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On a New Operator Based on a Primal and its Associated Topology

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Abstract. This paper aims to introduce and study two new operators $(.)^{\diamond}_{\omega}$ and $cl^{\diamond}_{\omega}(\cdot)$ by utilizing the notion of primal defined by Acharjee et al. Also, we investigate some fundamental properties of them. In addition, we showed that the operator $cl^{\diamond}_{\omega}(.)$ satisfied the Kuratowski closure axioms. Therefore, we obtain a new topology denoted by $\tau^{\diamond}_{\omega}$, which is finer than the original one. Moreover, the topology $\tau^{\diamond}_{\omega}$ obtained via the operator $cl^{\diamond}_{\omega}(\cdot)$ is finer than τ_{ω}, where τ_{ω} is the family of all ω-open subsets of a primal topological space $(X, \tau, \mathcal{P})$. Furthermore, we not only examine the fundamental properties of this class of sets but also provide some counterexamples.

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Key Words and Phrases: Primal, primal topological space, ω-open, the operator $(\cdot)^{\diamond}_{\omega}$, the operator $cl^{\diamond}_{\omega}$, the topology $\tau^{\diamond}_{\omega}$

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1. Introduction

One of the most popular ways of building topology is to add another structures such as filter [16], ideal [16], grill [12], and primal [1]. The concepts of filters, ideals, and grills are the structures studied for many years and among the most important concepts of topology. In 2014, Kuratowski introduced the concept of ideal [16] from filter [16]. The notion of ideal comes across as the dual structure of filter. The notion of grill [12] was defined and studied by Choquet in 1947; for more details, see [17, 18]. However, the dual of the notion of grill has not been introduced by any authors until 2022. In 2022, the concept of primal [1] was defined and studied by Acharjee et al. They introduced primal topological spaces via the notion of primal. The notion of primal is the dual of the notion of grill. This topic has won its importance aspects of interest. This concept has been analysed by many authors in a short period of time; for more details, see [2–9, 11, 20].

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Undoubtedly, another important concept in general topology is the types of open sets. Some of these types of sets are regular open sets [22], δ-open sets [23] and ω-open sets [15]. The notion of ω-open set comes across a weaker concept than the concept of open set while the notion of δ-open set is a stronger concept than the concept of open set. These types of sets have been studied by many authors in different directions.

Some other types of sets such as fuzzy sets, soft sets, and rough sets play an important role in pure and applied sciences. The notion of fuzzy sets was introduced by Zadeh [24] and studied in many directions in the recent past. After then, the notion of soft sets was defined by Molodtsov [19] and also investigated by many authors in many directions; for more details, see [13, 14].

Recently, Pawlak introduced and studied the concept of rough set in [21]. The concepts of fuzzy sets, soft sets, and rough sets have many applications in the literature. These kind of sets are very important in terms of having applications. Fuzzy sets, soft sets and especially the concept of rough sets are still intensively studied in the literature. Also, these kind of sets has been considered with different structures such as filter, ideal, grill, and primal as well.

In this study, we will define the operator $cl_\omega^\diamond$ with the help of the definition of primal topological space given by Acharjee et al. [1] in 2022 and the operator cl_ω or ω-cl [15] given by Hdeib in 1982. Accordingly, we will introduce the topology $\tau_\omega^\diamond$ and examine some important set theoretical properties. We also examined the relationship between the definitions given before and gave examples to the contrary. In addition, we showed that the operator $cl_\omega^\diamond$ is a Kuratowski closure operator. We also showed that the topology $\tau_\omega^\diamond$, given with the help of the operator $cl_\omega^\diamond$, is finer than both τ and τ_ω. Some examples related to the notions were given.

2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted $C(X)$ (resp. $O(X)$ or τ) and the family of all closed (resp. open) sets of X containing a point x of X is denoted by $C(X, x)$ (resp. $O(X, x)$ or $\tau(x)$).

Now, we recall some of the definitions in the literature and used in this study.

Definition 1. *Let A be a subset of a space X. A is said to be ω-open [10] if for every $x \in A$, there exists an open set U containing x such that $U \setminus A$ is countable. The complement of an ω-open set is called an ω-closed. The family of all ω-open (resp. ω-closed) sets of X will be denoted by $\omega O(X)$ or τ_ω (resp. $\omega C(X)$). The family of all ω-open (resp. ω-closed) sets of X containing a point x of X will be denoted by $\omega O(X, x)$ (resp. $\omega C(X, x)$). The intersection of all ω-closed sets containing A is called the ω-closure of A and is denoted by $cl_\omega(A)$ or ω-$cl(A)$.*

Definition 2. *Let X be a non-empty set. A collection $\mathcal{P} \subseteq 2^X$ is called a primal on X [1] if it satisfies the following conditions:*

(a) $X \notin \mathcal{P}$,

(b) *if* $A \in \mathcal{P}$ *and* $B \subseteq A$, *then* $B \in \mathcal{P}$,

(c) *if* $A \cap B \in \mathcal{P}$, *then* $A \in \mathcal{P}$ *or* $B \in \mathcal{P}$.

