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Fermatean Fuzzy Set Theory Applied to IUP-Algebras

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Abstract. In 1965, Zadeh introduced the foundational concept of fuzzy sets, followed by Atanassov's introduction of intuitionistic fuzzy sets in 1986. Yager expanded this field with Pythagorean fuzzy sets in 2013, and in 2020, Senapati and Yager further advanced the theory by proposing Fermatean fuzzy sets. This study applies Fermatean fuzzy sets to IUP-algebras, focusing on Fermatean fuzzy IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals. We examine their properties, including characteristic Fermatean fuzzy sets and upper and lower t-(strong) level subsets, offering deeper insights into their structural relationships.

2020 Mathematics Subject Classifications: 03G25, 03E72, 08A72

Key Words and Phrases: IUP-algebra, Fermatean fuzzy set, Fermatean fuzzy IUP-subalgebra, Fermatean fuzzy IUP-ideal, Fermatean fuzzy IUP-filter, Fermatean fuzzy strong IUP-ideal, upper t-(strong) level subset, lower t-(strong) level subset

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1. Introduction

The concept of fuzzy sets (FSs), introduced by Zadeh [15], revolutionized the handling of uncertainty by allowing elements to have varying degrees of membership. This foundational idea was extended by Atanassov [2] with intuitionistic fuzzy sets (IFSs), which added a degree of non-membership. Yager [14] further advanced this field with Pythagorean fuzzy sets (PFSs), where the square sum of membership and non-membership degrees is ≤ 1. The most recent development, Fermatean fuzzy sets (FFSs), was introduced by Senapati

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and Yager [11]. Fermatean fuzzy sets allow the sum of the cubes of membership and non-membership degrees to be ≤ 1, providing even greater flexibility and precision. These progressive enhancements have significantly enriched decision-making, medical diagnosis, and risk assessment, showcasing fuzzy logic's dynamic evolution and growing sophistication in modelling complex uncertainties.

After that, the concept of Fermatean fuzzy sets has been studied extensively and continuously in many spaces, such as Lalitha and Buvaneswari [10] identified and proved various properties, especially those involving the operation $A \rightharpoondown B$ defined as Fermatean fuzzy implication with other operations. Muhammad et al. [8] proposed a new type of fuzzy system known as the Fermatean fuzzy system. More precisely, they presented the notion of Fermatean fuzzy ideal theory and rough Fermatean fuzzy sets in semigroups and initiated the idea of lower and upper approximations in Fermatean fuzzy sets. They extended the study to rough Fermatean fuzzy left (resp., right, interior) ideals in semigroups. Balamurugan and Nagarajan [3] came up with the idea of a Fermatean fuzzy soft-covered generalized bi-ideal on a semigroup. This extends the idea of a Fermatean fuzzy soft bi-ideal and describes regular semigroups in terms of Fermatean fuzzy soft generalized bi-ideals. They framed the combining of fuzzy relations, composition relations, and compatible relations with Fermatean fuzzy sets. They also introduced the notions of a Fermatean fuzzy soft equivalence relation and a Fermatean fuzzy soft compatible relation on a semigroup. Finally, they provided a Fermatean fuzzy soft inverse relation and a Fermatean fuzzy soft congruence on a semigroup. Balamurugan and Nagarajan [4] first discussed bipolar Fermatean uncertainty subalgebras regarding R-ideals. They also discussed some exciting ideas and examined how bipolar Fermatean uncertainty soft ideals and bipolar Fermatean uncertainty soft R-ideals are related. Adak et al. [1] introduced the concept of Fermatean fuzzy semi-prime ideals and Fermatean fuzzy prime ideals of ordered semigroups. They illustrated some novel concepts to construct Fermatean fuzzy intra-regular and regular ideals and gave several relations for the family of Fermatean fuzzy regular ideals of ordered semigroups.

Iampan et al. [7] introduced the groundbreaking concept of IUP-algebras. This innovative theory defines four key subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. Each subset's fundamental properties were meticulously examined, unveiling new research avenues and applications in the mathematical world. Since its introduction, the mathematical structure of IUP-algebras has captivated numerous researchers, sparking extensive studies that continue to this day. Enthusiastic scholars have delved deep into the intricacies of IUP-algebras and applied its principles to various other concepts. This has led to the creation of many new definitions and theories, significantly expanding the field and demonstrating the far-reaching impact of IUP-algebras on modern mathematics. Chanmanee et al. [6] introduced the concept of the direct product of an infinite family of IUP-algebras. They explored the external direct product of specific subsets and introduced the weak direct product. Additionally, they presented fundamental theorems on (anti-)IUP-homomorphisms within this context. Their work significantly advances both the theoretical framework and practical understanding of IUP-algebras. Chanmanee et al. [5] pioneered the concept of the direct product for an infinite family of IUP-algebras, demonstrating that it forms a DIUP-algebra. Additionally, they introduced the innovative idea of weak direct product DIUP-algebras, further expanding the theoretical framework of IUP-algebras. Kuntama et al. [9] has revolutionized the application of fuzzy set theory to IUP-algebras by introducing four groundbreaking concepts: fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals. Their study delves deep into these innovative ideas, meticulously exploring their unique properties and intricate interrelationships. This work marks a significant advancement in the field, opening new avenues for research and application. Suayngam et al. [13] made significant steps forward in the study of IUP-algebras in 2024 by coming up with the ideas of intuitionistic fuzzy IUP-subalgebras, intuitionistic fuzzy IUP-ideals, intuitionistic fuzzy IUP-filters, and intuitionistic fuzzy strong IUP-ideals. This pioneering work expands the theoretical landscape of IUP-algebras, blending intuitionistic fuzzy set theory with algebraic structures in innovative ways.

Building on extensive research into Fermatean fuzzy sets, this paper aims to extend these concepts to IUP-algebras. We introduce and explore Fermatean fuzzy IUP-subalgebras, Fermatean fuzzy IUP-ideals, Fermatean fuzzy IUP-filters, and Fermatean fuzzy strong IUP-ideals. Our study investigates their properties, focusing on characteristic Fermatean fuzzy sets, upper t-(strong) level subsets, and lower t-(strong) level subsets.

2. Preliminaries

Before delving into our study, let's review the foundational concepts of IUP-algebras, including their various properties and pertinent definitions crucial to this research.

Definition 1. *[7] An algebra $X = (X; \cdot, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a non-empty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 \cdot x = x) \tag{IUP-1}$$

$$(\forall x \in X)(x \cdot x = 0) \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot (x \cdot z) = y \cdot z) \tag{IUP-3}$$

Example 1. *[7] Let $(G, \bullet, e)$ be a group such that all elements self-inverse. Then $(G, \bullet, e)$ is an IUP-algebra.*

Example 2. *[7] Let X be a set and $\mathcal{P}(X)$ means the power set of X. It follows from Example 1 that $(\mathcal{P}(X), \triangle, \emptyset)$ is an IUP-algebra where the binary operation $\triangle$ is defined as the symmetric difference of any two sets.*

Example 3. *[7] Let $(G, \bullet, e)$ be a group with the identity element e. Define a binary operation $\bullet$ on G by:*

$$(\forall x, y \in G)(x \bullet y = yx^{-1}) \tag{2.1}$$

Then $(G, \bullet, e)$ is an IUP-algebra.

For convenience, we refer to X as an IUP-algebra $X = (X; \cdot, 0)$ until otherwise specified.

Proposition 1. *[7] In X, the following assertions are valid (see [7]).*

$$(\forall x, y \in X)((x \cdot 0) \cdot (x \cdot y) = y) \tag{2.2}$$
$$(\forall x \in X)((x \cdot 0) \cdot (x \cdot 0) = 0) \tag{2.3}$$
$$(\forall x, y \in X)((x \cdot y) \cdot 0 = y \cdot x) \tag{2.4}$$
$$(\forall x \in X)((x \cdot 0) \cdot 0 = x) \tag{2.5}$$
$$(\forall x, y \in X)(x \cdot ((x \cdot 0) \cdot y) = y) \tag{2.6}$$
$$(\forall x, y \in X)(((x \cdot 0) \cdot y) \cdot x = y \cdot 0) \tag{2.7}$$
$$(\forall x, y, z \in X)(x \cdot y = x \cdot z \Leftrightarrow y = z) \tag{2.8}$$
$$(\forall x, y \in X)(x \cdot y = 0 \Leftrightarrow x = y) \tag{2.9}$$
$$(\forall x \in X)(x \cdot 0 = 0 \Leftrightarrow x = 0) \tag{2.10}$$
$$(\forall x, y, z \in X)(y \cdot x = z \cdot x \Leftrightarrow y = z) \tag{2.11}$$
$$(\forall x, y \in X)(x \cdot y = y \Rightarrow x = 0) \tag{2.12}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot 0 = (z \cdot y) \cdot (z \cdot x)) \tag{2.13}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (z \cdot x) \cdot (z \cdot y) = 0) \tag{2.14}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (x \cdot z) \cdot (y \cdot z) = 0) \tag{2.15}$$
$$\text{the right and the left cancellation laws hold} \tag{2.16}$$

In the realm of IUP-algebras, four key subsets are crucial: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets provide a nuanced framework essential for understanding and applying IUP-algebras in various mathematical contexts.

