_2' - \frac{tn}{(n-1)}\right) - \sqrt{\left(d_1' - d_2' - t - \frac{t}{(n-1)}\right)^2 + \frac{4t(d_1' - d_2')}{n_2}}\right\}.$$

Since $d_1' = \Delta$ and $d_2' \geq \delta$ we get,

$$\eta_2 \geq \frac{1}{2}\left\{\left(\Delta + \delta - \frac{tn}{(n-1)}\right) - \sqrt{\left(\Delta - \delta - t - \frac{t}{(n-1)}\right)^2 + \frac{4t(\Delta - \delta)}{(n-1)}}\right\}.$$

By eigenvalue interlacing, we have

$$\lambda_2 \geq \frac{1}{2}\left\{\left(\Delta+\delta-\frac{tn}{(n-1)}\right)-\sqrt{\left(\Delta-\delta-\frac{tn}{(n-1)}\right)^2+\frac{4t(\Delta-\delta)}{(n-1)}}\right\}.$$

The above inequality can be rewritten as

$$\lambda_2 \geq \frac{1}{2}\left\{\left[\Delta+\delta-t\left(1+\frac{1}{(n-1)}\right)\right]-\sqrt{\left[\Delta-\delta-t\left(1+\frac{1}{(n-1)}\right)\right]^2+\frac{4t(\Delta-\delta)}{(n-1)}}\right\}.$$

Since $t=\Delta$, the above inequality becomes

$$\lambda_2 \geq \frac{1}{2}\left\{\left(\delta-\frac{\Delta}{(n-1)}\right)-\sqrt{\left(\delta+\frac{\Delta}{(n-1)}\right)^2+\frac{4\Delta(\Delta-\delta)}{(n-1)}}\right\}.$$

By making use of inequality (1), the result is obtained.

Theorem 2. *Let G be a $d-$regular graph with $n \geq 3$. Then*

$$\lambda_2 \geq \frac{3d}{sk(G)-m-3}.$$

Proof. Let A be the adjacency matrix of G represented in the following block matrix form

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Let A_Q be the quotient matrix of A of G with respect to the partitions P_1 and P_2. Let $|P_1|=n_1=1$ and $|P_2|=n_2=(n-1)$. Then

$$A_Q=\begin{bmatrix} d-t & t \\ \frac{t}{(n-1)} & d-\frac{t}{(n-1)} \end{bmatrix}$$

where t is the number of edges between P_1 and P_2.
The characteristic equation of the above matrix is given by

$$\left[\eta-\left(d-t\right)\right]\left[\eta-\left(d-\frac{t}{(n-1)}\right)\right]-\frac{t^2}{(n-1)}=0.$$

The roots of the above equation are $\eta_1=d$ and $\eta_2=d-\frac{tn}{(n-1)}$.
From eigenvalue interlacing we have,

$$\lambda_2 \geq d-\frac{tn}{(n-1)}.$$

Since $t=d$, the above inequality can be rewritten as

$$\lambda_2 \geq \frac{d}{(1-n)}.$$

Now, using inequality (1), we get the final result.

Note 1. *If G is a $d-$regular graph except complete split graph and complete multipartite graph with $n \geq 3$ then*

$$sk(G) \leq \frac{3d}{\lambda_2} + (m+3).$$

Theorem 3. *Let G be a connected graph with skewness $sk(G)$. Then*

$$q_2 \geq \frac{1}{2}\left\{\left[2\delta + \Delta(1-3u)\right] - \sqrt{\left[\Delta(1-3u) - 2\delta\right]^2 + 24\Delta(\Delta-\delta)u}\right\}$$

where $u = \frac{1}{(m - sk(G)+3)}$.

