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Conservation Laws and Symmetry Multi-Reductions of Two $(2+1)$-Dimensional Equations: The Zakharov-Kuznetsov (ZK) Equation and a Nonlinear Wave Equation

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Abstract. The construction of invariant solutions is a key application of Lie symmetry analysis in studying partial differential equations. The generalised double reduction method, which uses both symmetries and conservation laws of a PDE or system of PDEs, provides a powerful framework for constructing such solutions. This paper contributes to the application of the generalised double reduction method by analysing two $(2+1)$-dimensional equations: the Zakharov-Kuznetsov (ZK) equation and a nonlinear wave equation. We extend the work of Bokhari et al. [6, 7] on the nonlinear wave equation by performing a second symmetry reduction using previously unused inherited symmetries. For the ZK equation, we identify its Lie point symmetries, construct four conservation laws using the multiplier method, and determine their associated Lie point symmetries. This allows for symmetry reductions using each conservation law. This paper provides a detailed account of the generalised double reduction method, including the exploitation of inherited symmetries at each reduction step.

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Key Words and Phrases: Double reduction, Zakharov-Kuznetsov equation, Lie symmetry analysis, conservation law, invariant solution

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1. Introduction

Partial differential equations (PDEs) are widely used as models of real-world physical phenomena. Analytical solutions to PDEs are highly desirable whenever possible. Lie symmetry analysis [4, 5, 9, 32, 33] provides powerful routines for seeking analytical solutions of PDEs known as group-invariant solutions. This approach has been successfully applied to find exact solutions of many PDEs, including those in physics, engineering, and other fields [1, 16, 23–27, 34, 36, 42, 43].

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Based on pioneering work by Kara et al. [18–20] (see also [40, 41]), Sjöberg [38, 39] showed that the association of conservation laws with symmetries provides a new avenue for obtaining invariant solutions of PDEs. This association results in double reduction of a PDE. For a PDE of order q with two independent variables and m dependent variables that admits a nontrivial conserved law, with at least one associated symmetry, Sjöberg [38, 39] developed a double reduction method that reduces the PDE to an ODE of order $(q-1)$. There are many articles on the application of the double reduction method involving two independent variables and one dependent variables [8, 14, 15, 17, 37].

Recently, Bokhari et al. [7], and also Anco and Gandarias [2] generalised the double reduction theory to the case involving several independent variables. According to Bokhari et al. [7], a nonlinear system of qth-order PDEs with n independent and m dependent variables can be reduced to a nonlinear system of $(q-1)$ th-order ODEs. The reduction is possible only if in every reduction, there is at least one symmetry associated with a nontrivial conservation law. Naz et al. [31] utilised the double reduction theory to find some exact solutions of a class of nonlinear regularised long wave equations. Other applications of the generalised the double reduction theory include Bokhari et al [6], Sait et al [35] and Muatjetjeja et al [28].

The first part of this paper is essentially the extension of the seminal paper by Bokhari et al [6] on the generalisation of the double reduction theory. In [6] the theory was applied on the nonlinear $(2+1)$-dimensional wave equation

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y = 0 \tag{1.1}$$

involving two arbitrary functions $f(u)$ and $g(u)$. We have presented an extended account of the application and included a second multi-reduction of the conservation law of the equation by finding and using inherited symmetries that were not determined. We have also included for illustrative purposes solutions of the wave equation for particular specifications of the arbitrary functions.

In the second part of the paper we consider another equation, the $(2+1)$-dimensional Zakharov-Kuznetsov (ZK) equation [3, 12, 13, 29]

$$u_t + \mu u_x + \nu u u_x + \alpha u_{xxx} + \beta u_{xyy} = 0, \tag{1.2}$$

where μ, ν, α, and β are arbitrary constants, are found by employing the generalised double reduction theory. The ZK equation originated from the study of weakly nonlinear ion-acoustic waves in a strongly magnetised plasma consisting of cold ions and hot isothermal electrons. It was first introduced by Vladimir Zakharov and Boris Kuznetsov in 1974 [45] and serves as a two-dimensional generalisation of the well-known Korteweg-de Vries (KdV) equation, alongside the Kadomtsev-Petviashvili (KP) equation [10, 44].

Over time, researchers have investigated various aspects of the ZK equation, including its local, global, and scattering properties, as well as seeking novel exact solutions through techniques such as the $\left(\frac{G'}{G}\right)^2$-expand method, Lie symmetry analysis, and the homotopy perturbation method [10, 11, 16, 22, 44]. Research to find analytical solutions of the ZK equation and its variants has continued aimed at contributing to the understanding of complex physical phenomena modelled by the ZK equation that arise in diverse fields [10, 16, 22, 44].

The main goal of our present work is to obtain reductions of the ZK equation by exploiting the generalised double reduction theory [7]. We obtain four non-trivial conservation laws of the ZK equation (when $\mu = 0$ and $\nu = 1$) by the multiplier method. The generalised double reduction theorem is then applied leading to second-order ODEs in each of the cases where multireduction is possible. Our application of the generalised double reduction method is instructive in that we demonstrate the use of "nonassociated" symmetries in the reduction routine. We show that an associated symmetry with a reduced conserved form may be inherited from a nonassociated symmetry with the original conserved form.

2. Fundamentals of the Double Reduction Theorem

In this section, we present the double reduction routine for a qth-order ($q \geq 1$) partial differential equation with n independent variables $x = (x^1, x^2, \dots, x^n)$ and one dependent variable $u = u(x)$, namely

$$F(x, u, u_{(1)}, u_{(2)}, \dots, u_{(q)}) = 0, \tag{2.1}$$

where $u_{(q)}$ denotes the collection $\{u_q\}$ of qth-order partial derivatives. In this connection, we first present the following well-known definitions and results (see, e.g., [7, 18, 21, 30]).

