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Bounds for Certain Determinants of Logarithmic Coefficients for the Class of Functions with Bounded Turning

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Abstract. This paper aims to estimate the logarithmic coefficients for the class of functions with bounded turning. Hence, the upper bounds of the second-order for three types of determinants (Hankel, Toeplitz, and Vandermonde) whose entries are logarithmic coefficients for this class of functions are obtained. Some interesting consequences of these results are also highlighted, offering new findings within the class of functions with bounded turning.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Univalent functions, bounded turning functions, logarithmic coefficients, Hankel determinant, Toeplitz determinant, Vandermonde determinant

1. Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. We denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E.

A typical problem in geometric function theory is to study a functional consisting of combinations of the Taylor coefficients $a_n, n \geq 2$ for the subclass of univalent functions such as Hankel and Toeplitz determinants, but this is not limited to this. The unknown upper bounds of these determinants for the class of univalent functions have attracted researchers, making this an open and intriguing topic for further study. The Hankel

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determinant is an extremely useful tool in the study of singularities. This is particularly important when analyzing power series with integral coefficients [6, 9]. Meanwhile, the Toeplitz determinant has a variety of applications in both pure and applied mathematics, statistics, and probability; for example, it is used in algebra, quantum mechanics, queuing networks, signal processing, partial differential equations, and time series analysis [49].

Pommerenke [38, 39] and Ali et al. [7] defined the Hankel determinant $H_{q,n}(f)$ and Toeplitz determinant $T_{q,n}(f)$, $n, q \geq 1$, whose elements are Taylor coefficients $a_n, n \geq 2$ for functions $f(z) \in \mathcal{A}$, respectively, as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, a_1 = 1 \tag{2}$$

and

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}. \tag{3}$$

A recent work delves into the interesting world of Hankel and Toeplitz determinants in the context of considering logarithmic coefficients as the entries. This idea generalizes the traditional concept of both determinants (2) and (3) by replacing their entries with the logarithmic coefficients of $f(z) \in \mathcal{A}$. Kowalczyk and Lecko [22, 23], as well as Giri, Kumar, and Mohamad et al. [15, 35], introduced the Hankel and Toeplitz determinants of logarithmic coefficients γ_n, $n \geqslant 1$ for functions $f(z) \in \mathcal{A}$, respectively, as follows:

$$H_{q,n}(\gamma_f) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}, \tag{4}$$

and

$$T_{q,n}(\gamma_f) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_n & \cdots & \gamma_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q-2} & \cdots & \gamma_n \end{vmatrix}. \tag{5}$$

The logarithmic coefficients γ_n, $n \geqslant 1$ of $f(z) \in \mathcal{A}$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{6}$$

Differentiating (6) and equating coefficients of z^n provides the logarithmic coefficients in terms of Taylor coefficients for $f(z) \in \mathcal{A}$, which specifically, for $n = 1, 2, 3, 4$:

$$\gamma_1 = \frac{1}{2}a_2, \tag{7}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right), \tag{8}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\right), \tag{9}$$

and

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + {a_2}^2a_3 - \frac{1}{2}{a_3}^2 - \frac{1}{4}{a_2}^4\right). \tag{10}$$

Milin [31–33] highlighted the importance of logarithmic coefficients for estimating the Taylor coefficients of univalent functions. Subsequently, this led to de Branges [4] establishing the Bieberbach conjecture. Logarithmic coefficients also play a significant role in conformal mapping, which helped Kayumov [20] solve Brennan's conjecture. Since then, numerous studies on logarithmic coefficients have continued, with examples found in [3, 12, 14, 42].