Definition 2. *[7] A non-empty subset S of X is called*

(*i*) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \cdot y \in S) \tag{2.17}$$

(*ii*) *an IUP-filter of X if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S \tag{2.18}$$
$$(\forall x, y \in X)(x \cdot y \in S \text{ and } x \in S \Rightarrow y \in S) \tag{2.19}$$

(*iii*) *an IUP-ideal of X if it satisfies the condition* (2.18) *and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S \text{ and } y \in S \Rightarrow x \cdot z \in S) \tag{2.20}$$

(*iv*) *a strong IUP-ideal of X if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S) \tag{2.21}$$

According to [7], the concept of IUP-filters serves as a generalization encompassing IUP-ideals and IUP-subalgebras. Both IUP-ideals and IUP-subalgebras, in turn, generalize strong IUP-ideals. In an IUP-algebra X, it is observed that strong IUP-ideals coincide with X itself. This relationship is illustrated in the diagram of special subsets of IUP-algebras, depicted in Figure 1.

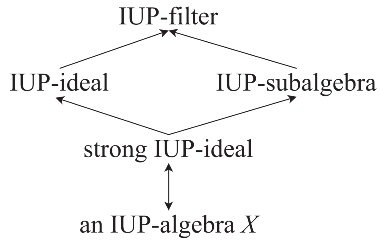

Figure 1: Special subsets of IUP-algebras

3. Main results

Before diving into the definition of Fermatean fuzzy sets, it's essential to revisit and understand the foundational concepts that underpin them. This background will provide the necessary context and enhance our comprehension of Fermatean fuzzy sets.

From now on, we will use abbreviations to represent the following technical terms.

Technical terms	Abbreviations
Fuzzy set	FS
Fermatean fuzzy set	FFS
Fermatean fuzzy IUP-subalgebra	FFIUP-subalgebra
Fermatean fuzzy IUP-ideal	FFIUP-ideal
Fermatean fuzzy IUP-filter	FFIUP-filter
Fermatean fuzzy strong IUP-ideal	FFSIUP-ideal

Definition 3. *[2] Let X be a universe of discourse. A Fermatean fuzzy set $\mathcal{F}$ (FFS) in X is an object having the form $\mathcal{F} = \{(x, \alpha_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x)) : x \in X\}$, where $\alpha_{\mathcal{F}}(x) : X \to [0,1]$ and $\beta_{\mathcal{F}}(x) : X \to [0,1]$, including the following condition:*

$$(\forall x \in X)(0 \leq (\alpha_{\mathcal{F}}(x))^3 + (\beta_{\mathcal{F}}(x))^3 \leq 1) \tag{3.1}$$

The numbers $\alpha_{\mathcal{F}}(x)$ and $\beta_{\mathcal{F}}(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set $\mathcal{F}$.

For any FFS $\mathcal{F}$ and $x \in X$, $\pi_{\mathcal{F}}(x) = \sqrt[3]{1 - (\alpha_{\mathcal{F}}(x))^3 - (\beta_{\mathcal{F}}(x))^3}$ is identified as the degree of indeterminacy of x to $\mathcal{F}$.

In the interest of simplicity, we shall mention the symbol $\mathcal{F} = (\alpha_{\mathcal{F}}, \beta_{\mathcal{F}})$ for the FFS $\mathcal{F} = \{(x, \alpha_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x)) : x \in X\}$.

For a subset G of a non-empty set X, the characteristic functions $\alpha_{\mathcal{F}_G}$ and $\beta_{\mathcal{F}_G}$ are functions of X into $\{0,1\}$ defined as follows:

$$\alpha_{\mathcal{F}_G}(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_{\mathcal{F}_G}(x) = \begin{cases} 0 & \text{if } x \in G \\ 1 & \text{otherwise} \end{cases}$$

By the definition of the characteristic function, $\alpha_{\mathcal{F}_G}$ and $\beta_{\mathcal{F}_G}$ are functions of X into $\{0,1\} \subset [0,1]$. Therefore, the FFS $\mathcal{F}_G = (\alpha_{\mathcal{F}_G}, \beta_{\mathcal{F}_G})$ is defined as the characteristic FFS of G in X.

Definition 4. *Let f be an FS in a non-empty set X. Then the FS $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in X$ is called the complement of f in X.*

Definition 5. *Let $\mathcal{F}$ be an FFS in a non-empty set X. Then the FFS $\overline{\mathcal{F}} = (\overline{\alpha}_{\mathcal{F}}, \overline{\beta}_{\mathcal{F}})$ is called the complement of $\mathcal{F}$ in X.*

We extend FFSs to IUP-algebras, introducing four innovative types: Fermatean fuzzy IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals. This application opens new dimensions in the study of IUP-algebras, enriching both their theoretical and practical frameworks.

Definition 6. *An FFS $\mathcal{F}$ in X is called a Fermatean fuzzy IUP-subalgebra (FFIUP-subalgebra) of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\alpha_{\mathcal{F}}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\}) \tag{3.2}$$

$$(\forall x, y \in X)(\beta_{\mathcal{F}}(x \cdot y) \leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}) \tag{3.3}$$

Definition 7. *An FFS $\mathcal{F}$ in X is called a Fermatean fuzzy IUP-ideal (FFIUP-ideal) of X if it satisfies the following properties:*

$$(\forall x \in X)(\alpha_{\mathcal{F}}(0) \geq \alpha_{\mathcal{F}}(x)) \tag{3.4}$$

$$(\forall x \in X)(\beta_{\mathcal{F}}(0) \leq \beta_{\mathcal{F}}(x)) \tag{3.5}$$

$$(\forall x, y, z \in X)(\alpha_{\mathcal{F}}(x \cdot z) \geq \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\}) \tag{3.6}$$

$$(\forall x, y, z \in X)(\beta_{\mathcal{F}}(x \cdot z) \leq \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\}) \tag{3.7}$$

Definition 8. *An FFS $\mathcal{F}$ in X is called a Fermatean fuzzy IUP-filter (FFIUP-filter) of X if it satisfies* (3.4), (3.5), *and the following properties:*

$$(\forall x, y \in X)(\alpha_{\mathcal{F}}(y) \geq \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\}) \tag{3.8}$$

$$(\forall x, y \in X)(\beta_{\mathcal{F}}(y) \leq \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\}) \tag{3.9}$$

Definition 9. *An FFS $\mathcal{F}$ in X is called a Fermatean fuzzy strong IUP-ideal (FFSIUP-ideal) of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\alpha_{\mathcal{F}}(x \cdot y) \geq \alpha_{\mathcal{F}}(y)) \tag{3.10}$$
$$(\forall x, y \in X)(\beta_{\mathcal{F}}(x \cdot y) \leq \beta_{\mathcal{F}}(y)) \tag{3.11}$$

Lemma 1. *Every FFIUP-subalgebra of X satisfies* (3.4) *and* (3.5)*.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned} \alpha_{\mathcal{F}}(0) &= \alpha_{\mathcal{F}}(x \cdot x) && \text{(by (IUP-2))} \\ &\geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(x)\} && \text{(by (3.2))} \\ &= \alpha_{\mathcal{F}}(x), \\ \beta_{\mathcal{F}}(0) &= \beta_{\mathcal{F}}(x \cdot x) && \text{(by (IUP-2))} \\ &\leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x)\}. && \text{(by (3.3))} \end{aligned}$$

Hence, $\mathcal{F}$ satisfies (3.4) and (3.5).

Theorem 1. *Every FFSIUP-ideal of X satisfies* (3.4) *and* (3.5)*.*

Proof. Assume that $\mathcal{F}$ is an FFSIUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned} \alpha_{\mathcal{F}}(0) &= \alpha_{\mathcal{F}}(x \cdot x) && \text{(by (IUP-2))} \\ &\geq \alpha_{\mathcal{F}}(x), && \text{(by (3.10))} \\ \beta_{\mathcal{F}}(0) &= \beta_{\mathcal{F}}(x \cdot x) && \text{(by (IUP-2))} \\ &\leq \beta_{\mathcal{F}}(x). && \text{(by (3.11))} \end{aligned}$$

Hence, $\mathcal{F}$ satisfies (3.4) and (3.5).

Theorem 2. *An FFSIUP-ideal and constant FFS coincide.*

Proof. Assume that $\mathcal{F}$ is an FFSIUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned} \alpha_{\mathcal{F}}(x) &= \alpha_{\mathcal{F}}((x \cdot 0) \cdot 0) && \text{(by (2.5))} \\ &\geq \alpha_{\mathcal{F}}(0), && \text{(by (3.10))} \\ \beta_{\mathcal{F}}(x) &= \beta_{\mathcal{F}}((x \cdot 0) \cdot 0) && \text{(by (2.5))} \\ &\leq \beta_{\mathcal{F}}(0). && \text{(by (3.11))} \end{aligned}$$

It follows from Theorem 1 that $\mathcal{F}$ is a constant FFS of X.

Conversely, it is obviously true that every constant FFS is an FFSIUP-ideal of X.

The following theorem is a direct consequence of Theorem 2.