Definition 2.1. *The total derivative operator with respect to x^i is*

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \dots, n, \tag{2.2}$$

where u_i denotes the derivative of u with respect to x^i. Similarly, u_{ij} denotes the derivative of u with respect to x^i and x^j.

Definition 2.2. *The Lie-Bäcklund operator is*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} \quad \xi^i, \eta \in A, \tag{2.3}$$

where A is the space of differential functions. The operator (2.3) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}}, \tag{2.4}$$

where the additional coefficients are determined uniquely by the prolongation formulae,

$$\begin{aligned} \zeta_i &= D_i\,(W) + \xi^j u_{ij} \\ \zeta_{i_1 \dots i_s} &= D_{i_1 \dots D_{i_s}}\,(W) + \xi^j u_{j i_1 \dots i_s}, \quad s > 1, \end{aligned} \tag{2.5}$$

in which W is the Lie characteristic function,

$$W = \eta - \xi^j u_j. \tag{2.6}$$

Definition 2.3. *An n-tuple* $T = \left(T^1, T^2, \ldots, T^n\right), \quad i = 1, 2, \ldots, n,$ *such that*

$$D_i T^i = 0 \tag{2.7}$$

holds for all solutions of (2.1) is known as a conservation law of (2.1).

Definition 2.4. *A multiplier* Λ *for Equation (2.1) is a non-singular function on the solution space of (2.1) with the property*

$$D_i T^i = \Lambda E \tag{2.8}$$

for arbitrary function $u\left(x^1, x^2, \ldots, x^n\right)$.

Definition 2.5. *The determining equations for multipliers are obtained by taking the variational derivative*

$$\frac{\delta}{\delta u}(\Lambda E) = 0, \tag{2.9}$$

where the Euler operator $\delta/\delta u$ *is defined by*

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_{ij} \frac{\partial}{\partial u_{ij}} - D_{ijk} \frac{\partial}{\partial u_{ijk}} + \cdots . \tag{2.10}$$

Definition 2.6. *A Lie-Bäcklund symmetry generator* X *of the form (2.3) is associated with a conserved vector* T *of the system (2.1) if* X *and* T *satisfy the relations*

$$X\left(T^i\right) + T^i D_j \xi^j - T^j D_j \xi^i, = 0, \quad i = 1, \ldots, n. \tag{2.11}$$

Theorem 2.1. *Suppose* $D_i T^i = 0$ *is a conservation law of the PDE system (2.1). Then under a similarity transformation of a symmetry* X *of the form (2.3) for the PDE, there exist functions* $\widetilde{T}^i$ *such that* X *is still symmetry for the PDE* $\widetilde{D}_i \widetilde{T}^i = 0$*, where* $\widetilde{T}^i$ *is given by*

$$\begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \cdot \\ \cdot \\ \cdot \\ T^n \end{pmatrix}, \tag{2.12}$$

where

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \ldots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \ldots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \ldots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \ldots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \ldots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \ldots & D_n \tilde{x}_n \end{pmatrix},$$

and $J = det(A)$.

Colollary 2.1. *(The necessary and sufficient condition for reduced conserved form [7]). The conserved form $D_iT^i = 0$ of the PDE system (2.1) can be reduced under a similarity transformation of a symmetry X to a reduced conserved form $\widetilde{D}_i\widetilde{T}^i = 0$ if and only if X is associated with the conservation law T.*

Colollary 2.2. *(see [7]). A nonlinear system of qth-order PDEs with n independent and m dependent variables which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a $(q-1)$ th-order nonlinear system of ODEs.*

Colollary 2.3. *(The Inherited Symmetries [7]). Any symmetry Y for the conserved form $D_iT^i = 0$ of PDE system (2.1) can be transformed under the similarity transformation of a symmetry X for the PDE to the symmetry $\tilde{Y}$ for the PDE $\tilde{D}_i\tilde{T}^i = 0$.*

3. Application of the generalised double reduction theory to the nonlinear wave equation (1.1)

It was established in [6, 7] that for arbitrary functions $f(u)$ and $g(u)$, the nonlinear $(2+1)$ wave equation (1.1) has the conserved vector

$$\left(T^t, T^x, T^y\right) = \left(-u_t, f(u)u_x, g(u)u_y\right), \tag{3.1}$$

and admits the Lie point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \\ X_3 &= \frac{\partial}{\partial y}, \quad X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{3.2}$$

Furthermore, the symmetries X_1, X_2 and X_3 are associated with the conserved vector (3.1), i.e., if $X = \kappa_1 X_1 + \kappa_2 X_2 + \kappa_3 X_3$, where κ_i's are arbitrary constants, then

$$\begin{aligned} X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} \\ + \left(D_t\xi^t + D_x\xi^x + D_y\xi^y\right)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0. \end{aligned} \tag{3.3}$$

So we can get a reduced conserved form by the combination of them $X = \frac{\partial}{\partial t} + \kappa_2\frac{\partial}{\partial x} + \kappa_3\frac{\partial}{\partial y}$, where the generator X has a canonical form $X = \frac{\partial}{\partial q}$. From the characteristic equations

$$\frac{dt}{1} = \frac{dx}{\kappa_2} = \frac{dy}{\kappa_3} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{3.4}$$

we obtain canonical coordinates

$$r = y - \kappa_3 t, \quad s = x - \kappa_2 t, \quad q = t, \quad w(r,s) = u. \tag{3.5}$$

The inverse canonical coordinates are

$$t = q, \quad x = \kappa_2 q + s, \quad y = \kappa_3 q + r, \quad u = w. \tag{3.6}$$