Proof. Let B_Q be the quotient matrix of the signless Laplacian matrix Q of G with respect to the partitions P_1 and P_2 where P_1 consists of one vertex of G with maximum degree Δ. We have $|P_1| = n_1 = 1$ and $|P_2| = n_2 = (n-1)$. Then

$$B_Q = \begin{bmatrix} 2d'_1 - t & t \\ \frac{t}{(n-1)} & 2d'_2 - \frac{t}{(n-1)} \end{bmatrix}$$

where d'_1, d'_2 are the average degrees of the two partitions P_1, P_2 and t is the number of edges between P_1 and P_2. The characteristic equation of the above matrix is given by

$$\zeta^2 - \left(2d'_1 + 2d'_2 - t - \frac{t}{(n-1)}\right)\zeta + \left(2d'_1 - t\right)\left(2d'_2 - \frac{t}{(n-1)}\right) - \frac{t^2}{(n-1)} = 0.$$

The roots of the above characteristic equation are

$$\zeta_1 = \tfrac{1}{2}\left\{\left(2d'_1 + 2d'_2 - \tfrac{tn}{(n-1)}\right) + \sqrt{r}\right\} \text{ and } \zeta_2 = \tfrac{1}{2}\left\{\left(2d'_1 + 2d'_2 - \tfrac{tn}{(n-1)}\right) - \sqrt{r}\right\}$$

where

$$\begin{aligned} r &= \left(2d'_1 - 2d'_2 - t + \frac{t}{(n-1)}\right)^2 + \frac{4t^2}{(n-1)} \\ &= \left(2d'_1 - 2d'_2 - t - \frac{t}{(n-1)}\right)^2 + \frac{8t(d'_1 - d'_2)}{(n-1)}. \end{aligned}$$

Therefore, we have

$$\zeta_2 = \frac{1}{2}\left\{\left(2d'_1 + 2d'_2 - \frac{tn}{(n-1)}\right) - \sqrt{\left(2d'_1 - 2d'_2 - \frac{tn}{(n-1)}\right)^2 + \frac{8t(d'_1 - d'_2)}{(n-1)}}\right\}.$$

By eigenvalue interlacing, we have

$$q_2 \geq \frac{1}{2}\left\{\left(2d'_1 + 2d'_2 - \frac{tn}{(n-1)}\right) - \sqrt{\left(2d'_1 - 2d'_2 - \frac{tn}{(n-1)}\right)^2 + \frac{8t(d'_1 - d'_2)}{(n-1)}}\right\}.$$

Since $d'_1 = \Delta$ and $d'_2 \geq \delta$ we get,

$$q_2 \geq \frac{1}{2}\left\{\left(2\Delta + 2\delta - \frac{tn}{(n-1)}\right) - \sqrt{\left(2\Delta - 2\delta - \frac{tn}{(n-1)}\right)^2 + \frac{8t(\Delta-\delta)}{(n-1)}}\right\}.$$

The above inequality can be rewritten as

$$q_2 \geq \frac{1}{2}\left\{\left[2\Delta + 2\delta - t\left(1 + \frac{1}{(n-1)}\right)\right] - \sqrt{\left[2\Delta - 2\delta - t\left(1 + \frac{1}{(n-1)}\right)\right]^2 + \frac{8t(\Delta-\delta)}{(n-1)}}\right\}.$$

Since $t = \Delta$, the above inequality becomes

$$q_2 \geq \frac{1}{2}\left\{\left(\Delta + 2\delta - \frac{\Delta}{(n-1)}\right) - \sqrt{\left(\Delta - 2\delta - \frac{\Delta}{(n-1)}\right)^2 + \frac{8\Delta(\Delta-\delta)}{(n-1)}}\right\}.$$

We obtain the result by making use of inequality (1).

Theorem 4. *Let G be a $d-$regular graph with $n \geq 3$. Then*

$$q_2 \geq d\left(1 - \frac{3}{m - sk(G) + 3}\right).$$

Proof. Let B_Q be the quotient matrix of the signless Laplacian matrix Q of G with respect to the partitions P_1 and P_2. Let $|P_1| = n_1 = 1$ and $|P_2| = n_2 = (n-1)$. Then

$$B_Q = \begin{bmatrix} 2d - t & t \\ \frac{t}{(n-1)} & 2d - \frac{t}{(n-1)} \end{bmatrix}$$

where t is the number of edges between P_1 and P_2.
The characteristic equation of the above matrix is given by

$$\left[\zeta - \left(2d - t\right)\right]\left[\zeta - \left(2d - \frac{t}{(n-1)}\right)\right] - \frac{t^2}{(n-1)} = 0.$$

The roots of the above equation are $\zeta_1 = 2d$ and $\zeta_2 = d - \frac{d}{(n-1)}$.
From eigenvalue interlacing we have,

$$q_2 \geq d - \frac{d}{(n-1)}.$$

Substituting $t = d$ in the above inequality we have,

$$q_2 \geq d - \frac{d}{(n-1)}.$$

The result is obtained by applying inequality (1) in the above inequality.