Theorem 3. *Every FFSIUP-ideal of X is an FFIUP-subalgebra of X.*

Example 4. *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	3	5	0	4	1	2
3	2	4	5	0	3	1
4	5	3	1	2	0	4
5	4	2	3	1	5	0

Then X *is an IUP-algebra. We define an FFS* $\mathcal{F}$ *in* X *as follows:*

$$\alpha_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.9 & 0.1 & 0.1 & 0.1 & 0.5 & 0.5 \end{pmatrix}$$

$$\beta_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.8 & 0.8 & 0.8 & 0.6 & 0.6 \end{pmatrix}$$

Then $\mathcal{F}$ *is an FFIUP-subalgebra of* X*. Since* $\alpha_{\mathcal{F}}(1 \cdot 4) = \alpha_{\mathcal{F}}(2) = 0.1 \ngeq 0.5 = \alpha_{\mathcal{F}}(4)$ *and* $\beta_{\mathcal{F}}(3 \cdot 4) = \beta_{\mathcal{F}}(3) = 0.8 \nleq 0.6 = \beta_{\mathcal{F}}(4)$*. Hence,* $\mathcal{F}$ *is not an FFSIUP-ideal of* X*.*

The following theorem is a direct consequence of Theorem 2.

Theorem 4. *Every FFSIUP-ideal of* X *is an FFIUP-ideal of* X*.*

Example 5. *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	3	1	5	2
2	2	5	0	4	3	1
3	5	4	1	0	2	3
4	1	3	5	2	0	4
5	3	2	4	5	1	0

Then X *is an IUP-algebra. We define an FFS* $\mathcal{F}$ *in* X *as follows:*

$$\alpha_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.1 & 0.1 & 0.3 & 0.1 & 0.3 \end{pmatrix}$$

$$\beta_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.9 & 0.9 & 0.7 & 0.9 & 0.7 \end{pmatrix}$$

Then $\mathcal{F}$ *is an FFIUP-ideal of* X*. Since* $\alpha_{\mathcal{F}}(5 \cdot 0) = \alpha_{\mathcal{F}}(3) = 0.3 \ngeq 0.5 = \alpha_{\mathcal{F}}(0)$ *and* $\beta_{\mathcal{F}}(1 \cdot 5) = \beta_{\mathcal{F}}(2) = 0.9 \nleq 0.7 = \beta_{\mathcal{F}}(5)$*. Hence,* $\mathcal{F}$ *is not an FFSIUP-ideal of* X*.*

Theorem 5. *Every FFIUP-ideal of* X *is an FFIUP-filter of* X*.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-ideal of X. By the assumption, it satisfies (3.4) and (3.5). Let $x, y \in X$. Then

$$\begin{aligned}
\alpha_{\mathcal{F}}(y) &= \alpha_{\mathcal{F}}(0 \cdot y) && \text{(by (IUP-1))} \\
&\geq \min\{\alpha_{\mathcal{F}}(0 \cdot (x \cdot y)), \alpha_{\mathcal{F}}(x)\} && \text{(by (3.6))} \\
&= \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\}, && \text{(by (IUP-1))} \\
\beta_{\mathcal{F}}(y) &= \beta_{\mathcal{F}}(0 \cdot y) && \text{(by (IUP-1))} \\
&\leq \max\{\beta_{\mathcal{F}}(0 \cdot (x \cdot y)), \beta_{\mathcal{F}}(x)\} && \text{(by (3.7))} \\
&= \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\}. && \text{(by (IUP-1))}
\end{aligned}$$

Hence, $\mathcal{F}$ is an FFIUP-filter of X.

Example 6. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	3	2
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	5	3	1	2	0	4
5	4	2	3	1	5	0

Then X is an IUP-algebra. We define an FFS $\mathcal{F}$ in X as follows:

$$\alpha_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.5 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

$$\beta_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.1 & 0.7 & 0.9 & 0.9 & 0.9 & 0.9 \end{pmatrix}$$

Then $\mathcal{F}$ is an FFIUP-filter of X. Since $\alpha_{\mathcal{F}}(2 \cdot 5) = \alpha_{\mathcal{F}}(3) = 0.2 \ngeq 0.5 = \min\{0.6, 0.5\} = \min\{\alpha_{\mathcal{F}}(0), \alpha_{\mathcal{F}}(1)\} = \min\{\alpha_{\mathcal{F}}(2 \cdot 2), \alpha_{\mathcal{F}}(1)\} = \min\{\alpha_{\mathcal{F}}(2 \cdot (1 \cdot 5)), \alpha_{\mathcal{F}}(1)\}$ and $\beta_{\mathcal{F}}(3 \cdot 4) = \beta_{\mathcal{F}}(2) = 0.9 \nleq 0.7 = \max\{0.1, 0.7\} = \max\{\beta_{\mathcal{F}}(0), \beta_{\mathcal{F}}(1)\} = \max\{\beta_{\mathcal{F}}(3 \cdot 3), \beta_{\mathcal{F}}(1)\} = \max\{\beta_{\mathcal{F}}(3 \cdot (1 \cdot 4)), \beta_{\mathcal{F}}(1)\}$. Hence, $\mathcal{F}$ is not an FFIUP-ideal of X.

Theorem 6. *Every FFIUP-subalgebra of X is an FFIUP-filter of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-subalgebra of X. By Lemma 1, we have $\mathcal{F}$ satisfies (3.4) and (3.5). Let $x, y \in X$. Then

$$\begin{aligned}
\alpha_{\mathcal{F}}(y) &= \alpha_{\mathcal{F}}(0 \cdot y) && \text{(by (IUP-1))} \\
&= \alpha_{\mathcal{F}}((x \cdot 0) \cdot (x \cdot y)) && \text{(by (IUP-3))} \\
&\geq \min\{\alpha_{\mathcal{F}}(x \cdot 0), \alpha_{\mathcal{F}}(x \cdot y)\} && \text{(by (3.2))} \\
&\geq \min\{\min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(0)\}, \alpha_{\mathcal{F}}(x \cdot y)\} && \text{(by (3.2))}
\end{aligned}$$

$$\begin{aligned}
&= \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(x \cdot y)\}, && \text{(by (3.4))}\\
\beta_{\mathcal{F}}(y) &= \beta_{\mathcal{F}}(0 \cdot y) && \text{(by (IUP-1))}\\
&= \beta_{\mathcal{F}}((x \cdot 0) \cdot (x \cdot y)) && \text{(by (IUP-3))}\\
&\leq \max\{\beta_{\mathcal{F}}(x \cdot 0), \beta_{\mathcal{F}}(x \cdot y)\} && \text{(by (3.3))}\\
&\leq \max\{\max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(0)\}, \beta_{\mathcal{F}}(x \cdot y)\} && \text{(by (3.3))}\\
&= \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x \cdot y)\}. && \text{(by (3.5))}
\end{aligned}$$

Hence, $\mathcal{F}$ is an FFIUP-filter of X.

Example 7. *[7] Let $\mathbb{R}^*$ be the set of all nonzero real numbers. Define a binary operation $\cdot$ on $\mathbb{R}^*$ by:*

$$(\forall x, y \in \mathbb{R}^*)(x \cdot y = \frac{y}{x}).$$

Thus, $(\mathbb{R}^*, \cdot, 1)$ is an IUP-algebra.

Example 8. *From Example 7, let $S = \{x \in \mathbb{R}^* \mid x \geq 1\}$. Then $1 \in S$. Next, let $x, y, z \in \mathbb{R}^*$ be such that $x \cdot (y \cdot z) \geq 1$ and $y \geq 1$. Then $\frac{z}{yx} \geq 1$. Thus, $x \cdot z = \frac{z}{x} = (\frac{z}{yx})y \geq 1$, that is, $x \cdot z \in S$. Hence, S is an IUP-ideal of $\mathbb{R}^*$. Then S is an IUP-filter of $\mathbb{R}^*$. By Theorems 9 and 10, we have $\mathcal{F}_S$ is an FFIUP-ideal and an FFIUP-filter of $\mathbb{R}^*$. Since $1, 3 \in S$ but $3 \cdot 1 = \frac{1}{3} \in S$, we have S is not an IUP-subalgebra of $\mathbb{R}^*$. By Theorem 8, we have $\mathcal{F}_S$ is not an FFIUP-subalgebra of $\mathbb{R}^*$.*

Example 9. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	4	5	3
2	1	2	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Then X is an IUP-algebra. We define an FFS $\mathcal{F}$ in X as follows:

$$\alpha_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.2 & 0.2 & 0.7 & 0.2 & 0.2 \end{pmatrix}$$

$$\beta_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.1 & 0.9 & 0.9 & 0.5 & 0.9 & 0.9 \end{pmatrix}$$

Then $\mathcal{F}$ is an FFIUP-subalgebra of X. Since $\alpha_{\mathcal{F}}(1 \cdot 4) = \alpha_{\mathcal{F}}(5) = 0.2 \ngeq 0.7 = \min\{0.8, 0.7\} = \min\{\alpha_{\mathcal{F}}(0), \alpha_{\mathcal{F}}(3)\} = \min\{\alpha_{\mathcal{F}}(1 \cdot (3 \cdot 4)), \alpha_{\mathcal{F}}(3)\}$ and $\beta_{\mathcal{F}}(1 \cdot 2) = \beta_{\mathcal{F}}(1) = 0.9 \nleq 0.5 = \max\{0.5, 0.5\} = \max\{\beta_{\mathcal{F}}(3), \beta_{\mathcal{F}}(3)\} = \max\{\beta_{\mathcal{F}}(1 \cdot (3 \cdot 2)), \beta_{\mathcal{F}}(3)\}$. Hence, $\mathcal{F}$ is not an FFIUP-ideal of X.