On the other hand, Vijayalakshmi et al. [44] introduced the Vandermonde determinant $V_{q,n}(f)$, where $n, q \geq 1$ and $a_n, n \geq 2$ are the coefficients of the Taylor series in (1):

$$V_{q,n}(f) = \begin{vmatrix} 1 & a_n & \cdots & {a_n}^{q-1} \\ 1 & a_{n+1} & \cdots & {a_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n+q-1} & \cdots & {a_{n+q-1}}^{q-1} \end{vmatrix}, a_1 = 1. \tag{11}$$

This determinant has many applications in a variety of domains. For example, it is used in digital signal processing to compute the discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT), and it also plays an important part in approximation problems [44]. The Vandermonde determinant, often known as a discriminant, is also an important tool in linear algebra; refer to [26] and the references therein for details.

Therefore, following the generalization of the Hankel and Toeplitz determinants in (2) and (3), where their entries are replaced by logarithmic coefficients, and acknowledging the significance of both the Vandermonde determinant and logarithmic coefficients, we now define the Vandermonde determinant of logarithmic coefficients for functions $f(z) \in \mathcal{A}$ as follows:

$$V_{q,n}(\gamma_f) = \begin{vmatrix} 1 & \gamma_n & \cdots & {\gamma_n}^{q-1} \\ 1 & \gamma_{n+1} & \cdots & {\gamma_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \gamma_{n+q-1} & \cdots & {\gamma_{n+q-1}}^{q-1} \end{vmatrix}. \tag{12}$$

Consequently, if $q = 2$ and $n = 2$, then from (4), (5), and (12), respectively, yield the second-order of three types of determinants, namely Hankel, Toeplitz, and Vandermonde, as follows:

$$H_{2,2}(\gamma_f) = \gamma_2\gamma_4 - {\gamma_3}^2, \tag{13}$$

$$T_{2,2}(\gamma_f) = {\gamma_2}^2 - {\gamma_3}^2, \tag{14}$$

and

$$V_{2,2}\left(\gamma_f\right) = \gamma_3 - \gamma_2. \tag{15}$$

In [2, 5, 8, 24, 28, 43, 46, 48], sharp bounds for the Hankel determinant of logarithmic coefficients were recently established for several subclasses of univalent functions. Works by [1, 35] also investigated both Hankel and Toeplitz determinants with logarithmic coefficients as entries, specifically for the subclass of starlike functions with respect to other points. While there has been limited study on Toeplitz determinants in this context, it is important to note that, in general, the upper bounds for both Hankel and Toeplitz determinants remain unknown for classes of functions. In fact, to the best of our knowledge, no one has yet studied the Vandermonde determinant of logarithmic coefficients.

Thus, motivated by the previous studies, in this paper, we aim to estimate the upper bounds of the logarithmic coefficients $|\gamma_n|$, specifically for $n = 1, 2, 3, 4$. Hence, we focus on estimating the upper bounds of the second-order Hankel, Toeplitz, and Vandermonde determinants whose entries are logarithmic coefficients, as given in (7)-(10), for functions belonging to the following class of bounded turning functions:

Definition 1. *A function $f(z)$ given by (1) is said to be in the class $G(\alpha, \delta)$ if the following condition is satisfied:*

$$\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > \delta, \; z \in E,$$

where $|\alpha| < \pi$, $0 \leqslant \delta < 1$, and $\cos\alpha > \delta$.

This class was introduced by Mohamad [34].

Remark 1. *Selecting specific values for the parameters α and δ in the class $G(\alpha, \delta)$ yields the following classes:*

(i) If we choose $\alpha = \delta = 0$, then $G(\alpha, \delta)$ reduces to R which satisfies $\operatorname{Re} f'(z) > 0$. The functions from R are said to be of bounded turning.

(ii) If we choose $\alpha = 0$, then $G(\alpha, \delta)$ reduces to $R(\delta)$ which satisfies $\operatorname{Re}\left(f'(z)\right) > \delta$. The class $R(\delta)$ is called the class of bounded turning functions of order δ.

(iii) If we choose $\delta = 0$, then $G(\alpha, \delta)$ reduces to $R(\alpha)$ which satisfies $\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$.