Definition 3. *[1] A topological space* (X, τ) *with a primal* $\mathcal{P}$ *on* X *is called a primal topological space and denoted by* $(X, \tau, \mathcal{P})$.

Definition 4. *[1] Let* $(X, \tau, \mathcal{P})$ *be a primal topological space. We consider a map* $(\cdot)^\diamond : 2^X \to 2^X$ *as* $A^\diamond(X, \tau, \mathcal{P}) = \{x \in X : (\forall U \in O(X, x))(A^c \cup U^c \in \mathcal{P})\}$ *for any subset* A *of* X. *We can also write* $A^\diamond$ *as* $A^\diamond(X, \tau, \mathcal{P})$ *to specify the primal as per our requirements.*

Definition 5. *[1] Let* $(X, \tau, \mathcal{P})$ *be a primal topological space. We consider a map* $cl^\diamond : 2^X \to 2^X$ *as* $cl^\diamond(A) = A \cup A^\diamond$, *where* A *is any subset of* X.

Definition 6. *[1] Let* $(X, \tau, \mathcal{P})$ *be a primal topological space. Then, the family* $\tau^\diamond = \{A \subseteq X | cl^\diamond(A^c) = A^c\}$ *is a topology on* X *induced by topology* τ *and primal* $\mathcal{P}$.

3. The operator $(.)^\diamond_\omega$ and its basic properties

Definition 7. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space. We consider a map* $(\cdot)^\diamond_\omega : 2^X \to 2^X$ *as* $A^\diamond_\omega(X, \tau, \mathcal{P}) = \{x \in X : (\forall U \in \omega O(X, x))(A^c \cup U^c \in \mathcal{P})\}$ *for any subset* A *of* X. *We can also write* $A^\diamond_\omega$ *as* $A^\diamond_\omega(X, \tau, \mathcal{P})$ *to specify the primal and the topology if necessary.*

Corollary 1. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space and* $A \subseteq X$. *Then,* $A^\diamond_\omega \subseteq A^\diamond$.

Remark 1. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space and* $A \subseteq X$. *There is no relationship between* $A^\diamond_\omega$ *and* A *as shown by the following examples.*

Example 1. *Let* $X = \{1, 2, 3\}$ *with the topology* $\tau = \{\emptyset, X\}$. *We consider the primal* $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ *on* X. *Now, if* $A = \{1\}$, *then* $A = \{1\} \nsubseteq \emptyset = A^\diamond_\omega$.

Example 2. *Let* $(\mathbb{R}, \tau)$ *be indiscrete topological space. Consider the primal* $\mathcal{P} = 2^{\mathbb{R}} \setminus \{\mathbb{R}\}$. *For the subset* $A = [0, \infty)$, *we have* $-1 \in A^\diamond_\omega$ *but* $-1 \notin A$. *Therefore,* $A^\diamond_\omega \nsubseteq A$.

Theorem 1. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space and* $A \subseteq X$. *If* A *is* ω*-closed, then* $A^\diamond_\omega \subseteq A$.

Proof. Let $A \in \omega C(X)$ and $x \in A^\diamond_\omega$. Suppose that $x \notin A$.

$$\left.\begin{array}{r} x \in A^\diamond_\omega \Rightarrow (\forall U \in \omega O(X, x))(A^c \cup U^c \in \mathcal{P}) \\ x \notin A \in \omega C(X) \Rightarrow A^c \in \omega O(X, x) \end{array}\right\} \Rightarrow A^c \cup (A^c)^c = A^c \cup A = X \in \mathcal{P}$$

This is a contradiction because any primal does not involve the set X.

Theorem 2. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the following statements hold for any two subsets A and B of X.*

(*a*) $\emptyset^\diamond_\omega = \emptyset$,

(*b*) $A^\diamond_\omega \in \omega C(X)$,

(*c*) $(A^\diamond_\omega)^\diamond_\omega \subseteq A^\diamond_\omega$,

(*d*) *If* $A \subseteq B$, *then* $A^\diamond_\omega \subseteq B^\diamond_\omega$,

(*e*) $A^\diamond_\omega \cup B^\diamond_\omega = (A\cup B)^\diamond_\omega$,

(*f*) $(A\cap B)^\diamond_\omega \subseteq A^\diamond_\omega \cap B^\diamond_\omega$.

Proof. (a) Suppose that $\emptyset^\diamond_\omega \neq \emptyset$. Then, there exists $x\in X$ such that $x \in \emptyset^\diamond_\omega$. Thus, we have $U^c \cup \emptyset^c = U^c \cup X = X \in \mathcal{P}$ for every $U \in \omega O(X,x)$ which is a contradiction.