The study revealed a relationship between the four concepts: FFIUP-ideals and FFIUP-subalgebras are generalizations of FFSIUP-ideals of IUP-algebras, where FFSIUP-ideals of IUP-algebras can only be a constant FFS. FFIUP-filters are a generalization of FFIUP-ideals and FFIUP-subalgebras. We summarize the relationship between these four concepts, shown in Figure 2.

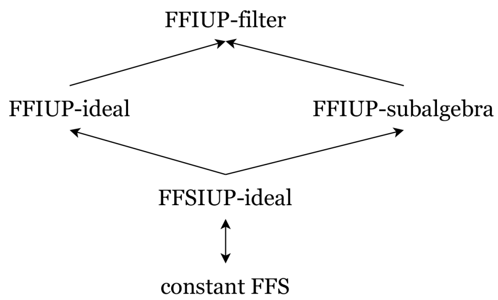

Figure 2: FFSs in IUP-algebras

Theorem 7. *If $\mathcal{F}$ is an FFIUP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X) \begin{pmatrix} \alpha_{\mathcal{F}}(y \cdot (x \cdot z)) = \alpha_{\mathcal{F}}(x \cdot (y \cdot z)) \\ \beta_{\mathcal{F}}(y \cdot (x \cdot z)) = \beta_{\mathcal{F}}(x \cdot (y \cdot z)) \end{pmatrix} \tag{3.12}$$

then $\mathcal{F}$ is an FFIUP-ideal of X.

Proof. Assume that $\mathcal{F}$ is an FFIUP-filter of X satisfying the condition (3.12). By the assumption, it satisfies (3.4) and (3.5). Let $x, y, z \in X$. Then

$$\begin{aligned} \alpha_{\mathcal{F}}(x \cdot z) &\geq \min\{\alpha_{\mathcal{F}}(y \cdot (x \cdot z)), \alpha_{\mathcal{F}}(y)\} && \text{(by (3.8))} \\ &= \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\}, && \text{(by (3.12))} \\ \beta_{\mathcal{F}}(x \cdot z) &\leq \max\{\beta_{\mathcal{F}}(y \cdot (x \cdot z)), \beta_{\mathcal{F}}(y)\} && \text{(by (3.9))} \\ &= \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\}. && \text{(by (3.12))} \end{aligned}$$

Hence, $\mathcal{F}$ is an FFIUP-ideal of X.

Lemma 2. *Let G be a non-empty subset of X. Then the constant 0 is in G if and only if the characteristic FFS $\mathcal{F}_G$ satisfies (3.4) and (3.5).*

Proof. Assume that the constant 0 is in G. Then $\alpha_{\mathcal{F}_G}(0) = 1$ and $\beta_{\mathcal{F}_G}(0) = 0$. Thus, $\alpha_{\mathcal{F}_G}(0) = 1 \geq \alpha_{\mathcal{F}_G}(x)$ and $\beta_{\mathcal{F}_G}(0) = 0 \leq \beta_{\mathcal{F}_G}(x)$ for all $x \in X$, that is, $\mathcal{F}_G$ satisfies (3.4) and (3.5).

Conversely, assume that the characteristic FFS $\mathcal{F}_G$ satisfies (3.4) and (3.5). Then $\alpha_{\mathcal{F}_G}(0) \geq \alpha_{\mathcal{F}_G}(x)$ for all $x \in X$. Since G is a non-empty subset of X, we let $a \in G$. Then $\alpha_{\mathcal{F}_G}(0) \geq \alpha_{\mathcal{F}_G}(a) = 1$, so $\alpha_{\mathcal{F}_G}(0) = 1$. Hence, the constant 0 is in G.

Theorem 8. *A non-empty subset G of X is an IUP-subalgebra of X if and only if the characteristic FFS $\mathcal{F}_G$ is an FFIUP-subalgebra of X.*

Proof. Assume that G is an IUP-subalgebra of X. Let $x, y \in X$.

Case 1: Suppose $x, y \in G$. Then $\alpha_{\mathcal{F}_G}(x) = 1$ and $\alpha_{\mathcal{F}_G}(y) = 1$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\alpha_{\mathcal{F}_G}(x \cdot y) = 1 \geq \min\{1, 1\} = \min\{\alpha_{\mathcal{F}_G}(x), \alpha_{\mathcal{F}_G}(y)\}$.

Case 2: Suppose $x \notin G$ or $y \notin G$. Then $\alpha_{\mathcal{F}_G}(x) = 0$ or $\alpha_{\mathcal{F}_G}(y) = 0$. Thus, $\alpha_{\mathcal{F}_G}(x \cdot y) \geq 0 = \min\{\alpha_{\mathcal{F}_G}(x), \alpha_{\mathcal{F}_G}(y)\}$.

Case 1': Suppose $x, y \in G$. Then $\beta_{\mathcal{F}_G}(x) = 0$ and $\beta_{\mathcal{F}_G}(y) = 0$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\beta_{\mathcal{F}_G}(x \cdot y) = 0 \leq 0 = \max\{\beta_{\mathcal{F}_G}(x), \beta_{\mathcal{F}_G}(y)\}$.

Case 2' : Suppose $x \notin G$ or $y \notin G$. Then $\beta_{\mathcal{F}_G}(x) = 1$ or $\beta_{\mathcal{F}_G}(y) = 1$. Thus, $\beta_{\mathcal{F}_G}(x \cdot y) \leq 1 = \max\{\beta_{\mathcal{F}_G}(x), \beta_{\mathcal{F}_G}(y)\}$.

Hence, the characteristic FFS $\mathcal{F}_G$ is an FFIUP-subalgebra of X.

Conversely, assume that the characteristic FFS $\mathcal{F}_G$ is an FFIUP-subalgebra of X. Let $x, y \in G$. Then $\alpha_{\mathcal{F}_G}(x) = 1$ and $\alpha_{\mathcal{F}_G}(y) = 1$. By (3.2), we have $\alpha_{\mathcal{F}_G}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}_G}(x), \alpha_{\mathcal{F}_G}(y)\} = \min\{1, 1\} = 1$. Thus, $\alpha_{\mathcal{F}_G}(x \cdot y) = 1$, that is, $x \cdot y \in G$. Hence, G is an IUP-subalgebra of X.

Theorem 9. *A non-empty subset G of X is an IUP-ideal of X if and only if the characteristic FFS $\mathcal{F}_G$ is an FFIUP-ideal of X.*

Proof. Assume that G is an IUP-ideal of X. Since $0 \in G$, it follows from Lemma 2 that $\alpha_{\mathcal{F}_G}$ and $\beta_{\mathcal{F}_G}$ satisfy (3.4) and (3.5), respectively. Next, let $x, y, z \in X$.

Case 1: Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\alpha_{\mathcal{F}_G}(x \cdot z) = 1 \geq 1 = \min\{1, 1\} = \min\{\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}_G}(y)\}$.

Case 2: Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)) = 0$ or $\alpha_{\mathcal{F}_G}(y) = 0$. Thus, $\alpha_{\mathcal{F}_G}(x \cdot z) \geq 0 = \min\{\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}_G}(y)\}$.

Case 1': Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\beta_{\mathcal{F}_G}(x \cdot z) = 0 \leq 0 = \max\{0, 0\} = \max\{\beta_{\mathcal{F}_G}(x \cdot (y \cdot z)), \beta_{\mathcal{F}_G}(y)\}$.

Case 2': Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\beta_{\mathcal{F}_G}(x \cdot (y \cdot z)) = 1$ or $\beta_{\mathcal{F}_G}(y) = 1$. Thus, $\beta_{\mathcal{F}_G}(x \cdot z) \leq 1 = \max\{\beta_{\mathcal{F}_G}(x \cdot (y \cdot z)), \beta_{\mathcal{F}_G}(y)\}$.

Hence, $\mathcal{F}_G$ is an FFIUP-ideal of X.

Conversely, assume that the characteristic FFS $\mathcal{F}_G$ is an FFIUP-ideal of X. Since $\alpha_{\mathcal{F}_G}$ satisfies (3.4), it follows from Lemma 2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)) = 1$ and $\alpha_{\mathcal{F}_G}(y) = 1$. Thus, $\min\{\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}_G}(y)\} = 1$. By (3.6), we have $\alpha_{\mathcal{F}_G}(x \cdot z) \geq \min\{\alpha_{\mathcal{F}_G}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}_G}(y)\} = 1$, that is, $\alpha_{\mathcal{F}_G}(x \cdot z) = 1$. Hence, $x \cdot z \in G$, so G is an IUP-ideal of X.

Theorem 10. *A non-empty subset G of X is an IUP-filter of X if and only if the characteristic FFS $\mathcal{F}_G$ is an FFIUP-filter of X.*

Proof. Assume that G is an IUP-filter of X. Since $0 \in G$, it follows from Lemma 2 that $\alpha_{\mathcal{F}_G}$ and $\beta_{\mathcal{F}_G}$ satisfy (3.4) and (3.5), respectively. Next, let $x, y \in X$.

Case 1: Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\alpha_{\mathcal{F}_G}(y) = 1 \geq 1 = \min\{1, 1\} = \min\{\alpha_{\mathcal{F}_G}(x \cdot y), \alpha_{\mathcal{F}_G}(x)\}$.

Case 2: Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\alpha_{\mathcal{F}_G}(x \cdot y) = 0$ or $\alpha_{\mathcal{F}_G}(x) = 0$. Thus, $\alpha_{\mathcal{F}_G}(y) \geq 0 = \min\{\alpha_{\mathcal{F}_G}(x \cdot y), \alpha_{\mathcal{F}_G}(x)\}$.