Pioneering researchers like Goel and Mehrok [16], Macgregor [30], Noshiro [37], Silverman and Silvia [45], and Warschawski [47] were among those who explored the classes R, $R(\delta)$, and $R(\alpha)$, and further investigation into the class of bounded turning functions has also been extensively studied by other researchers, see, for example, [13, 18, 19, 21, 25, 27, 36, 40], suggesting different directions than the current study.

2. Preliminary results

Let P denote the class of positive real part functions $p(z)$, also known as Carathéodory functions, of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{16}$$

which satisfy $\operatorname{Re} p(z) > 0$ for $z \in E$.
To verify our main findings, we require a few sharp estimates in the form of lemmas valid for functions with a positive real part, as follows:

Lemma 1. ([10]) *For a function $p(z) \in P$ of the form* (16)*, the sharp inequality $|p_n| \leqslant 2$ holds for each $n \geqslant 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2. ([11]) *Let $p(z) \in P$ be a function of the form* (16) *and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\}, \ 1 \leqslant k \leqslant n-1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

3. Main results

This section presents the proof of our main findings, primarily focusing on the upper bounds of logarithmic coefficients and three types of determinants (Hankel, Toeplitz, and Vandermonde) for the class $G(\alpha, \delta)$.

3.1. Logarithmic coefficients for $\mathbf{G}(\alpha, \delta)$

We now estimate the upper bounds of the logarithmic coefficients for functions belonging to $G(\alpha, \delta)$.

Theorem 1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G(\alpha, \delta)$, then*

$$|\gamma_1| \leq \frac{t_{\alpha\delta}}{2},$$

$$|\gamma_2| \leq \frac{t_{\alpha\delta}}{3},$$

$$|\gamma_3| \leq \frac{t_{\alpha\delta}}{4} + \frac{{t_{\alpha\delta}}^3}{6},$$

and

$$|\gamma_4| \leq \frac{t_{\alpha\delta}}{5} + \frac{{t_{\alpha\delta}}^2}{4} + \frac{{t_{\alpha\delta}}^4}{8},$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Let a function $f(z) \in G(\alpha, \delta)$ given by (1). Then there exists a function $p(z) \in P$ such that [34]

$$\frac{e^{i\alpha} f'(z) - i \sin \alpha - \delta}{t_{\alpha\delta}} = p(z),$$

where $t_{\alpha\delta} = \cos \alpha - \delta$, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, and $f'(z) = 1 + n \sum_{n=2}^{\infty} a_n z^{n-1}$.
Moreover, it can be observed that

$$a_n = \frac{t_{\alpha\delta} e^{-i\alpha} p_{n-1}}{n}, \ n \geqslant 2, \tag{17}$$

and specifically, for $n = 2, 3, 4, 5$, we get

$$\left. \begin{aligned} a_2 &= \tfrac{t_{\alpha\delta} e^{-i\alpha} p_1}{2}, \\ a_3 &= \tfrac{t_{\alpha\delta} e^{-i\alpha} p_2}{3}, \\ a_4 &= \tfrac{t_{\alpha\delta} e^{-i\alpha} p_3}{4}, \\ a_5 &= \tfrac{t_{\alpha\delta} e^{-i\alpha} p_4}{5}. \end{aligned} \right\} \tag{18}$$

Substituting (18) into (7)-(10) yields

$$\gamma_1 = \frac{t_{\alpha\delta} e^{-i\alpha} p_1}{4}, \tag{19}$$

$$\gamma_2 = \frac{t_{\alpha\delta} e^{-i\alpha}}{48} \left(8p_2 - 3t_{\alpha\delta} e^{-i\alpha} p_1^2\right), \tag{20}$$

$$\gamma_3 = \frac{t_{\alpha\delta} e^{-i\alpha}}{48} \left(6p_3 - 4t_{\alpha\delta} e^{-i\alpha} p_1 p_2 + {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3\right), \tag{21}$$

and

$$\gamma_4 = \frac{t_{\alpha\delta} e^{-i\alpha} p_4}{10} - \frac{{t_{\alpha\delta}}^2 e^{-2i\alpha} p_2^2}{36} - \frac{{t_{\alpha\delta}}^2 e^{-2i\alpha} p_1 p_3}{16} + \frac{{t_{\alpha\delta}}^3 e^{-3i\alpha} p_1^2 p_2}{24} - \frac{{t_{\alpha\delta}}^4 e^{-4i\alpha} p_1^4}{128}. \tag{22}$$