(b) We have always $A^\diamond_\omega \subseteq \omega\text{-}cl(A^\diamond_\omega)\ldots(1)$

Conversely, now let $x \in \omega\text{-}cl(A^\diamond_\omega)$.

$x \in \omega\text{-}cl(A^\diamond_\omega) \Rightarrow (\forall U \in \omega O(X,x))(U\cap A^\diamond_\omega \neq \emptyset)$
$\Rightarrow (\forall U \in \omega O(X,x))(\exists y \in X)(y\in U)(y \in A^\diamond_\omega)$

$$\left.\begin{array}{r}\Rightarrow (\forall U \in \omega O(X,x))(\exists y \in X)(y\in U)(\forall V \in \omega O(X,y))(V^c\cup A^c \in \mathcal{P})\\ V:=U\end{array}\right\}\Rightarrow$$

$\Rightarrow (\forall U \in \omega O(X,x))(U^c \cup A^c \in \mathcal{P})$
$\Rightarrow x \in A^\diamond_\omega$.

Then, we have $\omega\text{-}cl(A^\diamond_\omega) \subseteq A^\diamond_\omega\ldots(2)$

$(1),(2) \Rightarrow A^\diamond_\omega = \omega\text{-}cl(A^\diamond_\omega) \Rightarrow A^\diamond_\omega \in \omega C(X)$.

(c) Let $A \subseteq X$.

$$A \subseteq X \overset{(b)}{\Rightarrow} A^\diamond_\omega \in \omega C(X) \overset{\text{Theorem 1}}{\Rightarrow} (A^\diamond_\omega)^\diamond_\omega \subseteq A^\diamond_\omega.$$

(d) Let $A\subseteq B$ and $x \in A^\diamond_\omega$. We will prove that $x\in B^\diamond_\omega$.

$$\left.\begin{array}{r} x\in A^\diamond_\omega \Rightarrow (\forall U \in \omega O(X,x))(U^c\cup A^c\in\mathcal{P})\\ A\subseteq B\end{array}\right\}\Rightarrow (\forall U \in \omega O(X,x))(U^c \cup B^c \in \mathcal{P})$$

$\Rightarrow x \in B^\diamond_\omega$.

(e) Let $A,B\subseteq X$.

$$\left.\begin{array}{r} A\subseteq X \Rightarrow A\subseteq A\cup B \overset{(d)}{\Rightarrow} A^\diamond_\omega \subseteq (A\cup B)^\diamond_\omega\\ B\subseteq X \Rightarrow B\subseteq A\cup B \overset{(d)}{\Rightarrow} B^\diamond_\omega \subseteq (A\cup B)^\diamond_\omega\end{array}\right\}\Rightarrow A^\diamond_\omega \cup B^\diamond_\omega \subseteq (A\cup B)^\diamond_\omega\ldots(1)$$

Conversely, let $x\notin A^\diamond_\omega \cup B^\diamond_\omega$.

$$\left.\begin{array}{r} x\notin A^\diamond_\omega\cup B^\diamond_\omega \Rightarrow (x\notin A^\diamond_\omega)(x\notin B^\diamond_\omega) \Rightarrow (\exists U,V\in\omega O(X,x))(U^c\cup A^c\notin\mathcal{P})(V^c\cup B^c\notin\mathcal{P})\\ W:=U\cap V\end{array}\right\}\Rightarrow$$

$\Rightarrow (W\in\omega O(X,x))(W^c\cup A^c\notin\mathcal{P})(W^c\cup B^c\notin\mathcal{P})$

$\Rightarrow (W \in \omega O(X,x))(W^c \cup (A \cup B)^c = (W^c \cup A^c) \cap (W^c \cup B^c) \notin \mathcal{P})$
$\Rightarrow x \notin (A \cup B)^{\diamond}_{\omega}$

Then, we have $(A \cup B)^{\diamond}_{\omega} \subseteq A^{\diamond}_{\omega} \cup B^{\diamond}_{\omega} \ldots (2)$

$(1),(2) \Rightarrow (A \cup B)^{\diamond}_{\omega} = A^{\diamond}_{\omega} \cup B^{\diamond}_{\omega}$.

(f) It is clear from (d).

Theorem 3. *Let $(X,\tau,\mathcal{P})$ and $(X,\tau,\mathcal{Q})$ be two primal topological spaces and $A \subseteq X$. If $\mathcal{P} \subseteq \mathcal{Q}$, then $A^{\diamond}_{\omega}(\mathcal{P}) \subseteq A^{\diamond}_{\omega}(\mathcal{Q})$.*

Proof. Let $x \in A^{\diamond}_{\omega}(\mathcal{P})$ and $\mathcal{P} \subseteq \mathcal{Q}$.

$$\left.\begin{array}{r} x \in A^{\diamond}_{\omega}(\mathcal{P}) \Rightarrow (\forall U \in \omega O(X,x))(U^c \cup A^c \in \mathcal{P}) \\ \mathcal{P} \subseteq \mathcal{Q} \end{array}\right\} \Rightarrow (\forall U \in \omega O(X,x))(U^c \cup A^c \in \mathcal{Q})$$

$\Rightarrow x \in A^{\diamond}_{\omega}(\mathcal{Q})$.