Case 1': Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\beta_{\mathcal{F}_G}(y) = 0 \leq 0 = \max\{0, 0\} = \max\{\beta_{\mathcal{F}_G}(x \cdot y), \beta_{\mathcal{F}_G}(x)\}$.

Case 2': Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\beta_{\mathcal{F}_G}(x \cdot y) = 1$ or $\beta_{\mathcal{F}_G}(x) = 1$. Thus, $\beta_{\mathcal{F}_G}(y) \leq 1 = \max\{\beta_{\mathcal{F}_G}(x \cdot y), \beta_{\mathcal{F}_G}(x)\}$.

Hence, $\mathcal{F}_G$ is an FFIUP-filter of X.

Conversely, assume that the characteristic FFS $\mathcal{F}_G$ is an FFIUP-filter of X. Since $\alpha_{\mathcal{F}_G}$ satisfies (3.4), it follows from Lemma 2 that $0 \in G$. Next, let $x, y \in G$ be such that $x \cdot y \in G$ and $x \in G$. Then $\alpha_{\mathcal{F}_G}(x \cdot y) = 1$ and $\alpha_{\mathcal{F}_G}(x) = 1$. Thus, $\min\{\alpha_{\mathcal{F}_G}(x \cdot y), \alpha_{\mathcal{F}_G}(x)\} = 1$. By (3.8), we have $\alpha_{\mathcal{F}_G}(y) = \min\{\alpha_{\mathcal{F}_G}(x \cdot y), \alpha_{\mathcal{F}_G}(x)\} = 1$, that is, $\alpha_{\mathcal{F}_G}(y) = 1$. Hence, $y \in G$, so G is an IUP-filter of X.

The following theorem is a direct consequence of Theorem 2.

Theorem 11. *A non-empty subset G of X is a strong IUP-ideal of X if and only if the characteristic FFS $\mathcal{F}_G$ is an FFSIUP-ideal of X.*

Lemma 3. *[13] Let f be an FS in a non-empty set X. Then the following statements hold:*

$$(\forall x, y \in X)(1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}) \tag{3.13}$$

$$(\forall x, y \in X)(1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}) \tag{3.14}$$

Lemma 4. *[13] Let f be an FS in a non-empty set X. Then the following statements hold:*

$$(\forall x, y, z \in X)(f(z) \geq \min\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \leq \max\{\overline{f}(x), \overline{f}(y)\}) \tag{3.15}$$

$$(\forall x, y, z \in X)(f(z) \leq \max\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \geq \min\{\overline{f}(x), \overline{f}(y)\}) \tag{3.16}$$

Before presenting theorems on the relationship between FSSs and their complements, it's crucial to grasp their basic concept. FSSs extend traditional FSs by incorporating hesitation degrees. The following theorem highlights the key relationship between these sets and their complements.

Theorem 12. *An FFS $\mathcal{F}$ is an FFIUP-subalgebra of X if and only if the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy* (3.2)*, and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy* (3.3)*.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-subalgebra of X. Then

$$\alpha_{\mathcal{F}}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\},$$
$$\beta_{\mathcal{F}}(x \cdot y) \leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}.$$

Thus,

$$\overline{\alpha}_{\mathcal{F}}(x \cdot y) \leq \max\{\overline{\alpha}_{\mathcal{F}}(x), \overline{\alpha}_{\mathcal{F}}(y)\}, \qquad \text{(by (3.15))}$$
$$\overline{\beta}_{\mathcal{F}}(x \cdot y) \geq \min\{\overline{\beta}_{\mathcal{F}}(x), \overline{\beta}_{\mathcal{F}}(y)\},. \qquad \text{(by (3.16))}$$

Hence, the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.2), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.3).

Conversely, assume that the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.2), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.3). Then $\alpha_{\mathcal{F}}$ satisfies (3.2), and $\beta_{\mathcal{F}}$ satisfies (3.3). Hence, $\mathcal{F}$ is an FFIUP-subalgebra of X.

Theorem 13. *An FFS $\mathcal{F}$ is an FFIUP-ideal of X if and only if the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy* (3.4) *and* (3.6)*, and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy* (3.5) *and* (3.7)*.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-ideal of X. Then

$$\begin{aligned}
\alpha_{\mathcal{F}}(0) &\geq \alpha_{\mathcal{F}}(x),\\
\beta_{\mathcal{F}}(0) &\leq \beta_{\mathcal{F}}(x),\\
\alpha_{\mathcal{F}}(x \cdot z) &\geq \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\},\\
\beta_{\mathcal{F}}(x \cdot z) &\leq \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\alpha}_{\mathcal{F}}(0) &\leq \overline{\alpha}_{\mathcal{F}}(x),\\
\overline{\beta}_{\mathcal{F}}(0) &\geq \overline{\beta}_{\mathcal{F}}(x),\\
\overline{\alpha}_{\mathcal{F}}(x \cdot z) &\leq \max\{\overline{\alpha}_{\mathcal{F}}(x \cdot (y \cdot z)), \overline{\alpha}_{\mathcal{F}}(y)\},\\
\overline{\beta}_{\mathcal{F}}(x \cdot z) &\geq \min\{\overline{\beta}_{\mathcal{F}}(x \cdot (y \cdot z)), \overline{\beta}_{\mathcal{F}}(y)\}.
\end{aligned}$$

Hence, the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.4) and (3.6), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.5) and (3.7).

Conversely, assume that the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.4) and (3.6), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.5) and (3.7). Then $\alpha_{\mathcal{F}}$ satisfies (3.4) and (3.6), and $\beta_{\mathcal{F}}$ satisfies (3.5) and (3.7). Hence, $\mathcal{F}$ is an FFIUP-ideal of X.

Theorem 14. *An FFS $\mathcal{F}$ is an FFIUP-filter of X if and only if the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy* (3.4) *and* (3.8)*, and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy* (3.5) *and* (3.9)*.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-ideal of X. Then

$$\begin{aligned}
\alpha_{\mathcal{F}}(0) &\geq \alpha_{\mathcal{F}}(x),\\
\beta_{\mathcal{F}}(0) &\leq \beta_{\mathcal{F}}(x),\\
\alpha_{\mathcal{F}}(y) &\geq \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\},\\
\beta_{\mathcal{F}}(y) &\leq \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\alpha}_{\mathcal{F}}(0) &\leq \overline{\alpha}_{\mathcal{F}}(x),\\
\overline{\beta}_{\mathcal{F}}(0) &\geq \overline{\beta}_{\mathcal{F}}(x),\\
\overline{\alpha}_{\mathcal{F}}(y) &\leq \max\{\overline{\alpha}_{\mathcal{F}}(x \cdot y), \overline{\alpha}_{\mathcal{F}}(x)\},
\end{aligned}$$

$$\overline{\beta}_{\mathcal{F}}(y) \geq \min\{\overline{\beta}_{\mathcal{F}}(x \cdot y), \overline{\beta}_{\mathcal{F}}(x)\}.$$

Hence, the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.4) and (3.8), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.5) and (3.9).

Conversely, assume that the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy (3.4) and (3.8), and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy (3.5) and (3.9). Then $\alpha_{\mathcal{F}}$ satisfies (3.4) and (3.8), and $\beta_{\mathcal{F}}$ satisfies (3.5) and (3.9). Hence, $\mathcal{F}$ is an FFIUP-filter of X.

The following theorem is a direct consequence of Theorem 2.

Theorem 15. *An FFS $\mathcal{F}$ is an FFSIUP-ideal of X if and only if the FSs $\alpha_{\mathcal{F}}$ and $\overline{\beta}_{\mathcal{F}}$ satisfy* (3.10)*, and the FSs $\overline{\alpha}_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$ satisfy* (3.11)*.*

Theorem 16. *An FFS $\mathcal{F}$ is an FFIUP-subalgebra of X if and only if FFS $*\mathcal{F} = (\alpha_{\mathcal{F}}, \overline{\alpha}_{\mathcal{F}})$ and $\triangle\mathcal{F} = (\overline{\beta}_{\mathcal{F}}, \beta_{\mathcal{F}})$ are FFIUP-subalgebras of X.*

Proof. It is straightforward by Theorem 12.

Theorem 17. *An FFS $\mathcal{F}$ is an FFIUP-ideal of X if and only if FFS $*\mathcal{F} = (\alpha_{\mathcal{F}}, \overline{\alpha}_{\mathcal{F}})$ and $\triangle\mathcal{F} = (\overline{\beta}_{\mathcal{F}}\beta_{\mathcal{F}})$ are FFIUP-ideals of X.*

Proof. It is straightforward by Theorem 13.

Theorem 18. *An FFS $\mathcal{F}$ is an FFIUP-filter of X if and only if FFS $*\mathcal{F} = (\alpha_{\mathcal{F}}, \overline{\alpha}_{\mathcal{F}})$ and $\triangle\mathcal{F} = (\overline{\beta}_{\mathcal{F}}\beta_{\mathcal{F}})$ are FFIUP-filters of X.*

Proof. It is straightforward by Theorem 14.

Theorem 19. *An FFS $\mathcal{F}$ is an FFSIUP-ideal of X if and only if FFS $*\mathcal{F} = (\alpha_{\mathcal{F}}, \overline{\alpha}_{\mathcal{F}})$ and $\triangle\mathcal{F} = (\overline{\beta}_{\mathcal{F}}\beta_{\mathcal{F}})$ are FFSIUP-ideals of X.*

Proof. It is straightforward by Theorem 15.