It follows therefore that the partial derivatives u_t, u_x and u_y expressed in terms of the canonical variables (3.5) are

$$u_t = -\kappa_2 w_s - \kappa_3 w_r, \quad u_x = w_s, \quad u_y = w_r. \tag{3.7}$$

According to Theorem 2.1, we obtain the reduced conserved form

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}, \tag{3.8}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \kappa_2 & \kappa_3 \end{pmatrix}, \tag{3.9}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\kappa_3 & -\kappa_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

with $J = \det(A) = -1$. Substituting for the partial derivatives in (3.8) using (3.7), we obtain

$$\begin{aligned} T^r &= \kappa_3^2 w_r + \kappa_2\kappa_3 w_s - g(w) w_r, \\ T^s &= \kappa_2\kappa_3 w_r + \kappa_2^2 w_s - f(w) w_s, \\ T^q &= -\kappa_3 w_r - \kappa_2 w_s. \end{aligned} \tag{3.11}$$

Therefore, the reduced conserved form is

$$D_r T^r + D_s T^s = 0. \tag{3.12}$$

After writing the symmetries (3.2) in the canonical variables (3.5), we obtain

$$\widetilde{X}_1 = \kappa_3 \frac{\partial}{\partial r} + \kappa_2 \frac{\partial}{\partial s}, \quad \widetilde{X}_2 = \frac{\partial}{\partial s}, \quad \widetilde{X}_3 = \frac{\partial}{\partial r}, \quad \widetilde{X}_4 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}, \tag{3.13}$$

and it turns out that they are all inherited by (3.12). Furthermore, all the inherited symmetries (3.13) are associated with the conservation law (3.12), i.e., if $\widetilde{X} = \delta_1 \widetilde{X}_1 + \delta_2 \widetilde{X}_2 + \delta_3 \widetilde{X}_3 + \delta_4 \widetilde{X}_4$, then

$$\widetilde{X}\begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} + \left(D_r\xi^r + D_s\xi^s\right)\begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0. \tag{3.14}$$

We consider reduction of the conservation law (3.12) under two cases: under the inherited symmetry $Y_1 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$, and under the inherited symmetry $Y_2 = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s}$, where γ is an arbitrary constant.

3.1. Reduction of (T^r, T^s) under $Y_1 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$

The generator $Y_1 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$ has a canonical form $Y = \frac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{3.15}$$

or

$$n = \frac{s}{r}, \quad m = \ln r, \quad v(n) = w. \tag{3.16}$$

The inverse canonical coordinates are given by

$$r = e^m, \quad s = e^m n, \quad w = v, \tag{3.17}$$

and so the partial derivatives in the conserved vector (T^r, T^s) in terms of the canonical coordinates (3.16) are given by

$$w_r = -e^{-m} n v_n, \quad w_s = -e^{-m} n v_n. \tag{3.18}$$

By using the formula (2.12) we get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{3.19}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & e^m \\ e^m & e^m n \end{pmatrix},$$
$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\frac{s}{r^2} & \frac{1}{r} \\ \frac{1}{r} & 0 \end{pmatrix},$$

and $J = \det(A) = -e^{2m}$.

From (3.18) and (3.19) we obtain that

$$\begin{aligned} T^n &= v_n\left(2\kappa_2\kappa_3 n - \kappa_3^2 n^2 + n^2 g(v) - \kappa_2^2 + f(v)\right), \\ T^m &= v_n\left(\kappa_3^2 n - \kappa_2\kappa_3 - n g(v)\right). \end{aligned} \tag{3.20}$$

Therefore, the reduced conservation law is

$$D_n T^n = 0, \tag{3.21}$$

or

$$v_n\left(2\kappa_2\kappa_3 n - \kappa_3^2 n^2 + n^2 g(v) - \kappa_2^2 + f(v)\right) = k, \tag{3.22}$$

where k is a constant. The solution of the nonlinear wave equation (1.1) follows from the solution of (3.22) via (3.16) and (3.5).

For illustrative purposes, let us consider particular choices of the arbitrary functions in equation (1.1). If we let $f(u) = 1$, $g(u) = u$ and set $\kappa_2 = 1$ and $\kappa_3 = 0$, then the ODE (3.22) reduces to

$$n^2 v(n) v'(n) = k, \tag{3.23}$$

the solution of which is

$$\frac{2k}{n} + v^2 = \lambda, \tag{3.24}$$

where λ is an arbitrary constant. In light of the change of variables (3.16) and (3.5), with $\kappa_2 = 1$, $\kappa_3 = 0$, $n = \frac{x-t}{y}$ and $v = u$, the solution (3.24) translates into the following solution of the nonlinear wave equation (1.1):

$$\frac{2ky}{x-t} + u^2 = \lambda. \tag{3.25}$$

3.2. Reduction of (T^r, T^s) under $Y_2 = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}$

From the generator Y_2 the canonical coordinates are

$$n = s - \gamma r, \quad m = r, \quad v(n) = w, \tag{3.26}$$

and the inverse canonical coordinates are given by

$$r = m, \quad s = \gamma m + n, \quad w = v. \tag{3.27}$$

Therefore, the partial derivatives in (T^r, T^s) in terms of the canonical coordinates (3.26) are given by

$$w_r = -\gamma v_n, \quad w_s = v_n. \tag{3.28}$$

By using the formula (2.12) we get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{3.29}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix},$$

and $J = \det(A) = -1$.