Theorem 4. *Let $(X,\tau,\mathcal{P})$ and $(X,\sigma,\mathcal{P})$ be two primal topological spaces and $A \subseteq X$. If $\tau \subseteq \sigma$, then $A^{\diamond}_{\omega}(X,\sigma,\mathcal{P}) \subseteq A^{\diamond}_{\omega}(X,\tau,\mathcal{P})$.*

Proof. Let $x \in A^{\diamond}_{\omega}(X,\sigma,\mathcal{P})$ and $\tau \subseteq \sigma$.

$$\left.\begin{array}{r} x \in A^{\diamond}_{\omega}(X,\sigma,\mathcal{P}) \Rightarrow (\forall U \in \omega O_{\sigma}(X,x))(U^c \cup A^c \in \mathcal{P}) \\ \tau \subseteq \sigma \end{array}\right\} \Rightarrow$$

$\Rightarrow (\forall U \in \omega O_{\tau}(X,x))(U^c \cup A^c \in \mathcal{P})$
$\Rightarrow x \in A^{\diamond}_{\omega}(X,\tau,\mathcal{P})$.

Theorem 5. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the following statements hold for any two subsets A and B of X.*

(a) $A^{\diamond}_{\omega} \subseteq cl(A)$,

(b) $cl(A^{\diamond}_{\omega}) \subseteq cl(A)$,

(c) $A^{\diamond}_{\omega} \setminus B^{\diamond}_{\omega} \subseteq (A \setminus B)^{\diamond}_{\omega}$,

(d) $A^{\diamond}_{\omega} \setminus B^{\diamond}_{\omega} = (A \setminus B)^{\diamond}_{\omega} \setminus B^{\diamond}_{\omega}$.

Proof. (a) Let $x \notin cl(A)$. Our aim is to show that $x \notin A^{\diamond}_{\omega}$.

$$\left.\begin{array}{r} x \notin cl(A) \Rightarrow (\exists U \in O(X,x))(U \cap A = \emptyset) \\ O(X,x) \subseteq \omega O(X,x) \end{array}\right\} \Rightarrow (\exists U \in \omega O(X,x))(A \subseteq U^c)$$

$\Rightarrow (\exists U \in \omega O(X,x))(X = A \cup A^c \subseteq U^c \cup A^c \notin \mathcal{P})$
$\Rightarrow x \notin A^{\diamond}_{\omega}$.

(b) Let $A \subseteq X$.

$$A \subseteq X \overset{(a)}{\Rightarrow} A^{\diamond}_{\omega} \subseteq cl(A) \Rightarrow cl(A^{\diamond}_{\omega}) \subseteq cl(cl(A)) = cl(A).$$

(c) Let $A, B \subseteq X$.

$$\begin{array}{rcl} A,B\subseteq X & \Rightarrow & A\subseteq (A\setminus B)\cup B \\ & \Rightarrow & A^\diamond_\omega \subseteq [(A\setminus B)\cup B]^\diamond_\omega = (A\setminus B)^\diamond_\omega \cup B^\diamond_\omega \\ & \Rightarrow & A^\diamond_\omega \setminus B^\diamond_\omega \subseteq (A\setminus B)^\diamond_\omega. \end{array}$$

(d) Let $A, B \subseteq X$.

$$\left.\begin{array}{r} A,B\subseteq X \Rightarrow A\setminus B\subseteq A \Rightarrow (A\setminus B)^\diamond_\omega \subseteq A^\diamond_\omega \Rightarrow (A\setminus B)^\diamond_\omega \setminus B^\diamond_\omega \subseteq A^\diamond_\omega \setminus B^\diamond_\omega \\ A,B\subseteq X \overset{(c)}{\Rightarrow} A^\diamond_\omega \setminus B^\diamond_\omega \subseteq (A\setminus B)^\diamond_\omega \Rightarrow (A^\diamond_\omega \setminus B^\diamond_\omega)\setminus B^\diamond_\omega = A^\diamond_\omega \setminus B^\diamond_\omega \subseteq (A\setminus B)^\diamond_\omega \setminus B^\diamond_\omega \end{array}\right\} \Rightarrow$$

$$\Rightarrow (A\setminus B)^\diamond_\omega \setminus B^\diamond_\omega = A^\diamond_\omega \setminus B^\diamond_\omega.$$

Theorem 6. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If A is ω-open in X, then $A \cap B^\diamond_\omega \subseteq (A \cap B)^\diamond_\omega$.*

Proof. Let $x \in A \cap B^\diamond_\omega$.