Definition 10. *[12] Let f be an FS in a non-empty set X. For any $t \in [0, 1]$, the sets*

$$U(f; t) = \{x \in X \mid f(x) \geq t\}, \tag{3.17}$$

$$L(f; t) = \{x \in X \mid f(x) \leq t\} \tag{3.18}$$

are called an upper t-level subset and a lower t-level subset of f, respectively. The sets

$$U^{+}(f; t) = \{x \in X \mid f(x) > t\}, \tag{3.19}$$

$$L^{-}(f; t) = \{x \in X \mid f(x) < t\} \tag{3.20}$$

are called an upper t-strong level subset and a lower t-strong level subset of f, respectively.

Before presenting theorems on the relationship between level subsets and their corresponding FFSs, it's essential to grasp the key concepts. Level subsets characterize FFSs by detailing the distribution of membership degrees. The following theorem formalizes this relationship, providing insights into the structure of FFSs.

Theorem 20. *An FFS $\mathcal{F}$ is an FFIUP-subalgebra of X if and only if for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-subalgebra of X. Let $t \in [0, 1]$ be such that $U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. Let $x, y \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(x) \geq t$ and $\alpha_{\mathcal{F}}(y) \geq t$. Thus, $\min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\} \geq t$. By (3.2), we have $\alpha_{\mathcal{F}}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\} \geq t$, that is, $\alpha_{\mathcal{F}}(x \cdot y) \geq t$. Thus, $x \cdot y \in U(\alpha_{\mathcal{F}}; t)$. Hence, $U(\alpha_{\mathcal{F}}; t)$ is an IUP-subalgebra of X.

Let $s \in [0, 1]$ be such that $L(\beta_{\mathcal{F}}; s) \neq \emptyset$. Let $x, y \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(x) \leq s$ and $\beta_{\mathcal{F}}(y) \leq s$. Thus, $\max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\} \leq s$. By (3.3), we have $\beta_{\mathcal{F}}(x \cdot y) \leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\} \leq s$, that is, $\beta_{\mathcal{F}}(x \cdot y) \leq s$. Thus, $x \cdot y \in L(\beta_{\mathcal{F}}; s)$. Hence, $L(\beta_{\mathcal{F}}; s)$ is an IUP-subalgebra of X.

Conversely, assume that for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Let $t = \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\}$. Then $\alpha_{\mathcal{F}}(x) \geq t$ and $\alpha_{\mathcal{F}}(y) \geq t$. Thus, $x, y \in U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. By the assumption, we have $U(\alpha_{\mathcal{F}}; t)$ is an IUP-subalgebra of X. By (2.17), we have $x \cdot y \in U(\alpha_{\mathcal{F}}; t)$. Thus, $\alpha_{\mathcal{F}}(x \cdot y) \geq t = \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\}$.

Let $x, y \in X$. Let $s = \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}$. Then $\beta_{\mathcal{F}}(x) \leq s$ and $\beta_{\mathcal{F}}(y) \leq s$. Thus, $x, y \in L(\beta_{\mathcal{F}}; s) \neq \emptyset$. By the assumption, we have $L(\beta_{\mathcal{F}}; s)$ is an IUP-subalgebra of X. By (2.17), we have $x \cdot y \in L(\beta_{\mathcal{F}}; s)$. Thus, $\beta_{\mathcal{F}}(x \cdot y) \leq s = \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}$.

Hence, $\mathcal{F}$ is an FFIUP-subalgebra of X.

Theorem 21. *An FFS $\mathcal{F}$ in X is an FFIUP-ideal of X if and only if for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-ideal of X. Let $t \in [0, 1]$ be such that $U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. Let $r \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(r) \geq t$. By (3.4), we have $\alpha_{\mathcal{F}}(0) \geq \alpha_{\mathcal{F}}(r) \geq t$. Thus, $0 \in U(\alpha_{\mathcal{F}}; t)$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(\alpha_{\mathcal{F}}; t)$ and $y \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(x \cdot (y \cdot z)) \geq t$ and $\alpha_{\mathcal{F}}(y) \geq t$. Thus, $\min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\} \geq t$. By (3.6), we have $\alpha_{\mathcal{F}}(x \cdot z) \geq \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\} \geq t$. Thus, $x \cdot z \in U(\alpha_{\mathcal{F}}; t)$. Hence, $U(\alpha_{\mathcal{F}}; t)$ is an IUP-ideal of X.

Let $s \in [0, 1]$ be such that $L(\beta_{\mathcal{F}}; s) \neq \emptyset$. Let $\beta \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(l) \leq s$. By (3.5), we have $\beta_{\mathcal{F}}(0) \leq \beta_{\mathcal{F}}(l) \leq s$. Thus, $0 \in L(\beta_{\mathcal{F}}; s)$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L(\beta_{\mathcal{F}}; s)$ and $y \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(x \cdot (y \cdot z)) \leq s$ and $\beta_{\mathcal{F}}(y) \leq s$. Thus, $\max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\} \leq s$. By (3.7), we have $\beta_{\mathcal{F}}(x \cdot z) \leq \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\} \leq s$. Thus, $x \cdot z \in L(\beta_{\mathcal{F}}; s)$. Hence, $L(\beta_{\mathcal{F}}; s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-ideals of X. Let $x \in X$. Let $t = \alpha_{\mathcal{F}}(x)$. Then $\alpha_{\mathcal{F}}(x) \geq t$. Thus, $x \in U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. By the assumption, we have $U(\alpha_{\mathcal{F}}; t)$ is an IUP-ideal of X. By (2.18), we have $0 \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(0) \geq t = \alpha_{\mathcal{F}}(x)$. Let $x, y, z \in X$. Let $t = \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot$

$z)), \alpha_{\mathcal{F}}(y)\}$. Then $\alpha_{\mathcal{F}}(x \cdot (y \cdot z)) \geq t$ and $\alpha_{\mathcal{F}}(y) \geq t$. Thus, $x \cdot (y \cdot z), y \in U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. By the assumption, we have $U(\alpha_{\mathcal{F}}; t)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U(\alpha_{\mathcal{F}}; t)$. Thus, $\alpha_{\mathcal{F}}(x \cdot z) \geq t = \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\}$.

Let $x \in X$. Let $s = \beta_{\mathcal{F}}(x)$. Then $\beta_{\mathcal{F}}(x) \leq s$. Thus, $x \in L(\beta_{\mathcal{F}}; s) \neq \emptyset$. By the assumption, we have $L(\beta_{\mathcal{F}}; s)$ is an IUP-ideal of X. By (2.18), we have $0 \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(0) \leq s = \beta_{\mathcal{F}}(x)$. Let $x, y, z \in X$. Let $s = \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\}$. Then $\beta_{\mathcal{F}}(x \cdot (y \cdot z)) \leq s$ and $\beta_{\mathcal{F}}(y) \leq s$. Thus, $x \cdot (y \cdot z), y \in L(\beta_{\mathcal{F}}; s) \neq \emptyset$. By the assumption, we have $U(\beta_{\mathcal{F}}; s)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in L(\beta_{\mathcal{F}}; s)$. Thus, $\beta_{\mathcal{F}}(x \cdot z) \leq s = \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\}$.

Hence, $\mathcal{F}$ is an FFIUP-ideal of X.

Theorem 22. *An FFS $\mathcal{F}$ in X is an FFIUP-filter of X if and only if for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-filter of X. Let $t \in [0, 1]$ be such that $U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. Let $r \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(r) \geq t$. By (3.4), we have $\alpha_{\mathcal{F}}(0) \geq \alpha_{\mathcal{F}}(r) \geq t$. Thus, $0 \in U(\alpha_{\mathcal{F}}; t)$. Let $x, y \in X$ be such that $x \cdot y \in U(\alpha_{\mathcal{F}}; t)$ and $x \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(x \cdot y) \geq t$ and $\alpha_{\mathcal{F}}(x) \geq t$. Thus, $\min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\} \geq t$. By (3.8), we have $\alpha_{\mathcal{F}}(y) \geq \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\} \geq t$. Thus, $y \in U(\alpha_{\mathcal{F}}; t)$. Hence, $U(\alpha_{\mathcal{F}}; t)$ is an IUP-filter of X.

Let $s \in [0, 1]$ be such that $L(\beta_{\mathcal{F}}; s) \neq \emptyset$. Let $l \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(l) \leq s$. By (3.5), we have $\beta_{\mathcal{F}}(0) \leq \beta_{\mathcal{F}}(l) \leq s$. Thus, $0 \in L(\beta_{\mathcal{F}}; s)$. Let $x, y \in X$ be such that $x \cdot y \in L(\beta_{\mathcal{F}}; s)$ and $x \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(x \cdot y) \leq s$ and $\beta_{\mathcal{F}}(x) \leq s$. Thus, $\max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\} \leq s$. By (3.9), we have $\beta_{\mathcal{F}}(y) \leq \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\} \leq s$. Thus, $y \in L(\beta_{\mathcal{F}}; s)$. Hence, $L(\beta_{\mathcal{F}}; s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or IUP-filters of X. Let $x \in X$. Let $t = \alpha_{\mathcal{F}}(x)$. Then $\alpha_{\mathcal{F}}(x) \geq t$. Thus, $x \in U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. By the assumption, we have $U(\alpha_{\mathcal{F}}; t)$ is an IUP-filter of X. By (2.18), we have $0 \in U(\alpha_{\mathcal{F}}; t)$. Then $\alpha_{\mathcal{F}}(0) \geq t = \alpha_{\mathcal{F}}(x)$. Let $x, y \in X$. Let $t = \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\}$. Then $\alpha_{\mathcal{F}}(x \cdot y) \geq t$ and $\alpha_{\mathcal{F}}(x) \geq t$. Thus, $x \cdot y, x \in U(\alpha_{\mathcal{F}}; t) \neq \emptyset$. By the assumption, we have $U(\alpha_{\mathcal{F}}; t)$ is an IUP-filter of X. By (2.19), we have $y \in U(\alpha_{\mathcal{F}}; t)$. Thus, $\alpha_{\mathcal{F}}(y) \geq t = \min\{\alpha_{\mathcal{F}}(x \cdot y), \alpha_{\mathcal{F}}(x)\}$.