From (3.28) and (3.29) we obtain that

$$\begin{aligned} T^n &= v_n \left(f(v) + \gamma^2 g(v) - (\kappa_2 - \gamma\kappa_3)^2 \right), \\ T^m &= v_n(\kappa_3(\gamma\kappa_3 - \kappa_2) - \gamma g(v)). \end{aligned} \tag{3.30}$$

Therefore, the reduced conservation law is

$$D_n T^n = 0, \tag{3.31}$$

or

$$v_n \left(f(v) + \gamma^2 g(v) - (\kappa_2 - \gamma\kappa_3)^2 \right) = k, \tag{3.32}$$

where k is a constant. The solution of the nonlinear wave equation (1.1) follows from the solution of (3.32) via (3.26) and (3.5).

If for illustrative purposes we let $f(u) = g(u) = u$ in the nonlinear wave equation (1.1), and set $\gamma = \frac{\kappa_3}{\kappa_2}$, the ODE (3.32) reduces to

$$(Mv(n) + v(n) - L)v'(n) = k, \tag{3.33}$$

where

$$M = \frac{\kappa_3^2}{\kappa_2^2}, \quad L = \left(\kappa_2 - \frac{\kappa_3^2}{\kappa_2}\right)^2.$$

The solution of (3.33) is

$$\lambda - 2kn - 2Lw + (M+1)w^2 = 0, \tag{3.34}$$

where λ is an arbitrary constant. In light of (3.5) and (3.26), and if we set $\gamma = \frac{\kappa_3}{\kappa_2}$, we obtain

$$n = \frac{\kappa_3^2 t}{\kappa_2} - \frac{\kappa_3 y}{\kappa_2} - \kappa_2 t + x \quad \text{and} \quad w = u. \tag{3.35}$$

Therefore (3.34) becomes

$$k\left(\sqrt{L}\,t + \frac{\kappa_3}{\kappa_2}y - x\right) + \frac{\lambda}{2} - Lu + \frac{1}{2}(M+1)u^2 = 0, \tag{3.36}$$

which is the solution in implicit form of the non-linear wave equation (1.1).

4. Symmetries and Conservation Laws of the ZK Equation

First, we will derive the Lie point symmetries of (1.2), in the case $\mu = 0, \nu = 1$, $\alpha\beta \neq 0$, i.e.,

$$u_t + uu_x + \alpha u_{xxx} + \beta u_{xyy} = 0. \tag{4.1}$$

If the operator

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}$$

is a generator of a Lie point symmetry of (4.1), then it must satisfy the invariance condition

$$X^{[3]}\left[u_t + uu_x + \alpha u_{xxx} + \beta u_{xyy}\right]\Big|_{(4.1)} = 0, \tag{4.2}$$

where $X^{(3)}$ is the third prolongation of X and can be computed according to (2.5). Equation (4.2), after expansion and separation, yields an overdetermined system of linear first-order partial differential equations for the unknown coefficients ξ^1, ξ^2, ξ^3, and η. The solution of the system leads to the following Lie point symmetries of (4.1):

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \\ &X_5 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{aligned} \tag{4.3}$$

The conservation laws for (4.1) are constructed by the multiplier approach. We consider multipliers of the form $\Lambda(x,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{tx},u_{yy},u_{tt},u_{ty})$ for (4.1). The determining equation for the multipliers is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_t+uu_x+\alpha u_{xxx}+\beta u_{xyy}\right)\right]=0, \tag{4.4}$$

where $\frac{\delta}{\delta u}$ is the standard Euler operator (2.10). Expanding and then separating (4.4) with respect to different combinations of partial derivatives of u results in the following overdetermined system for the multipliers:

$$\begin{aligned}
&\Lambda_{tt}=0, && \Lambda_{tu}-\frac{\Lambda_t}{u}=0, && \Lambda_{tu_{yy}}=0,\\
&\Lambda_{uu}-\frac{\Lambda_{u_{yy}}}{\beta}=0, && \Lambda_{u,u_{yy}}=0, && \Lambda_{u_{yy}u_{yy}}=0,\\
&\Lambda_x+\frac{\Lambda_t}{u}=0, && \Lambda_{ux}=0, && \Lambda_{uy}=0,\\
&\Lambda_{ut}=0, && \Lambda_{u_{xx}}-\frac{\alpha\Lambda_{u_{yy}}}{\beta}=0, && \Lambda_{u_{xy}}=0,\\
&\Lambda_{u_{tx}}=0, && \Lambda_{u_{tt}}=0, && \Lambda_{u_{ty}}=0,
\end{aligned} \tag{4.5}$$

provided that $\alpha\beta\neq 0$. The solution of system (4.5) is

$$\Lambda=c_1tu-c_1x+c_2u_{yy}+\frac{\alpha c_2}{\beta}u_{xx}+\frac{c_2}{2\beta}u^2+c_3u+c_4, \tag{4.6}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. From (4.6), we obtain single parameter multipliers

$$\Lambda_1=tu-x,\quad \Lambda_2=u_{yy}+\frac{1}{2\beta}u^2+\frac{\alpha}{\beta}u_{xx},\quad \Lambda_3=u,\quad \Lambda_4=1. \tag{4.7}$$

According to (2.8), the multipliers in (4.7) satisfy

$$\Lambda\left(u_t+uu_x+\alpha u_{xxx}+\beta u_{xyy}\right)=D_tT^t+D_xT^x+D_yT^y \tag{4.8}$$

for arbitrary functions $u(t,x,y)$. We obtain four nontrivial conserved vectors

$$T_1=\left(T_1^t,T_1^x,T_1^y\right),\quad T_2=\left(T_2^t,T_2^x,T_2^y\right),\quad T_3=\left(T_3^t,T_3^x,T_3^y\right),\quad T_4=\left(T_4^t,T_4^x,T_4^y\right), \tag{4.9}$$