$$\left.\begin{array}{r} x\in A\cap B^\diamond_\omega \Rightarrow (x\in A)(x\in B^\diamond_\omega) \Rightarrow (x\in A)(\forall U\in \omega O(X,x))(U^c\cup B^c\in\mathcal{P}) \\ A\in\omega O(X) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\forall U\in\omega O(X,x))(U\cap A\in\omega O(X,x))((U\cap A)^c\cup B^c = U^c\cup (A\cap B)^c\in\mathcal{P})$$

$$\Rightarrow x\in (A\cap B)^\diamond_\omega.$$

Corollary 2. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If A is open in X, then $A \cap B^\diamond_\omega \subseteq (A \cap B)^\diamond_\omega$.*

Theorem 7. *Let $(X, \tau, \mathcal{P})$ be a primal topological space.*

(a) *If $\omega C(X) \setminus \{X\} \subseteq \mathcal{P}$, then $X^\diamond_\omega = X$;*

(b) *If $\omega C(X) \setminus \{X\} \subseteq \mathcal{P}$, then $A \subseteq A^\diamond_\omega$ for all $A \in \omega O(X)$.*

Proof. (a) Let $x \in X$ and $U \in \omega O(X, x)$.

$$\left.\begin{array}{r} U\in\omega O(X,x) \Rightarrow U^c\in\omega C(X)\setminus\{X\} \\ \omega C(X)\setminus\{X\}\subseteq\mathcal{P} \end{array}\right\} \Rightarrow U^c\cup X^c = U^c\cup\emptyset = U^c\in\mathcal{P}$$

Then, we have $x \in X^\diamond_\omega$. Thus, $X \subseteq X^\diamond_\omega$ which means $X^\diamond_\omega = X$.

(b) Let $A \in \omega O(X)$.

$$\left.\begin{array}{r} A\in\omega O(X) \overset{\text{Theorem 6}}{\Rightarrow} A\cap X^\diamond_\omega \subseteq (A\cap X)^\diamond_\omega = A^\diamond_\omega \\ \omega C(X)\setminus\{X\}\subseteq\mathcal{P} \overset{(a)}{\Rightarrow} X^\diamond_\omega = X \end{array}\right\} \Rightarrow A\subseteq A^\diamond_\omega.$$

Theorem 8. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If $B \in \mathcal{P}$, then $(A \cup B)^\diamond_\omega = A^\diamond_\omega = (A \setminus B)^\diamond_\omega$.*

Proof. Let $A, B \subseteq X$.

$$\left.\begin{array}{r} A,B\subseteq X \overset{\text{Theorem 5}}{\Rightarrow} A^\diamond_\omega\setminus B^\diamond_\omega = (A\setminus B)^\diamond_\omega\setminus B^\diamond_\omega \\ B\in\mathcal{P}\Rightarrow B^\diamond_\omega=\emptyset \end{array}\right\} \Rightarrow A^\diamond_\omega = (A\setminus B)^\diamond_\omega \ldots (1)$$

$$\left.\begin{array}{r} A,B\subseteq X \overset{\text{Theorem 2}}{\Rightarrow} A^\diamond_\omega\cup B^\diamond_\omega = (A\cup B)^\diamond_\omega \\ B\in\mathcal{P}\Rightarrow B^\diamond_\omega=\emptyset \end{array}\right\} \Rightarrow A^\diamond_\omega = (A\cup B)^\diamond_\omega \ldots (2)$$

$(1),(2) \Rightarrow (A\cup B)^{\diamond}_{\omega} = A^{\diamond}_{\omega} = (A\setminus B)^{\diamond}_{\omega}$.

Theorem 9. *Let $\mathcal{P}$ be a primal on topological space (X,τ) and $A\subseteq X$. If $A^c\notin\mathcal{P}$, then $A^{\diamond}_{\omega}=\emptyset$.*

Proof. Suppose that $A^{\diamond}_{\omega}\neq\emptyset$.

$$\left.\begin{array}{r} A^{\diamond}_{\omega}\neq\emptyset \Rightarrow (\exists x\in X)(x\in A^{\diamond}_{\omega}) \Rightarrow (\forall U\in\omega O(X,x))(A^c\subseteq U^c\cup A^c\in\mathcal{P}) \\ \mathcal{P} \text{ is a primal on } X \end{array}\right\} \Rightarrow A^c\in\mathcal{P}$$

This contradicts with the hypothesis.

4. The operator $cl^{\diamond}_{\omega}$ and its associated topology

Definition 8. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. We consider a map $cl^{\diamond}_{\omega}: 2^X\to 2^X$ as $cl^{\diamond}_{\omega}(A)=A\cup A^{\diamond}_{\omega}$, where A is any subset of X.*

Theorem 10. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and $A,B\subseteq X$. Then, the following statements hold:*

(a) $cl^{\diamond}_{\omega}(\emptyset)=\emptyset$,

(b) $cl^{\diamond}_{\omega}(X)=X$,

(c) $A\subseteq cl^{\diamond}_{\omega}(A)\subseteq cl^{\diamond}(A)$,

(d) *If* $A\subseteq B\subseteq X$, *then* $cl^{\diamond}_{\omega}(A)\subseteq cl^{\diamond}_{\omega}(B)$,

(e) $cl^{\diamond}_{\omega}(A)\cup cl^{\diamond}_{\omega}(B)=cl^{\diamond}_{\omega}(A\cup B)$,

(f) $cl^{\diamond}_{\omega}(cl^{\diamond}_{\omega}(A))=cl^{\diamond}_{\omega}(A)$.