Let $x \in X$. Let $s = \beta_{\mathcal{F}}(x)$. Then $\beta_{\mathcal{F}}(x) \leq s$. Thus, $x \in L(\beta_{\mathcal{F}}; s) \neq \emptyset$. By the assumption, we have $L(\beta_{\mathcal{F}}; s)$ is an IUP-filter of X. By (2.18), we have $0 \in L(\beta_{\mathcal{F}}; s)$. Then $\beta_{\mathcal{F}}(0) \leq s = \beta_{\mathcal{F}}(x)$. Let $x, y \in X$. Let $s = \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\}$. Then $\beta_{\mathcal{F}}(x \cdot y) \leq s$ and $\beta_{\mathcal{F}}(x) \leq s$. Thus, $x \cdot y, x \in L(\beta_{\mathcal{F}}; s) \neq \emptyset$. By the assumption, we have $L(\beta_{\mathcal{F}}; s)$ is an IUP-filter of X. By (2.19), we have $y \in L(\beta_{\mathcal{F}}; s)$. Thus, $\beta_{\mathcal{F}}(y) \leq s = \max\{\beta_{\mathcal{F}}(x \cdot y), \beta_{\mathcal{F}}(x)\}$.

Hence, $\mathcal{F}$ is an FFIUP-filter of X.

The following theorem is a direct consequence of Theorem 2.

Theorem 23. *An FFS $\mathcal{F}$ in X is an FFSIUP-ideal of X if and only if for all $t, s \in [0, 1]$, the sets $U(\alpha_{\mathcal{F}}; t)$ and $L(\beta_{\mathcal{F}}; s)$ are either empty or strong IUP-ideal of X.*

Theorem 24. *An FFS $\mathcal{F}$ in X is an FFIUP-subalgebra of X if and only if for all $t, s \in [0, 1]$, the sets $U^{+}(\alpha_{\mathcal{F}}; t)$ and $L^{-}(\beta_{\mathcal{F}}; s)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-subalgebra of X. Let $t \in [0,1]$ be such that $U^+(\alpha_{\mathcal{F}};t) \neq \emptyset$. Let $x, y \in U^+(\alpha_{\mathcal{F}};t)$. Then $\alpha_{\mathcal{F}}(x) > t$ and $\alpha_{\mathcal{F}}(y) > t$. Thus, $\min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\} > t$. By (3.2), we have $\alpha_{\mathcal{F}}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\} > t$. Thus, $x \cdot y \in U^+(\alpha_{\mathcal{F}};t)$. Hence, $U^+(\alpha_{\mathcal{F}};t)$ is an IUP-subalgebra of X.

Let $s \in [0,1]$ be such that $L^-(\beta_{\mathcal{F}};s) \neq \emptyset$. Let $x, y \in L^-(\beta_{\mathcal{F}};s)$. Then $\beta_{\mathcal{F}}(x) < s$ and $\beta_{\mathcal{F}}(y) < s$. Thus, $\max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\} < s$. By (3.3), we have $\beta_{\mathcal{F}}(x \cdot y) \leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\} < s$. Thus, $x \cdot y \in L^-(\beta_{\mathcal{F}};s)$. Hence, $L^-(\beta_{\mathcal{F}};s)$ is an IUP-subalgebra of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U^+(\alpha_{\mathcal{F}};t)$ and $L^-(\beta_{\mathcal{F}};s)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Assume that $\alpha_{\mathcal{F}}(x \cdot y) < \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\}$. Let $t = \alpha_{\mathcal{F}}(x \cdot y)$. Then $\alpha_{\mathcal{F}}(x) > t$ and $\alpha_{\mathcal{F}}(y) > t$. Thus, $x, y \in U^+(\alpha_{\mathcal{F}};t)$. By the assumption, we have $U^+(\alpha_{\mathcal{F}};t)$ is an IUP-subalgebra. By (2.17), we have $x \cdot y \in U^+(\alpha_{\mathcal{F}};t)$. So $\alpha_{\mathcal{F}}(x \cdot y) > t = \alpha_{\mathcal{F}}(x \cdot y)$, which is a contradiction. Thus, $\alpha_{\mathcal{F}}(x \cdot y) \geq \min\{\alpha_{\mathcal{F}}(x), \alpha_{\mathcal{F}}(y)\}$.

Let $x, y \in X$. Assume that $\beta_{\mathcal{F}}(x \cdot y) > \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}$. Let $s = \beta_{\mathcal{F}}(x \cdot y)$. Then $\beta_{\mathcal{F}}(x) < s$ and $\beta_{\mathcal{F}}(y) < s$. Thus, $x, y \in L^-(\beta_{\mathcal{F}};s)$. By the assumption, we have $L^-(\beta_{\mathcal{F}};s)$ is an IUP-subalgebra. By (2.17), we have $x \cdot y \in L^-(\beta_{\mathcal{F}};s)$. So $\beta_{\mathcal{F}}(x \cdot y) < s = \beta_{\mathcal{F}}(x \cdot y)$, which is a contradiction. Thus, $\beta_{\mathcal{F}}(x \cdot y) \leq \max\{\beta_{\mathcal{F}}(x), \beta_{\mathcal{F}}(y)\}$.

Hence, $\mathcal{F}$ is an FFIUP-subalgebra of X.

Theorem 25. *An FFS $\mathcal{F}$ in X is an FFIUP-ideal of X if and only if for all $t, s \in [0,1]$, the sets $U^+(\alpha_{\mathcal{F}};t)$ and $L^-(\beta_{\mathcal{F}};s)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-ideal of X. Let $t \in [0,1]$ be such that $U^+(\alpha_{\mathcal{F}};t) \neq \emptyset$. Let $a \in U^+(\alpha_{\mathcal{F}};t)$. Then $\alpha_{\mathcal{F}}(a) > t$. By (3.4), we have $\alpha_{\mathcal{F}}(0) \geq \alpha_{\mathcal{F}}(a) > t$. Thus, $0 \in U^+(\alpha_{\mathcal{F}};t)$. Let $x, y, z \in U^+(\alpha_{\mathcal{F}};t)$ be such that $x \cdot (y \cdot z), y \in U^+(\alpha_{\mathcal{F}};t)$. Then $\alpha_{\mathcal{F}}(x \cdot (y \cdot z)) > t$ and $\alpha_{\mathcal{F}}(y) > t$. Thus, $\min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\} > t$. By (3.6). we have $\alpha_{\mathcal{F}}(x \cdot z) \geq \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\} > t$. Thus, $x \cdot z \in U^+(\alpha_{\mathcal{F}};t)$. Hence, $U^+(\alpha_{\mathcal{F}};t)$ is an IUP-ideal of X.

Let $s \in [0,1]$ be such that $L^-(\beta_{\mathcal{F}};s) \neq \emptyset$. Let $\beta \in L^-(\beta_{\mathcal{F}};s)$. Then $\beta_{\mathcal{F}}(\beta) < s$. By (3.5), we have $\beta_{\mathcal{F}}(0) \leq \beta_{\mathcal{F}}(\beta) < s$. Thus, $0 \in L^-(\beta_{\mathcal{F}};s)$. Let $x, y, z \in L^-(\beta_{\mathcal{F}};s)$ be such that $x \cdot (y \cdot z), y \in L^-(\beta_{\mathcal{F}};s)$. Then $\beta_{\mathcal{F}}(x \cdot (y \cdot z)) < s$ and $\beta_{\mathcal{F}}(y) < s$. Thus, $\max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\} < s$. By (3.7). we have $\beta_{\mathcal{F}}(x \cdot z) \leq \max\{\beta_{\mathcal{F}}(x \cdot (y \cdot z)), \beta_{\mathcal{F}}(y)\} > s$. Thus, $x \cdot z \in L^-(\beta_{\mathcal{F}};s)$. Hence, $L^-(\beta_{\mathcal{F}};s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U^+(\alpha_{\mathcal{F}};t)$ and $L^-(\beta_{\mathcal{F}};s)$ are either empty or IUP-ideals of X. Let $x \in X$. Assume that $\alpha_{\mathcal{F}}(0) < \alpha_{\mathcal{F}}(x)$. Let $t = \alpha_{\mathcal{F}}(0)$. Then $x \in U^+(\alpha_{\mathcal{F}};t) \neq \emptyset$. By the assumption, we have $U^+(\alpha_{\mathcal{F}};t)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\alpha_{\mathcal{F}};t)$. So $\alpha_{\mathcal{F}}(0) > t = \alpha_{\mathcal{F}}(0)$, which is a contradiction. Thus, $\alpha_{\mathcal{F}}(0) \geq \alpha_{\mathcal{F}}(x)$. Let $x, y, z \in X$. Assume that $\alpha_{\mathcal{F}}(x \cdot z) < \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\}$. Let $t = \alpha_{\mathcal{F}}(x \cdot z)$. Then $x \cdot (y \cdot z), y \in U^+(\alpha_{\mathcal{F}};t) \neq \emptyset$. By the assumption, we have $U^+(\alpha_{\mathcal{F}};t)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U^+(\alpha_{\mathcal{F}};t)$. So $\alpha_{\mathcal{F}}(x \cdot z) > t = \alpha_{\mathcal{F}}(x \cdot z)$, which is a contradiction. Thus, $\alpha_{\mathcal{F}}(x \cdot z) \geq \min\{\alpha_{\mathcal{F}}(x \cdot (y \cdot z)), \alpha_{\mathcal{F}}(y)\}$.