Hence, we can express (19)-(22) as follows:

$$|\gamma_1| = \left| \frac{t_{\alpha\delta} e^{-i\alpha} p_1}{4} \right|, \tag{23}$$

$$|\gamma_2| = \left| \frac{t_{\alpha\delta} e^{-i\alpha}}{48} \left(8 \left(p_2 - \frac{3t_{\alpha\delta} e^{-i\alpha}}{8} p_1^2 \right) \right) \right|, \tag{24}$$

$$|\gamma_3| = \left| \frac{t_{\alpha\delta} e^{-i\alpha}}{48} \left(6 \left(p_3 - \frac{2t_{\alpha\delta} e^{-i\alpha}}{3} p_1 p_2 \right) + {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3 \right) \right|, \tag{25}$$

$$|\gamma_4| = \left| t_{\alpha\delta} e^{-i\alpha} \left(-\frac{1}{10} \left(p_4 - \frac{10 t_{\alpha\delta} e^{-i\alpha}}{36} p_2^2 \right) + \frac{t_{\alpha\delta} e^{-i\alpha} p_1}{16} \left(p_3 - \frac{2 t_{\alpha\delta} e^{-i\alpha}}{3} p_1 p_2 \right) + \frac{{t_{\alpha\delta}}^3 e^{-3i\alpha} p_1^4}{128} \right) \right|. \tag{26}$$

Applying Lemma 2, it can be observed that

$$\left.\begin{aligned} \left| p_2 - \tfrac{3 t_{\alpha\delta} e^{-i\alpha}}{8} p_1^2 \right| &\le 2 \max\left\{ 1, \left| \tfrac{3 t_{\alpha\delta} e^{-i\alpha} - 4}{4} \right| \right\} = 2, \\ \left| p_3 - \tfrac{2 t_{\alpha\delta} e^{-i\alpha}}{3} p_1 p_2 \right| &\le 2 \max\left\{ 1, \left| \tfrac{4 t_{\alpha\delta} e^{-i\alpha} - 3}{3} \right| \right\} = 2, \\ \left| p_4 - \tfrac{10 t_{\alpha\delta} e^{-i\alpha}}{36} p_2^2 \right| &\le 2 \max\left\{ 1, \left| \tfrac{5 t_{\alpha\delta} e^{-i\alpha} - 9}{9} \right| \right\} = 2, \\ \left| p_3 - \tfrac{2 t_{\alpha\delta} e^{-i\alpha}}{3} p_1 p_2 \right| &\le 2 \max\left\{ 1, \left| \tfrac{4 t_{\alpha\delta} e^{-i\alpha} - 3}{3} \right| \right\} = 2. \end{aligned}\right\} \tag{27}$$

Thus, the upper bounds of $|\gamma_1|$ and $|\gamma_2|$ result from applying Lemma 1 and Lemma 2, respectively. Meanwhile, the upper bounds of $|\gamma_3|$ and $|\gamma_4|$ result from using both Lemma 1 and Lemma 2, as well as triangle inequality. This completes the proof of Theorem 1.

3.2. Second-Order Hankel Determinant of Logarithmic Coefficients for $\mathbf{G}(\alpha, \delta)$

Now, in this subsection, using the results from Theorem 1, we estimate the upper bound of the second-order Hankel determinant of logarithmic coefficients, specifically for $n = 2$ and $q = 2$, for functions belonging to $G(\alpha, \delta)$.