Note 2. *If G is a $d-$regular graph except complete split graph and complete multipartite graph then*

$$sk(G) \leq m - \frac{3q_2}{(d-q_2)}.$$

Example 1. *The graph illustrated in Figure 5 has 8 vertices, 20 edges with $\Delta = 7$, $\delta = 3$ and $\lambda_2 = 0.828$. Its skewness is given by $sk(G) = 2$. Also its crossing number $cr(G) = 5$. Lower bound for λ_2 from Theorem 1 is $\lambda_2 \geq -1.828$.*

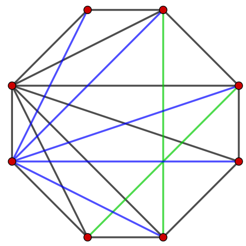

Figure 5: Graph with $sk(G) = 2$ and $cr(G) = 5$.

Example 2. *The graph illustrated in Figure 6 has 6 vertices, 9 edges with $d = 3$, $\lambda_2 = 0$ and $q_2 = 3$. Also it has $sk(G) = 1$. Lower bound from Theorem 2 is $\lambda_2 \geq -0.818$ and from Theorem 4 we have $q_2 \geq 2.182$.*

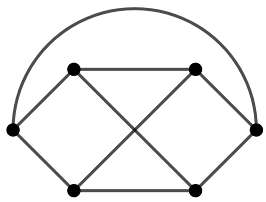

Figure 6: A $3-$regular graph with $sk(G) = cr(G) = 1$

3.2. Thickness

This section presents the theorems connecting a graph's thickness $\tau(G)$ and second largest adjacency and signless Laplacian eigenvalues. Lower bounds are established for λ_2 and q_2 in terms of $\tau(G)$. Also, lower bounds for the thickness $\tau(G)$ of regular graphs G are presented in terms of λ_2 and q_2.

Theorem 5. *Let G be a connected graph with thickness $\tau(G)$. Then*

$$\lambda_2 \geq \frac{1}{2}\left\{\left[\delta - \frac{3\tau\Delta}{m+3\tau}\right] - \sqrt{\left[\delta + \frac{3\tau\Delta}{m+3\tau}\right]^2 + \frac{12\tau\Delta(\Delta-\delta)}{m+3\tau}}\right\}.$$

Proof. The proof is similar to the proof of Theorem 1. By making use of inequality (2), we get the result.

Theorem 6. *Let G be a $d-$regular graph with $n \geq 3$.*

$$\lambda_2 \geq \frac{-3\tau d}{m+3\tau}.$$

Proof. The proof is on the same lines as the proof of Theorem 2. Inequality (2) is used to get the final result.

Note 3. *If G is a $d-$regular graph with $n \geq 3$ then*

$$\tau \geq \frac{-m\lambda_2}{3(\lambda_2+d)}.$$

Theorem 7. *Let G be a connected graph. Then*

$$q_2 \geq \frac{1}{2}\left\{[\Delta(1-3\tau u)+2\delta] - \sqrt{[\Delta(1-3\tau u)-2\delta]^2 + 24\tau\Delta(\Delta-\delta)u}\right\}$$

where $u = \frac{1}{m+3\tau}$.

Proof. The proof is similar to the proof of Theorem 3. By using inequality (2), we get the result.

Theorem 8. *Let G be a $d-$regular graph with $n \geq 3$. Then*

$$q_2 \geq \frac{dm}{m+3\tau}.$$

Proof. The proof is similar to the proof of Theorem 4. By using inequality (2), we get the result.