Let $x \in X$. Assume that $\beta_{\mathcal{F}}(0) > \beta_{\mathcal{F}}(x)$. Let $s = \beta_{\mathcal{F}}(0)$. Then $x \in L^-(\beta_{\mathcal{F}};s) \neq \emptyset$. By the assumption, we have $L^-(\beta_{\mathcal{F}};s)$ is an IUP-ideal of X. By (2.18), we have $0 \in$

$L^-(\beta_\mathcal{F}; s)$. So $\beta_\mathcal{F}(0) < s = \beta_\mathcal{F}(0)$, which is a contradiction. Thus, $\beta_\mathcal{F}(0) \leq \beta_\mathcal{F}(x)$. Let $x, y, z \in X$. Assume that $\beta_\mathcal{F}(x \cdot z) > \max\{\beta_\mathcal{F}(x \cdot (y \cdot z)), \beta_\mathcal{F}(y)\}$. Let $s = \beta_\mathcal{F}(x \cdot z)$. Then $x \cdot (y \cdot z), y \in L^-(\beta_\mathcal{F}; s) \neq \emptyset$. By the assumption, we have $L^-(\beta_\mathcal{F}; s)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in L^-(\beta_\mathcal{F}; s)$. So $\beta_\mathcal{F}(x \cdot z) < s = \beta_\mathcal{F}(x \cdot z)$, which is a contradiction. Thus, $\beta_\mathcal{F}(x \cdot z) \leq \max\{\beta_\mathcal{F}(x \cdot (y \cdot z)), \beta_\mathcal{F}(y)\}$.

Hence, $\mathcal{F}$ is an FFIUP-ideal of X.

Theorem 26. *An FFS $\mathcal{A}$ in X is an FFIUP-filter of X if and only if for all $t, s \in [0, 1]$, the sets $U^+(\alpha_\mathcal{F}; t)$ and $L^-(\beta_\mathcal{F}; s)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{F}$ is an FFIUP-filter of X. Let $t \in [0, 1]$ be such that $U^+(\alpha_\mathcal{F}; t) \neq \emptyset$. Let $a \in U^+(\alpha_\mathcal{F}; t)$. Then $\alpha_\mathcal{F}(a) > t$. By (3.4), we have $\alpha_\mathcal{F}(0) \geq \alpha_\mathcal{F}(a) > t$. Thus, $0 \in U^+(\alpha_\mathcal{F}; t)$. Let $x, y \in U^+(\alpha_\mathcal{F}; t)$ be such that $x \cdot y, x \in U^+(\alpha_\mathcal{F}; t)$. Then $\alpha_\mathcal{F}(x \cdot y) > t$ and $\alpha_\mathcal{F}(x) > t$. Thus, $\min\{\alpha_\mathcal{F}(x \cdot y), \alpha_\mathcal{F}(x)\} > t$. By (3.8). we have $\alpha_\mathcal{F}(y) \geq \min\{\alpha_\mathcal{F}(x \cdot y), \alpha_\mathcal{F}(x)\} > t$. Thus, $y \in U^+(\alpha_\mathcal{F}; t)$. Hence, $U^+(\alpha_\mathcal{F}; t)$ is an IUP-filter of X.

Let $s \in [0, 1]$ be such that $L^-(\beta_\mathcal{F}; s) \neq \emptyset$. Let $\beta \in L^-(\beta_\mathcal{F}; s)$. Then $\beta_\mathcal{F}(\beta) < s$. By (3.5), we have $\beta_\mathcal{F}(0) \leq \beta_\mathcal{F}(\beta) < s$. Thus, $0 \in L^-(\beta_\mathcal{F}; s)$. Let $x, y \in L^-(\beta_\mathcal{F}; s)$ be such that $x \cdot y, x \in L^-(\beta_\mathcal{F}; s)$. Then $\beta_\mathcal{F}(x \cdot y) < s$ and $\beta_\mathcal{F}(x) < s$. Thus, $\max\{\beta_\mathcal{F}(x \cdot y), \beta_\mathcal{F}(x)\} < s$. By (3.9). we have $\beta_\mathcal{F}(y) \leq \max\{\beta_\mathcal{F}(x \cdot y), \beta_\mathcal{F}(x)\} > s$. Thus, $y \in L^-(\beta_\mathcal{F}; s)$. Hence, $L^-(\beta_\mathcal{F}; s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0, 1]$, the sets $U^+(\alpha_\mathcal{F}; t)$ and $L^-(\beta_\mathcal{F}; s)$ are either empty or IUP-filters of X. Let $x \in X$. Assume that $\alpha_\mathcal{F}(0) < \alpha_\mathcal{F}(x)$. Let $t = \alpha_\mathcal{F}(0)$. Then $x \in U^+(\alpha_\mathcal{F}; t) \neq \emptyset$. By the assumption, we have $U^+(\alpha_\mathcal{F}; t)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\alpha_\mathcal{F}; t)$. So $\alpha_\mathcal{F}(0) > t = \alpha_\mathcal{F}(0)$, which is a contradiction. Thus, $\alpha_\mathcal{F}(0) \geq \alpha_\mathcal{F}(x)$. Let $x, y \in X$. Assume that $\alpha_\mathcal{F}(y) < \min\{\alpha_\mathcal{F}(x \cdot y), \alpha_\mathcal{F}(x)\}$. Let $t = \alpha_\mathcal{F}(y)$. Then $x \cdot y, x \in U^+(\alpha_\mathcal{F}; t) \neq \emptyset$. By the assumption, we have $U^+(\alpha_\mathcal{F}; t)$ is an IUP-filter of X. By (2.19), we have $y \in U^+(\alpha_\mathcal{F}; t)$. So $\alpha_\mathcal{F}(y) > t = \alpha_\mathcal{F}(y)$, which is a contradiction. Thus, $\alpha_\mathcal{F}(y) \geq \min\{\alpha_\mathcal{F}(x \cdot y), \alpha_\mathcal{F}(x)\}$.

Let $x \in X$. Assume that $\beta_\mathcal{F}(0) > \beta_\mathcal{F}(x)$. Let $s = \beta_\mathcal{F}(0)$. Then $x \in L^-(\beta_\mathcal{F}; s) \neq \emptyset$. By the assumption, we have $L^-(\beta_\mathcal{F}; s)$ is an IUP-filter of X. By (2.18), we have $0 \in L^-(\beta_\mathcal{F}; s)$. So $\beta_\mathcal{F}(0) < s = \beta_\mathcal{F}(0)$, which is a contradiction. Thus, $\beta_\mathcal{F}(0) \leq \beta_\mathcal{F}(x)$. Let $x, y \in X$. Assume that $\beta_\mathcal{F}(y) > \max\{\beta_\mathcal{F}(x \cdot y), \beta_\mathcal{F}(x)\}$. Let $s = \beta_\mathcal{F}(y)$. Then $x \cdot y, x \in L^-(\beta_\mathcal{F}; s) \neq \emptyset$. By the assumption, we have $L^-(\beta_\mathcal{F}; s)$ is an IUP-filter of X. By (2.19), we have $y \in L^-(\beta_\mathcal{F}; s)$. So $\beta_\mathcal{F}(y) < s = \beta_\mathcal{F}(y)$, which is a contradiction. Thus, $\beta_\mathcal{F}(y) \leq \max\{\beta_\mathcal{F}(x \cdot y), \beta_\mathcal{F}(x)\}$.

Hence, $\mathcal{F}$ is an FFIUP-filter of X.

The following theorem is a direct consequence of Theorem 2.

Theorem 27. *An FFS $\mathcal{F}$ in X is an FFSIUP-ideal of X if and only if for all $t, s \in [0, 1]$, the sets $U^+(\alpha_\mathcal{F}; t)$ and $L^-(\beta_\mathcal{F}; s)$ are either empty or strong IUP-ideals of X.*

4. Conclusions and future work

Our paper introduces pioneering concepts like FFIUP-subalgebras, FFIUP-ideals, FFIUP-filters, and FFSIUP-ideals. We explore their crucial properties, examining their relationships to complements, characteristic functions, and level subsets. These insights reveal complex structures within FFSs and IUP-algebras, offering new perspectives and practical applications in mathematical theory.

The study of FFSs represents a dynamic evolution in FS theory, extending beyond academic curiosity. This expanding research inspires scientists and scholars to explore the synergy between FFSs and IUP-algebras. This convergence is set to drive innovations, shaping the future of fuzzy mathematics and its diverse applications.

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