where $T_i^t,T_i^x,T_i^y,\ i=1,\dots,4$, are given by

$$T_1\ :\ \begin{cases}
T_1^t=\dfrac{tu^2}{2}-ux,\\
T_1^x=\dfrac{tu^3}{3}-\dfrac{u^2x}{2}+\alpha u_x-\dfrac{\alpha}{2}tu_x^2+\alpha tuu_{xx}-\alpha xu_{xx}+\dfrac{\beta}{2}tuu_{yy}-\beta xu_{yy},\\
T_1^y=\dfrac{\beta tuu_{xy}}{2}+\beta u_y-\dfrac{\beta tu_xu_y}{2},
\end{cases} \tag{4.10}$$

$$T_2 \quad : \quad \begin{cases} T_2^t = \dfrac{uu_{yy}}{2} + \dfrac{u^3}{6\beta} + \dfrac{\alpha u u_{xx}}{2\beta}, \\ T_2^x = \alpha u_{xx}u_{yy} + \dfrac{u^2 u_{yy}}{2} + \dfrac{u^4}{8\beta} + \dfrac{\alpha u^2 u_{xx}}{2\beta} - \dfrac{\alpha u u_{tx}}{2\beta} + \dfrac{\alpha u_x u_t}{2\beta} + \dfrac{\alpha^2 u_{xx}^2}{2\beta} + \dfrac{\beta u_{yy}^2}{2}, \\ T_2^y = \dfrac{u_t u_y}{2} - \dfrac{u u_{ty}}{2}, \end{cases} \tag{4.11}$$

$$T_3 \quad : \quad \begin{cases} T_3^t = \dfrac{u^2}{2}, \\ T_3^x = \dfrac{u^3}{3} - \dfrac{\alpha u_x^2}{2} + \alpha u u_{xx} + \dfrac{\beta u u_{yy}}{2}, \\ T_3^y = \dfrac{\beta u u_{xy}}{2} - \dfrac{\beta u_x u_y}{2}, \end{cases} \tag{4.12}$$

$$T_4 \quad : \quad \begin{cases} T_4^t = u, \\ T_4^x = \dfrac{u^2}{2} + \alpha u_{xx} + \beta u_{yy}, \\ T_4^y = 0, \end{cases} \tag{4.13}$$

5. Double Reduction of the ZK equation

To determine symmetries associated with the conserved vectors (4.9), we set a linear combination of the symmetries (4.3), i.e., $X = \sum_{i=1}^{5} \kappa_i X_i$, where κ_i's are arbitrary constants, and then apply the association condition

$$X \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^1 & D_x\xi^1 & D_y\xi^1 \\ D_t\xi^2 & D_x\xi^2 & D_y\xi^2 \\ D_t\xi^3 & D_x\xi^3 & D_y\xi^3 \end{pmatrix} \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + \left(D_t\xi^1 + D_x\xi^2 + D_y\xi^3\right) \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0 \tag{5.1}$$

for each of conserved vector T_i in (4.9). We obtain that T_1 is associated with only X_3, T_2 and T_3 are associated with the linear combination $\kappa_1 X_1 + \kappa_2 X_2 + \kappa_4 X_4$, and T_4 is associated with the linear combination $\kappa_1 X_1 + \kappa_2 X_2 + \kappa_3 X_3 + \kappa_5 X_5$. For the multi-reductions that follow we use these associations:

$$\begin{aligned} X_3 &\longrightarrow T_1 \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_2 \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_3 \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_4 \\ X_5 &\longrightarrow T_4. \end{aligned} \tag{5.2}$$

5.1. Multi-Reduction of the ZK equation by T_3

We obtain the first reduction of the conserved vector T_3 by writing the vector in canonical variables determined from the associated symmetry

$$X = X_1 + \kappa_2 X_2 + \kappa_3 X_3. \tag{5.3}$$

Writing (5.3) in the canonical form $X = \frac{\partial}{\partial q}$, we obtain from the corresponding characteristic equations

$$\frac{dt}{1} = \frac{dx}{\kappa_2} = \frac{dy}{\kappa_3} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{5.4}$$

the canonical coordinates

$$r = y - \kappa_3 t, \quad s = x - \kappa_2 t, \quad q = t, \quad w = u, \tag{5.5}$$

where $w = w(r, s)$. Inverse canonical coordinates are given by

$$t = q, \quad x = \kappa_2 q + s, \quad y = \kappa_3 q + r, \quad u = w. \tag{5.6}$$

From Theorem 2.1, we obtain the reduced conserved form

$$\begin{pmatrix} T_3^r \\ T_3^s \\ T_3^q \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T_3^t \\ T_3^x \\ T_3^y \end{pmatrix}, \tag{5.7}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \kappa_2 & \kappa_3 \end{pmatrix}, \tag{5.8}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\kappa_3 & -\kappa_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5.9}$$

and

$$J = \det(A) = -1. \tag{5.10}$$

Expressing the first and second partial derivatives u_t, u_x, u_{tt}, u_{xy} and u_{xx} in terms of the canonical coordinates (5.5), we obtain

$$\begin{aligned} u_t &= -\kappa_2 w_s - \kappa_3 w_r, & u_x &= w_s, & u_y &= w_r, \\ u_{xx} &= w_{ss}, & u_{tx} &= -\kappa_2 w_{ss} - \kappa_3 w_{rs}, & u_{yy} &= w_{rr}. \end{aligned} \tag{5.11}$$