Note 4. *If G is a $d-$regular graph except complete split graph and complete multipartite graph with $n \geq 3$ then*

$$\tau \geq \frac{m}{3}\left(\frac{d}{q_2} - 1\right).$$

Example 3. *The hypercube Q_4 is a $4-$regular graph on 16 vertices and 32 edges with thickness $\tau(G) = 2$, $\lambda_2 = 2$ and $q_2 = 6$. The lower bounds from Theorems 6 and 8 are given by $\lambda_2 \geq -0.631$ and $q_2 \geq 3.368$.*

In Table 1, numerical values of the bounds that we have proved in Theorem 1 and Theorem 5 are presented for Complete bipartite graph $K_{m.n}$ with $m \neq n$. The graphical illustration of these bounds are given in Figure 7 and Figure 8.

S.No.	Graph	λ_2	Theorem 1 bound	Theorem 5 bound
1	$K_{3,5}$	0	-1.4494	-2.0717
2	$K_{4,6}$	0	-1.3042	-1.626
3	$K_{5,7}$	0	-1.2072	-1.3472
4	$K_{6,8}$	0	-1.138	-1.138
5	$K_{7,9}$	0	-1.0863	-1.3931
6	$K_{8,10}$	0	-1.0466	-1.2303
7	$K_{9,11}$	0	-1.0151	-1.0982
8	$K_{10,12}$	0	-0.9895	-0.9896
9	$K_{11,13}$	0	-0.9683	-1.1714
10	$K_{12,14}$	0	-0.9503	-1.0702
11	$K_{13,15}$	0	-0.9355	-0.9938

Table 1: Lower bounds of λ_2 for $K_{m,n}$

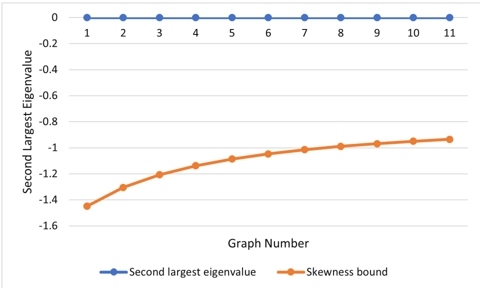

Figure 7: Skewness

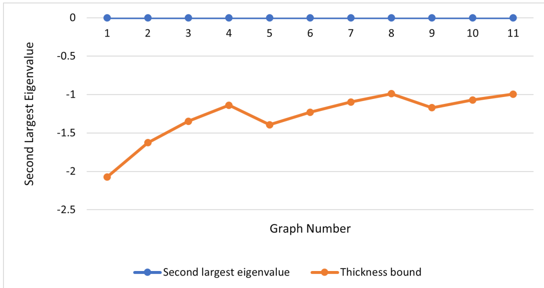

Figure 8: Thickness

Numerical values for the lower bounds of λ_2 proved in Theorems 2 and 6 are presented for Complete bipartite graphs $K_{n,n}$ in Table 2. The graphical representation of these bounds are given in Figure 9 and Figure 10.

S.No.	Graph	λ_2	Theorem 2 bound	Theorem 6 bound
1	$K_{3,3}$	0	-0.8181	-1.2
2	$K_{4,4}$	0	-0.8	-1.0909
3	$K_{5,5}$	0	-0.7895	-0.9677
4	$K_{6,6}$	0	-0.783	-0.8571
5	$K_{7,7}$	0	-0.7778	-1.0862
6	$K_{8,8}$	0	-0.7742	-0.9863
7	$K_{9,9}$	0	-0.7714	-0.9
8	$K_{10,10}$	0	-0.7692	-0.8257
9	$K_{11,11}$	0	-0.7674	-0.9925
10	$K_{12,12}$	0	-0.7659	-0.9231