Proof.
(a) Since $\emptyset^{\diamond}_{\omega}=\emptyset$, we have $cl^{\diamond}_{\omega}(\emptyset)=\emptyset\cup\emptyset^{\diamond}_{\omega}=\emptyset$.

(b) Since $X^{\diamond}_{\omega}\subseteq X$, we have $cl^{\diamond}_{\omega}(X)=X\cup X^{\diamond}_{\omega}=X$.

(c) Let $A\subseteq X$.

$$A\subseteq X \Rightarrow A^{\diamond}_{\omega}\subseteq A^{\diamond} \Rightarrow A\subseteq A\cup A^{\diamond}_{\omega}=cl^{\diamond}_{\omega}(A)\subseteq A\cup A^{\diamond}=cl^{\diamond}(A).$$

(d) Let $A\subseteq B\subseteq X$.

$$A\subseteq B \Rightarrow A^{\diamond}_{\omega}\subseteq B^{\diamond}_{\omega} \Rightarrow cl^{\diamond}_{\omega}(A)=A\cup A^{\diamond}_{\omega}\subseteq B\cup B^{\diamond}_{\omega}=cl^{\diamond}_{\omega}(B).$$

(e) Let $A,B\subseteq X$.

$$\begin{array}{rcl} cl^{\diamond}_{\omega}(A\cup B) & = & (A\cup B)\cup(A\cup B)^{\diamond}_{\omega} \\ & = & (A\cup B)\cup(A^{\diamond}_{\omega}\cup B^{\diamond}_{\omega}) \\ & = & (A\cup A^{\diamond}_{\omega})\cup(B\cup B^{\diamond}_{\omega}) \\ & = & cl^{\diamond}_{\omega}(A)\cup cl^{\diamond}_{\omega}(B). \end{array}$$

(f) Let $A \subseteq X$. It is obvious from (c) and (d) that $cl_\omega^\diamond(A) \subseteq cl_\omega^\diamond(cl_\omega^\diamond(A)) \ldots (1)$

$$\left.\begin{array}{r} cl_\omega^\diamond(cl_\omega^\diamond(A)) = cl_\omega^\diamond(A) \cup (cl_\omega^\diamond(A))_\omega^\diamond = cl_\omega^\diamond(A) \cup (A \cup A_\omega^\diamond)_\omega^\diamond = cl_\omega^\diamond(A) \cup A_\omega^\diamond \cup (A_\omega^\diamond)_\omega^\diamond \\ A \subseteq X \Rightarrow A_\omega^\diamond \in \omega C(X) \Rightarrow (A_\omega^\diamond)_\omega^\diamond \subseteq A_\omega^\diamond \end{array}\right\} \Rightarrow$$

$\Rightarrow cl_\omega^\diamond(cl_\omega^\diamond(A)) \subseteq cl_\omega^\diamond(A) \ldots (2)$

$(1),(2) \Rightarrow cl_\omega^\diamond(A) = cl_\omega^\diamond(cl_\omega^\diamond(A))$.

Corollary 3. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the operator $cl_\omega^\diamond : 2^X \to 2^X$ defined by $cl_\omega^\diamond(A) = A \cup A_\omega^\diamond$, where A is any subset of X, is a Kuratowski closure operator.*

Definition 9. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\tau_\omega^\diamond = \{A \subseteq X | cl_\omega^\diamond(A^c) = A^c\}$ is a topology on X induced by topology τ and primal $\mathcal{P}$.*

Theorem 11. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, we have $\tau \subseteq \tau^\diamond \subseteq \tau_\omega^\diamond$.*

Proof. We have $\tau \subseteq \tau^\diamond$ from Theorem 3.6 in [1]. Now, let $A \in \tau^\diamond$. We will prove that $A \in \tau_\omega^\diamond$.

$$\left.\begin{array}{r} A \in \tau^\diamond \Rightarrow cl^\diamond(A^c) = A^c \\ A \subseteq X \Rightarrow (A^c)_\omega^\diamond \subseteq (A^c)^\diamond \Rightarrow cl_\omega^\diamond(A^c) \subseteq cl^\diamond(A^c) \end{array}\right\} \Rightarrow \left.\begin{array}{r} cl_\omega^\diamond(A^c) \subseteq A^c \\ A^c \subseteq cl_\omega^\diamond(A^c) \end{array}\right\} \Rightarrow A^c = cl_\omega^\diamond(A^c)$$

$\Rightarrow A \in \tau_\omega^\diamond$.