Substituting for the partial derivatives u_{xx} and u_{yy} in (5.7) using (5.11), we obtain

$$\begin{aligned} T_3^r &= \frac{\kappa_3 w^2}{2} - \frac{\beta w w_{rs}}{2} + \frac{\beta w_r w_s}{2}, \\ T_3^s &= \frac{\kappa_2 w^2}{2} - \frac{w^3}{3} - \frac{\beta w w_{rr}}{2} - \alpha w w_{ss} + \frac{\alpha w_s^2}{2}, \\ T_3^q &= -\frac{w^2}{2}, \end{aligned} \tag{5.12}$$

leading to the reduced conservation law

$$D_r T_3^r + D_s T_3^s = 0. \tag{5.13}$$

Equation (5.13) inherits the symmetries X_1, X_2 and X_3 from (4.3), which when written in terms of the canonical variables (5.5), are:

$$\widetilde{X}_1 = \kappa_3 \frac{\partial}{\partial r} + \kappa_2 \frac{\partial}{\partial s}, \quad \widetilde{X}_2 = \frac{\partial}{\partial s}, \quad \widetilde{X}_3 = \frac{\partial}{\partial r}. \tag{5.14}$$

It turns out that all the symmetries in (5.14) are associated with the conservation law (5.13), i.e.

$$\widetilde{X} \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix} \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} + (D_r\xi^r + D_s\xi^s) \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} = 0, \tag{5.15}$$

if $\widetilde{X} = \delta_1\widetilde{X}_1 + \delta_2\widetilde{X}_2 + \delta_3\widetilde{X}_3$, where $\delta_i's$ are arbitrary constants.

So, we can get a further reduction of the conserved vector (T^r, T^s) by

$$Y = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}, \tag{5.16}$$

where γ is an arbitrary constant. The generator Y has a canonical form $Y = \dfrac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{5.17}$$

which results in canonical coordinates

$$n = s - \gamma r, \quad m = r, \quad v = w, \tag{5.18}$$

where $v = v(n)$. The inverse canonical coordinates are given by

$$r = m, \quad s = \gamma m + n, \quad w = v. \tag{5.19}$$

Therefore, the partial derivatives in the components (5.12) in terms of the canonical coordinates (5.18) are given by

$$w_r = -\gamma v_n, \quad w_s = v_n, \quad w_{rr} = \gamma^2 v_{nn}, \quad w_{rs} = -\gamma v_{nn}, \quad w_{ss} = v_{nn}. \tag{5.20}$$

According to Theorem 2.1, we have that

$$\begin{pmatrix} T_3^n \\ T_3^m \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix}, \tag{5.21}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix}, \tag{5.22}$$

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.23}$$

and

$$J = \det(A) = -1. \tag{5.24}$$

Substituting the partial derivatives in (5.20) into (5.21) results in a reduced conserved vector with the following components:

$$\begin{aligned} T_3^r &= (\alpha+\beta\gamma^2)\left(vv_{nn} - \frac{v_n^2}{2}\right) + \frac{1}{2}v^2(\gamma\kappa_3 - \kappa_2) + \frac{v^3}{3}, \\ T_3^s &= \frac{1}{2}\beta\gamma v_n^2 - \frac{\kappa_3 v^2}{2} - \frac{1}{2}\beta\gamma v v_{nn}. \end{aligned} \tag{5.25}$$

This leads to the reduced conservation law $D_r T_3^r = 0$, from which we obtain the second-order ODE

$$(\alpha+\beta\gamma^2)\left(vv_{nn} - \frac{v_n^2}{2}\right) + \frac{1}{2}v^2(\gamma\kappa_3 - \kappa_2) + \frac{v^3}{3} = k, \tag{5.26}$$

where k is an arbitrary constant.

5.2. Multi-Reduction of the ZK equation by T_4

We see from (5.1) that the symmetries associated with T_4 are X_1, X_2, X_3 and X_5. We perform multi-reduction by T_4 under two cases, namely reduction from using the linear combination $X = X_1 + \kappa_2 X_2 + \kappa_3 X_3$ and reduction from using X_5.

5.2.1. Reduction via $X = X_1 + \kappa_2 X_2 + \kappa_3 X_3$

The canonical variables (5.5) obtained earlier under T_3 apply here. Therefore, the reduced conserved form resulting from T_4 is

$$\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix}, \tag{5.27}$$

where A, A^{-1}, J and the partial derivatives are the same as the ones computed earlier in (5.8)– (5.11). Therefore, we obtain

$$\begin{aligned} T_4^r &= \kappa_3 w, \\ T_4^s &= \kappa_2 w - \frac{w^2}{2} - \beta w_{rr} - \alpha w_{ss} \\ T_4^q &= -w, \end{aligned} \tag{5.28}$$

and the reduced conservation law

$$D_r T_4^r + D_s T_4^s = 0. \tag{5.29}$$

Like in the T_3 case, the symmetries X_1, X_2, and X_3, in (4.3) written in terms of the canonical variables (5.5), become

$$\widetilde{X}_1 = \kappa_3\frac{\partial}{\partial r} + \kappa_2\frac{\partial}{\partial s}, \quad \widetilde{X}_2 = \frac{\partial}{\partial s}, \quad \widetilde{X}_3 = \frac{\partial}{\partial r}, \tag{5.30}$$

and are inherited by (5.29). Also, they are all associated with the conservation law (5.29). Using $Y = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}$, where γ is an arbitrary constant, as in the T_3 case, we find canonical coordinates

$$n = s - \gamma r, \quad m = r, \quad v = w, \tag{5.31}$$

where $v = v(n)$. Taking advantage of the calculations done in the T_3 case, in which the same canonical variables were used, the reduced conserved vector is given by

$$\begin{pmatrix} T_4^n \\ T_4^m \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix}, \tag{5.32}$$

where A, A^{-1}, and J are given by (5.22), (5.23) and (5.24), respectively.