Theorem 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G(\alpha, \delta)$, *then*

$$|H_{2,2}(\gamma_f)| \le \frac{{t_{\alpha\delta}}^2}{2160} \left(36 \left| 5 t_{\alpha\delta} e^{-i\alpha} + 4 \right| + 9 t_{\alpha\delta} \left| 5 t_{\alpha\delta} e^{-i\alpha} + 12 \right| + 30 {t_{\alpha\delta}}^3 + 80 t_{\alpha\delta} + 135 \right),$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Using (8)–(10), we can establish

$$\begin{aligned} {\gamma_3}^2 &= \tfrac{{t_{\alpha\delta}}^2 e^{-2i\alpha}}{2304} \left(6 p_3 - 4 t_{\alpha\delta} e^{-i\alpha} p_1 p_2 + {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3 \right)^2 \\ &= \tfrac{{t_{\alpha\delta}}^2 e^{-2i\alpha}}{2304} \begin{pmatrix} 36 p_3^2 - 48 t_{\alpha\delta} e^{-i\alpha} p_1 p_2 p_3 + 16 {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^2 p_2^2 \\ +12 {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3 p_3 - 8 {t_{\alpha\delta}}^3 e^{-3i\alpha} p_1^4 p_2 + {t_{\alpha\delta}}^4 e^{-4i\alpha} p_1^6 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \gamma_2 \gamma_4 &= \tfrac{{t_{\alpha\delta}}^2 e^{-2i\alpha} \left(8 p_2 - 3 t_{\alpha\delta} p_1^2 e^{-i\alpha} \right)}{48} \left(\tfrac{p_4}{10} - \tfrac{t_{\alpha\delta} p_2^2 e^{-i\alpha}}{36} - \tfrac{t_{\alpha\delta} p_1 p_3 e^{-i\alpha}}{16} + \tfrac{{t_{\alpha\delta}}^2 p_1^2 p_2 e^{-2i\alpha}}{24} - \tfrac{{t_{\alpha\delta}}^3 p_1^4 e^{-3i\alpha}}{128} \right) \\ &= \tfrac{{t_{\alpha\delta}}^2 e^{-2i\alpha}}{2304} \begin{pmatrix} \tfrac{192 p_2 p_4}{5} - \tfrac{32 t_{\alpha\delta} e^{-i\alpha} p_2^3}{3} - 24 t_{\alpha\delta} e^{-i\alpha} p_1 p_2 p_3 + 20 {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^2 p_2^2 \\ - 9 {t_{\alpha\delta}}^3 e^{-3i\alpha} p_1^4 p_2 - \tfrac{72 t_{\alpha\delta} e^{-i\alpha} p_1^2 p_4}{5} + 9 {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3 p_3 + \tfrac{9 {t_{\alpha\delta}}^4 e^{-4i\alpha} p_1^6}{8} \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$H_{2,2}(\gamma_f) = \frac{{t_{\alpha\delta}}^2 e^{-2i\alpha}}{2304}\left(\begin{array}{c} \frac{192p_2p_4}{5} + 24t_{\alpha\delta}e^{-i\alpha}p_1p_2p_3 - \frac{32t_{\alpha\delta}e^{-i\alpha}p_2^3}{3} + 4{t_{\alpha\delta}}^2 e^{-2i\alpha}p_1^2{p_2}^2 - 36p_3^2 \\ -\frac{72t_{\alpha\delta}e^{-i\alpha}p_1^2p_4}{5} - 3{t_{\alpha\delta}}^2 e^{-2i\alpha}p_1^3p_3 - {t_{\alpha\delta}}^3 e^{-3i\alpha}p_1^4p_2 + \frac{{t_{\alpha\delta}}^4 e^{-4i\alpha}p_1^6}{8} \end{array}\right). \tag{28}$$