Theorem 12. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, we have $\tau \subseteq \tau_\omega \subseteq \tau_\omega^\diamond$.*

Proof. We have $\tau \subseteq \tau_\omega$ from [10]. Now, let $A \in \tau_\omega$. We will prove that $A \in \tau_\omega^\diamond$.

$$\left.\begin{array}{r} A \in \tau_\omega \overset{\text{Theorem 1}}{\Rightarrow} (A^c)_\omega^\diamond \subseteq A^c \Rightarrow cl_\omega^\diamond(A^c) = A^c \cup (A^c)_\omega^\diamond \subseteq A^c \cup A^c = A^c \\ A \subseteq X \Rightarrow A^c \subseteq cl_\omega^\diamond(A^c) \end{array}\right\} \Rightarrow$$

$\Rightarrow A^c = cl_\omega^\diamond(A^c)$

$\Rightarrow A \in \tau_\omega^\diamond$.

Corollary 4. *We have the following diagram from Definitions 1, 6, 9.*

$$\begin{array}{ccccc} \tau_R^\diamond\text{-}open & \rightarrow & \tau^\diamond\text{-}open & \rightarrow & \tau_\omega^\diamond\text{-}open \\ \uparrow & & \uparrow & & \uparrow \\ \tau_\delta\text{-}open & \rightarrow & \tau\text{-}open & \rightarrow & \tau_\omega\text{-}open \end{array}$$

Remark 2. *The converses of the implications given in the above diagram need not to be true as shown by the following examples.*

Example 3. *Consider the topology $\tau = \{U | 0 \notin U\} \cup \{\mathbb{R}\}$ with the primal $\mathcal{P} = 2^{\mathbb{R} \setminus \{0\}}$ on $\mathbb{R}$. Then, $[0, \infty) \in \tau_\omega^\diamond$ but $[0, \infty) \notin \tau_\omega$.*

Example 4. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}\}$. We consider the primal $\mathcal{P} = 2^X \setminus \{X, \{a, b\}\}$ on X. Then, $\{a, c\} \in \tau_\omega^\diamond = \tau_\omega = 2^X$ but $\{a, c\} \notin \tau^\diamond = \tau$.*

Theorem 13. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold:*

(a) *if $\mathcal{P} = \emptyset$, then $\tau_\omega^\diamond = 2^X$,*

(b) *if $\mathcal{P} = 2^X \setminus \{X\}$, then $\tau_\omega = \tau_\omega^\diamond$.*

Proof. (a) We have always $\tau_\omega^\diamond \subseteq 2^X \dots (1)$. Now, let $A \in 2^X$.

$$\left.\begin{array}{r} A \in 2^X \Rightarrow cl_\omega^\diamond(A^c) = (A^c)_\omega^\diamond \cup A^c \\ \mathcal{P} = \emptyset \Rightarrow (A^c)_\omega^\diamond = \emptyset \end{array}\right\} \Rightarrow cl_\omega^\diamond(A^c) = A^c \Rightarrow A \in \tau_\omega^\diamond$$

Then, we have $2^X \subseteq \tau_\omega^\diamond \dots (2)$

$(1), (2) \Rightarrow \tau_\omega^\diamond = 2^X$.

(b) We have $\tau_\omega \subseteq \tau_\omega^\diamond \dots (1)$. Now, let $A \in \tau_\omega^\diamond$. We will prove that $A \in \tau_\omega$.

$A \in \tau_\omega^\diamond \Rightarrow cl_\omega^\diamond(A^c) = A^c \Rightarrow A^c \cup (A^c)_\omega^\diamond = A^c \Rightarrow (A^c)_\omega^\diamond \subseteq A^c \dots (2)$

Now, let $x \notin (A^c)_\omega^\diamond$.

$$\left.\begin{array}{r} x \notin (A^c)_\omega^\diamond \Rightarrow (\exists U \in \omega O(X, x))(U^c \cup A \notin \mathcal{P}) \\ \mathcal{P} = 2^X \setminus \{X\} \end{array}\right\} \Rightarrow (\exists U \in \omega O(X, x))(U^c \cup A = X)$$

$\Rightarrow (\exists U \in \omega O(X, x))(U \cap A^c = \emptyset)$

$\Rightarrow x \notin \omega\text{-}cl(A^c)$

Then, we get $\omega\text{-}cl(A^c) \subseteq (A^c)_\omega^\diamond \dots (3)$. Thus, we have $\omega\text{-}cl(A^c) \subseteq A^c$ from (2) and (3). Therefore, $\omega\text{-}cl(A^c) = A^c$. Hence, A is ω-open.