Substituting the partial derivatives in (5.20) into (5.32) results in the following components of the reduced conserved vector:

$$\begin{aligned} T_4^r &= -\kappa_2 v + \gamma \kappa_3 v + \frac{v^2}{2} + v_{nn}\left(\alpha + \beta\gamma^2\right), \\ T_4^s &= -\kappa_3 v. \end{aligned} \tag{5.33}$$

This leads to the reduced conservation law $D_r T_4^r = 0$, from which it follows that

$$v(\gamma\kappa_3 - \kappa_2) + \frac{v^2}{2} + v_{nn}\left(\alpha + \beta\gamma^2\right) = k, \tag{5.34}$$

where k is an arbitrary constant.

5.2.2. Reduction via $X = X_5$

We obtain the first reduction of the conservation law T_4 by writing it in canonical variables determined from writing the associated symmetry $X = X_5$ in the form $X = \frac{\partial}{\partial q}$. From the associated characteristic equations

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{dy}{y} = \frac{du}{-2u} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{5.35}$$

we obtain the canonical coordinates

$$r = \frac{y}{\sqrt[3]{t}}, \quad s = \frac{x}{\sqrt[3]{t}}, \quad q = \frac{\ln t}{3}, \quad w = t^{2/3} u, \tag{5.36}$$

where $w = w(r, s)$. Inverse canonical coordinates are given by

$$t = e^{3q}, \quad x = e^q s, \quad y = e^q r, \quad u = e^{-2q} w, \tag{5.37}$$

and the partial derivatives in the conserved vector T_4, in terms of the canonical coordinates, are

$$u_{xx} = e^{-4q} w_{ss}, \quad u_{yy} = e^{-4q} w_{rr}. \tag{5.38}$$

The reduced conserved vector is therefore

$$\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix}, \tag{5.39}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^q \\ 0 & e^q & 0 \\ 3e^{3q} & e^q s & e^q r \end{pmatrix} \tag{5.40}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\frac{y}{3t^{4/3}} & -\frac{x}{3t^{4/3}} & \frac{1}{3t} \\ 0 & \frac{1}{\sqrt[3]{t}} & 0 \\ \frac{1}{\sqrt[3]{t}} & 0 & 0 \end{pmatrix}, \tag{5.41}$$

and

$$J = \det(A) = -3e^{5q}. \tag{5.42}$$

Substituting the partial derivatives (5.38) into (5.39), we obtain

$$\begin{aligned} T_4^r &= rw, \\ T_4^s &= sw - \frac{3w^2}{2} - 3\beta w_{rr} - 3\alpha w_{ss} \\ T_4^q &= -w. \end{aligned} \tag{5.43}$$

Then the reduced conservation law is

$$D_r T_4^r + D_s T_4^s = 0. \tag{5.44}$$

None of the symmetries from (4.3) are inherited by (5.44), and therefore no further reduction of (5.44) is performed in this case.

5.3. Multi-Reduction of the ZK equation by T_2

We see from (5.1) that the symmetry associated with T_2 is $X = \frac{\partial}{\partial t} + \kappa_2 \frac{\partial}{\partial x} + \kappa_3 \frac{\partial}{\partial y}$, the same symmetry used in the multi-reduction by T_3. Therefore, the first reduction can be achieved through the canonical coordinates

$$r = y - \kappa_3 t, \quad s = x - \kappa_2 t, \quad q = t, \quad w = u, \tag{5.45}$$

where $w = w(r, s)$. The resulting reduced conserved form is

$$\begin{pmatrix} T_2^r \\ T_2^s \\ T_2^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^t \\ T_2^x \\ T_2^y \end{pmatrix}, \tag{5.46}$$

where A, A^{-1}, and J are given by (5.8), (5.9) and (5.10), respectively. Substituting for the partial derivatives in the conserved vector T_2 using (5.11), we obtain

$$\begin{aligned} T_2^r &= -\frac{\kappa_2 w w_{rs}}{2} + \frac{\kappa_2 w_r w_s}{2} + \frac{\kappa_3 w^3}{6\beta} + \frac{\alpha\kappa_3 w w_{ss}}{2\beta} + \frac{\kappa_3 w_r^2}{2}, \\ T_2^s &= \frac{\kappa_2 w^3}{6\beta} + \frac{\kappa_2 w w_{rr}}{2} + \frac{\alpha\kappa_2 w_s^2}{2\beta} - \frac{\alpha\kappa_3 w w_{rs}}{2\beta} - \frac{w^4}{8\beta} - \frac{w^2 w_{rr}}{2} \\ &\quad + \frac{\alpha\kappa_3 w_r w_s}{2\beta} - \frac{\alpha w^2 w_{ss}}{2\beta} - \frac{\beta w_{rr}^2}{2} - \alpha w_{rr} w_{ss} - \frac{\alpha^2 w_{ss}^2}{2\beta}, \\ T_2^q &= -\frac{w^3}{6\beta} - \frac{w w_{rr}}{2} - \frac{\alpha w w_{ss}}{2\beta}. \end{aligned} \tag{5.47}$$

Then the reduced conservation law is

$$D_r T_2^r + D_s T_2^s = 0. \tag{5.48}$$

Like in the T_3 case, the symmetries X_1, X_2, and X_3, in (4.3) written in terms of the canonical variables (5.5), become

$$\widetilde{X}_1 = \kappa_3\frac{\partial}{\partial r} + \kappa_2\frac{\partial}{\partial s}, \quad \widetilde{X}_2 = \frac{\partial}{\partial s}, \quad \widetilde{X}_3 = \frac{\partial}{\partial r}, \tag{5.49}$$

and are inherited by (5.48). Also, they are all associated with the conservation law (5.48). Using $Y = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s}$, where γ is an arbitrary constant, we find canonical coordinates

$$n = s - \gamma r, \quad m = r, \quad v = w, \tag{5.50}$$

where $v = v(n)$. Taking advantage of the calculations done in the T_3 case, in which the same canonical variables were used, the reduced conserved vector is given by

$$\begin{pmatrix} T_2^n \\ T_2^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix}, \tag{5.51}$$

where A, A^{-1}, and J are given by (5.22), (5.23) and (5.24), respectively.