Table 2: Lower bounds of λ_2 for $K_{n,n}$

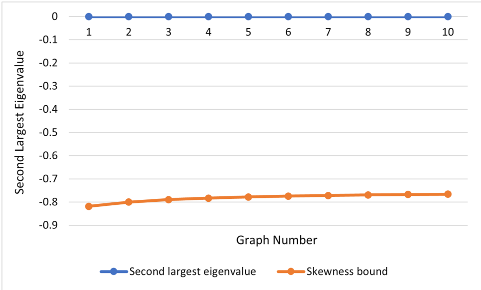

Figure 9: Skewness

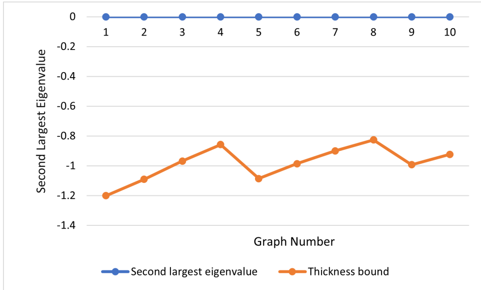

Figure 10: Thickness

Numerical values of the lower bounds of q_2 proved in Theorems 3 and 7 are presented for Complete bipartite graphs $K_{m,n}$ with $m \neq n$ in Table 3. The graphical illustration of these bounds are represented in Figure 11 and Figure 12.

S.No.	Graph	q_2	Theorem 3 bound	Theorem 7 bound
1	$K_{3,5}$	5	2.764	2.1045
2	$K_{4,6}$	6	4.0848	3.6871
3	$K_{5,7}$	7	5.3086	5.1336
4	$K_{6,8}$	8	6.4683	6.4684
5	$K_{7,9}$	9	7.5859	7.2121
6	$K_{8,10}$	10	8.6749	8.4528
7	$K_{9,11}$	11	9.7439	9.6445
8	$K_{10,12}$	12	10.799	10.7989
9	$K_{11,13}$	13	11.8584	11.6062
10	$K_{12,14}$	14	12.8802	12.7353

Table 3: Lower bounds of q_2 for $K_{m,n}$

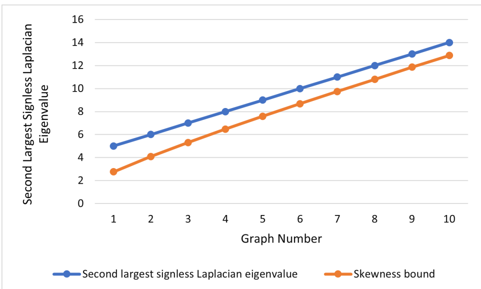

Figure 11: Skewness

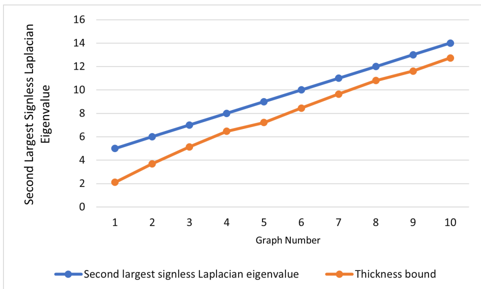

Figure 12: Thickness

Numerical values of the lower bounds of q_2 proved in Theorems 4 and 8 are presented for Complete bipartite graphs $K_{n,n}$ in Table 4. Its graphical representations are given in Figure 13 and Figure 14.

S.No.	Graph	q_2	Theorem 4 bound	Theorem 8 bound
1	$K_{3,3}$	3	2.1818	1.8
2	$K_{4,4}$	4	3.2	2.909
3	$K_{5,5}$	5	4.2105	4.032
4	$K_{6,6}$	6	5.2174	5.143
5	$K_{7,7}$	7	6.222	5.914
6	$K_{8,8}$	8	7.22	7.014
7	$K_{9,9}$	9	8.2286	8.1
8	$K_{10,10}$	10	9.2307	9.174
9	$K_{11,11}$	11	10.2326	10.008
10	$K_{12,12}$	12	11.234	11.0769