Taking the modulus of both sides of equation (28) and rearranging the terms according to Lemma 2, we obtain

$$|H_{2,2}(\gamma_f)| = \frac{{t_{\alpha\delta}}^2}{2304}\left|\begin{array}{c} -\frac{192p_2}{5}(p_4 - \nu^* p_1p_3) + \frac{32t_{\alpha\delta}e^{-i\alpha}p_2^2}{3}\left(p_2 - \nu^{**}p_1^2\right) + 36p_3^2 \\ +\frac{72t_{\alpha\delta}e^{-i\alpha}p_1^2}{5}(p_4 - \nu^{***}p_1p_3) + {t_{\alpha\delta}}^3 e^{-3i\alpha}p_1^4\left(p_2 - \nu^{****}p_1^2\right) \end{array}\right|, \tag{29}$$

where $\nu^* = \frac{-5t_{\alpha\delta}e^{-i\alpha}}{8}, \nu^{**} = \frac{3t_{\alpha\delta}e^{-i\alpha}}{8}, \nu^{***} = \frac{-5t_{\alpha\delta}e^{-i\alpha}}{24}$, and $\nu^{****} = \frac{t_{\alpha\delta}e^{-i\alpha}}{8}$.
We see that

$$\left.\begin{array}{c} |p_4 - \nu^* p_1p_3| \le \left|\frac{5t_{\alpha\delta}e^{-i\alpha}+4}{2}\right|, \\ \left|p_2 - \nu^{**}p_1^2\right| \le 2, \\ |p_4 - \nu^{***}p_1p_3| \le \left|\frac{5t_{\alpha\delta}e^{-i\alpha}+12}{6}\right|, \\ \left|p_2 - \nu^{****}p_1^2\right| \le 2. \end{array}\right\} \tag{30}$$

Thus, from (29), considering the triangle inequality, Lemma 1, and (30), we obtain the desired inequality. This concludes the proof of Theorem 2.

3.3. Second-Order Toeplitz Determinant of Logarithmic Coefficients for $\mathbf{G}(\alpha, \delta)$

In this subsection, using the results from Theorem 1, we determine the upper bound of the second-order Toeplitz determinant of logarithmic coefficients, specifically for $n = 2$ and $q = 2$, for functions belonging to $G(\alpha, \delta)$.

Theorem 3. *If* $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in G(\alpha, \delta)$, *then*

$$|T_{2,2}(\gamma_f)| \le \frac{{t_{\alpha\delta}}^2}{144}\left(16 + 37{t_{\alpha\delta}}^2 + 4\,{t_{\alpha\delta}}^4 + 3\left|8t_{\alpha\delta}e^{-i\alpha} - 3\right|\right),$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. In light of (8) and (9) give

$$\begin{aligned} {\gamma_2}^2 &= \tfrac{1}{2304}\left(8t_{\alpha\delta}e^{-i\alpha}p_2 - 3{t_{\alpha\delta}}^2 e^{-2i\alpha}p_1^2\right)^2 \\ &= \tfrac{{t_{\alpha\delta}}^2 e^{-2i\alpha}}{2304}\left(64p_2^2 - 48t_{\alpha\delta}e^{-i\alpha}p_1^2p_2 + 9{t_{\alpha\delta}}^2 e^{-2i\alpha}p_1^4\right) \end{aligned}$$

and

$$\begin{aligned}\gamma_3{}^2 &= \tfrac{1}{2304}\left(6t_{\alpha\delta}p_3e^{-i\alpha} - 4t_{\alpha\delta}{}^2p_1p_2e^{-2i\alpha} + t_{\alpha\delta}{}^3p_1^3e^{-3i\alpha}\right)^2 \\ &= \tfrac{t_{\alpha\delta}{}^2e^{-2i\alpha}}{2304}\left(\begin{array}{l} 36p_3^2 - 48t_{\alpha\delta}p_1p_2p_3e^{-i\alpha} + 12t_{\alpha\delta}{}^2p_1^3p_3e^{-2i\alpha} \\ +16t_{\alpha\delta}{}^2p_1^2p_2^2e^{-2i\alpha} - 8t_{\alpha\delta}{}^3p_1^4p_2e^{-3i\alpha} + t_{\alpha\delta}{}^4p_1^6e^{-4i\alpha}\end{array}\right).\end{aligned}$$