Remark 3. *The converse of Theorem 13(b) need not to be true as shown by the following example.*

Example 5. *Let $X = \{a, b, c\}$ with the discrete topology τ and $\mathcal{P} = 2^X \setminus \{X, \{b, c\}\}$. Then, $\tau_\omega = \tau_\omega^\diamond$ but $\mathcal{P} \neq 2^X \setminus \{X\}$.*

Theorem 14. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. Then, $A \in \tau_\omega^\diamond$ if and only if for all x in A, there exists an ω-open set U containing x such that $U^c \cup A \notin \mathcal{P}$.*

Proof. Let $A \in \tau_\omega^\diamond$.

$$\begin{array}{rcl} A \in \tau_\omega^\diamond & \Leftrightarrow & cl_\omega^\diamond(A^c) = A^c \\ & \Leftrightarrow & A^c \cup (A^c)_\omega^\diamond = A^c \\ & \Leftrightarrow & (A^c)_\omega^\diamond \subseteq A^c \\ & \Leftrightarrow & A \subseteq ((A^c)_\omega^\diamond)^c \\ & \Leftrightarrow & (\forall x \in A)(x \notin (A^c)_\omega^\diamond) \\ & \Leftrightarrow & (\forall x \in A)(\exists U \in \omega O(X, x))(U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{P}). \end{array}$$

Theorem 15. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A \subseteq X$. If $A \notin \mathcal{P}$, then $A \in \tau_\omega^\diamond$.*

Proof. Let $A \notin \mathcal{P}$ and $x \in A$.

$$\left.\begin{array}{r}(U := X)(x \in A) \Rightarrow (U \in \omega O(X,x))(A = U^c \cup A) \\ A \notin \mathcal{P}\end{array}\right\} \Rightarrow U^c \cup A \notin \mathcal{P}$$

Therefore, we get $A \in \tau_\omega^\diamond$ from Theorem 14.

Theorem 16. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\mathcal{B} = \{T \cap P \mid T \in \tau_\omega$ and $P \notin \mathcal{P}\}$ is a base for the topology $\tau_\omega^\diamond$ on X.*

Proof. Let $B \in \mathcal{B}$.

$$\left.\begin{array}{r}B \in \mathcal{B} \Rightarrow (\exists T \in \tau_\omega)(\exists P \notin \mathcal{P})(B = T \cap P) \\ \tau_\omega \subseteq \tau_\omega^\diamond\end{array}\right\} \overset{\text{Theorem 15}}{\Rightarrow} (T, P \in \tau_\omega^\diamond)(B = T \cap P)$$

$\Rightarrow B \in \tau_\omega^\diamond$

Then, we have $\mathcal{B} \subseteq \tau_\omega^\diamond \ldots (1)$

Now, let $A \in \tau_\omega^\diamond$ and $x \in A$.

$$\left.\begin{array}{r}x \in A \in \tau_\omega^\diamond \Rightarrow (\exists U \in \omega O(X,x))(U^c \cup A \notin \mathcal{P}) \\ B := U \cap (U^c \cup A)\end{array}\right\} \Rightarrow (B \in \mathcal{B})(x \in B \subseteq A) \ldots (2)$$

Therefore, $\mathcal{B}$ is a base for the topology $\tau_\omega^\diamond$ on X due to (1) and (2).

Theorem 17. *Let $(X, \tau, \mathcal{P})$ and $(X, \tau, \mathcal{Q})$ be two primal topological spaces. If $\mathcal{P} \subseteq \mathcal{Q}$, then $\tau_{\omega(\mathcal{Q})}^\diamond \subseteq \tau_{\omega(\mathcal{P})}^\diamond$.*

Proof. Let $A \in \tau_{\omega(\mathcal{Q})}^\diamond$.

$$\left.\begin{array}{r}A \in \tau_{\omega(\mathcal{Q})}^\diamond \Rightarrow (\forall x \in A)(\exists U \in \omega O(X,x))(U^c \cup A \notin \mathcal{Q}) \\ \mathcal{P} \subseteq \mathcal{Q}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\forall x \in A)(\exists U \in \omega O(X,x))(U^c \cup A \notin \mathcal{P})$

$\Rightarrow A \in \tau_{\omega(\mathcal{P})}^\diamond$.

5. Conclusion

In this article, we introduced and studied two new operators, denoted by $(\cdot)_\omega^\diamond$ and $cl_\omega^\diamond(\cdot)$, via the notions of primal and ω-open set. Also, we revealed their fundamental properties. Although the first one is not a Kuratowski closure operator, the second one appears as a Kuratowski closure operator. Thus, we obtained a new topology $\tau_\omega^\diamond$ which is finer than both $\tau^\diamond$ and τ_ω. Also, we built a basis for this new topology $\tau_\omega^\diamond$ and revealed several fundamental results. Moreover, we obtained some relationships between this new topology and the other topologies existed in the literature. We hope that this paper will stimulate further research on primals and rough sets as ideals.

In future work, we will study different operators by utilizing soft sets and rough sets via primals. Also, we will generate new topologies from primals and other types of sets in the literature.

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