Substituting the partial derivatives in (5.20) into (5.51) results in the following components of the reduced conserved vector:

$$\begin{aligned} T_2^n &= \frac{Q^2 v_{nn}^2}{2\beta} - \frac{(\kappa_2 - \gamma\kappa_3)\left(Q v_n^2 + v^3/3\right)}{2\beta} + \frac{Q v^2 v_{nn}}{2\beta} + \frac{v^4}{8\beta}, \\ T_2^m &= \frac{1}{2}\gamma\kappa_2 v_n^2 - \frac{1}{2}\gamma\kappa_2 v v_{nn} - \frac{\kappa_3 v^3}{6\beta} - \frac{\alpha\kappa_3 v v_{nn}}{2\beta} - \frac{1}{2}\gamma^2\kappa_3 v_n^2, \end{aligned} \tag{5.52}$$

where $Q = \alpha + \beta\gamma^2$. This leads to the reduced conservation law $D_n T_2^n = 0$, from which it follows that

$$\frac{Q^2 v_{nn}^2}{2\beta} - \frac{(\kappa_2 - \gamma\kappa_3)\left(Q v_n^2 + v^3/3\right)}{2\beta} + \frac{Q v^2 v_{nn}}{2\beta} + \frac{v^4}{8\beta} = k, \tag{5.53}$$

where k is an arbitrary constant.

5.4. Multi-Reduction of the ZK equation by T_1

The first reduction of the conserved vector T_1 is obtained from writing the vector in canonical variables determined from writing the associated symmetry $X = X_3$ in the form $X = \frac{\partial}{\partial q}$. From the corresponding characteristic equations

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{5.54}$$

we obtain the canonical coordinates

$$r = t, \quad s = x, \quad q = y, \quad w = u, \tag{5.55}$$

where $w = w(r, s)$. Inverse canonical coordinates are given by

$$t = r, \quad x = s, \quad y = q, \quad u = w. \tag{5.56}$$

Therefore, the partial derivatives in the conserved vector T_1 in terms of the canonical coordinates (5.55), are

$$u_x = w_s, \quad u_y = 0, \quad u_{xx} = w_{ss}, \quad u_{xy} = 0, \quad u_{yy} = 0. \tag{5.57}$$

The reduced conserved vector is

$$\begin{pmatrix} T_1^r \\ T_1^s \\ T_1^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_1^t \\ T_1^x \\ T_1^y \end{pmatrix}, \tag{5.58}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.59}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.60}$$

and

$$J = \det(A) = 1. \tag{5.61}$$

Substituting for the partial derivatives in (5.58) using (5.57), we obtain the components

$$\begin{aligned} T_1^r &= \frac{rw^2}{2} - sw, \\ T_1^s &= \frac{rw^3}{3} + \alpha r w w_{ss} - \frac{1}{2}\alpha r w_s^2 - \frac{sw^2}{2} - \alpha s w_{ss} + \alpha w_s, \\ T_1^q &= 0. \end{aligned} \tag{5.62}$$

The corresponding reduced conservation law

$$D_r T_1^r + D_s T_1^s = 0, \tag{5.63}$$

inherits the symmetries X_1, X_2, X_4, and X_5 from (4.3), or in terms of the canonical variables (5.55)

$$\widetilde{X}_1 = \frac{\partial}{\partial r}, \quad \widetilde{X}_2 = \frac{\partial}{\partial s}, \quad \widetilde{X}_4 = r\frac{\partial}{\partial s} + \frac{\partial}{\partial w}, \quad \widetilde{X}_5 = -\frac{3}{2}r\frac{\partial}{\partial r} - \frac{1}{2}s\frac{\partial}{\partial s} + w\frac{\partial}{\partial w}. \tag{5.64}$$

It turns out that none of the symmetries (5.64) are associated with the conservation law (5.63). Therefore, no further reduction of (5.63) is performed.

6. Concluding remarks

The work presented in this study significantly enhances the existing body of literature concerning the application of the generalised double reduction theory. The theory presents a powerful tool for obtaining invariant solutions of systems of PDEs from the association of symmetries of the system with its nontrivial conservation laws. Given a system of qth-order PDEs with n independent and m dependent variables, the theory provides for the reduction of the system to a system of $(q-1)$th-order ODEs.

We have applied the generalised double reduction theory to two (2+1)-dimensional partial differential equations, a nonlinear wave equation and the Zakharov-Kuznetsov equation. For the nonlinear wave equation, we extended the application presented in the seminal paper by Bokhari et al [6] on the generalised double reduction theory. We presented a more detailed account of the application and also determined and used additional inherited symmetries. We furthermore presented exact solutions of the equation for specific prescriptions of the arbitrary functions in the equation. As for the ZK equation, five Lie point symmetries of the equation were determined along with four nontrivial conservation laws. Upon establishing association of the conservation laws with the symmetries, multi-reductions were performed for each of the conservation laws.

Overall, our study has provided a comprehensive guide to using the generalised double reduction method. We have presented clear illustrative examples and included each key step of the method.

Author Contributions

Conceptualization, M.C.K. and W.S.; methodology, M.C.K., W.S. and P.M.; software, M.C.K. and W.S.; validation, W.S. and P.M.; formal analysis, M.C.K., W.S. and P.M.; writing—original draft preparation, M.C.K. and W.S.; writing—review and editing, M.C.K., W.S. and P.M. All authors have read and agreed to the published version of the manuscript.

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The authors declare no conflict of interest.

Data Availability Statement

No data were created or analyzed in this study.

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