Table 4: Lower bounds of q_2 for $K_{n,n}$

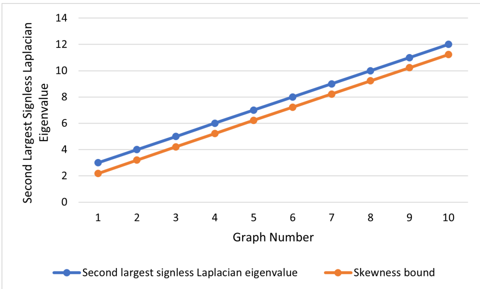

Figure 13: Skewness

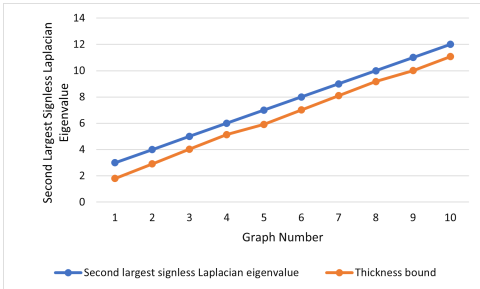

Figure 14: Thickness

3.3. Crossing number

This section presents the theorems connecting a graph's crossing number $cr(G)$ and second largest adjacency and signless Laplacian eigenvalues. Lower bounds are established for λ_2 and q_2 in terms of $cr(G)$ and lower bounds for the crossing number of regular graphs are determined in terms of λ_2 and q_2.

Theorem 9. *Let G be a connected graph with crossing number $cr(G)$. Then*

$$\lambda_2 \geq \frac{1}{2}\left\{[\delta + \Delta(1-u)] - \sqrt{[\delta - \Delta(1-u)]^2 - 4\Delta(\Delta-\delta)(1-u)}\right\}$$

where $u = \frac{m\sqrt{m}}{m\sqrt{m} - \sqrt{29cr}}$.

Proof. The proof is similar to the proof of Theorem 1. The result is obtained by using inequality (3).

Theorem 10. *Let G be a $d-$regular graph with $n \geq 3$. Then*

$$\lambda_2 \geq d\left(1 - \frac{m\sqrt{m}}{m\sqrt{m} - \sqrt{29cr}}\right).$$

Proof. The proof is on the same lines as the proof of Theorem 2. By using inequality (3), we get the result.

Note 5. *If G is a $d-$regular graph with $n \geq 3$ then*

$$cr \leq \frac{m^3 \lambda_2^2}{29(d-\lambda_2)^2}.$$

Theorem 11. *Let G be a connected graph. Then*

$$q_2 \geq \frac{1}{2}\left\{[2(\Delta+\delta)-\Delta u]-\sqrt{[2(\Delta-\delta)-\Delta u]^2-8\Delta(\Delta-\delta)(1-u)}\right\}$$

where $u = \frac{m\sqrt{m}}{m\sqrt{m}-\sqrt{29cr}}$.

Proof. The proof is similar to the proof of Theorem 3. Inequality (3) is used to get the result.

Theorem 12. *Let G be a $d-$regular graph with $n \geq 3$. Then*

$$q_2 \geq 2d - \frac{dm\sqrt{m}}{m\sqrt{m}-\sqrt{29cr}}.$$

Proof. The proof is on the same lines as the proof of Theorem 4. Inequality (3) is used to obtain the result.

Note 6. *If G is a $d-$regular graph with $n \geq 3$ then*

$$cr \leq \frac{m^3}{29}\left(\frac{d-q_2}{2d-q_2}\right)^2.$$

Example 4. *For the graph represented in Figure 5 with crossing number $cr(G) = 5$, the lower bound from Theorem 9 is given by $\lambda_2 \geq -1.971$.*

Example 5. *For the graph illustrated in Figure 6 with crossing number $cr(G) = 1$, the lower bound obtained using Theorem 10 is $\lambda_2 \geq -0.747$ and lower bound obtained using Theorem 12 is $q_2 \geq 2.253$.*

Example 6. *Let us consider the Petersen graph. It has $n = 10$, $m = 15$, $d = 3$, $\lambda_2 = 1$ and $q_2 = 4$. Then the lower bounds from Theorems 10 and 12 are given by $\lambda_2 \geq -0.7844$ and $q_2 \geq 2.2156$.*

4. Conclusion

Planarity is an important field of study in graph theory since it is essential for both theoretical understanding and real-world applications. In this work, the relationship between the parameters associated with planar graphs and a graph's second largest adjacency, Laplacian and Signless Laplacian eigenvalues have been determined. Lower bounds on the second largest adjacency and signless Laplacian eigenvalues of a graph are determined in terms of skewness $sk(G)$, thickness $\tau(G)$, and crossing number $cr(G)$ of graphs. In the future, we will investigate how the other graph parameters are related with the second largest adjacency and signless Laplacian eigenvalues of graphs. Furthermore, using λ_2, we will focus on characterising graphs.

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