Therefore, we obtain

$$T_{2,2}(\gamma_f) = \frac{t_{\alpha\delta}{}^2e^{-2i\alpha}}{2304}\left(\begin{array}{l} 64p_2^2 - 48t_{\alpha\delta}e^{-i\alpha}p_1^2p_2 - 36p_3^2 + 48t_{\alpha\delta}e^{-i\alpha}p_1p_2p_3 + 9t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^4 \\ -12t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^3p_3 + 8t_{\alpha\delta}{}^3e^{-3i\alpha}p_1^4p_2 - 16t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^2p_2^2 - t_{\alpha\delta}{}^4e^{-4i\alpha}p_1^6\end{array}\right), \tag{31}$$

and we can express (31) as follows:

$$|T_{2,2}(\gamma_f)| = \left|\frac{t_{\alpha\delta}{}^2e^{-2i\alpha}}{2304}\left(\begin{array}{l} -64p_2\left(p_2 - \kappa^*p_1^2\right) + 12t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^3\left(p_3 - \kappa^{**}p_1p_2\right) - 9t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^4 \\ +36p_3\left(p_3 - \kappa^{***}p_1p_2\right) + 16t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^2p_2^2 + t_{\alpha\delta}{}^4e^{-4i\alpha}p_1^6\end{array}\right)\right|, \tag{32}$$

where $\kappa^* = \frac{48t_{\alpha\delta}e^{-i\alpha}}{64}, \kappa^{**} = \frac{8t_{\alpha\delta}e^{-i\alpha}}{12}$, and $\kappa^{***} = \frac{48t_{\alpha\delta}e^{-i\alpha}}{36}$.
According to Lemma 2, we can conclude that

$$\left.\begin{array}{r} \left|p_2 - \kappa^*p_1^2\right| \le 2\max\left\{1, \left|\frac{3t_{\alpha\delta}e^{-i\alpha}-2}{2}\right|\right\} = 2, \\ \left|p_3 - \kappa^{**}p_1p_2\right| \le 2\max\left\{1, \left|\frac{4t_{\alpha\delta}e^{-i\alpha}-3}{3}\right|\right\} = 2, \\ \left|p_3 - \kappa^{***}p_1p_2\right| \le 2\max\left\{1, \left|\frac{8t_{\alpha\delta}e^{-i\alpha}-3}{3}\right|\right\} = 2\left|\frac{8t_{\alpha\delta}e^{-i\alpha}-3}{3}\right|. \end{array}\right\} \tag{33}$$

Using Lemma 1, (33), and the triangle inequality, we obtain the desired bound from (32). This concludes the proof of Theorem 3.

3.4. Second-Order Vandermonde Determinant of Logarithmic Coefficients for $\mathbf{G}(\alpha, \delta)$

In this subsection, we obtain the upper bound of the second-order Vandermonde determinant of logarithmic coefficients, specifically for $n = 2$ and $q = 2$, for functions belonging to $G(\alpha, \delta)$.

Theorem 4. *If* $f(z) = z + \sum\limits_{n=2}^{\infty} a_nz^n \in G(\alpha, \delta)$, *then*

$$|V_{2,2}(\gamma_f)| \le \frac{t_{\alpha\delta}\left(7 + 2t_{\alpha\delta}{}^2\right)}{12},$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Through (8) and (9) yield

$$V_{2,2}(\gamma_f) = \frac{t_{\alpha\delta}e^{-i\alpha}}{48}\left(6p_3 - 4t_{\alpha\delta}e^{-i\alpha}p_1p_2 + t_{\alpha\delta}{}^2e^{-2i\alpha}p_1^3 - 8p_2 + 3t_{\alpha\delta}e^{-i\alpha}p_1^2\right). \tag{34}$$

By rearranging the terms in (34) according to Lemma 2, we obtain

$$|V_{2,2}(\gamma_f)| = \left| \frac{t_{\alpha\delta} e^{-i\alpha}}{48} \left(6 \left(p_3 - \eta^* p_1 p_2 \right) - 8 \left(p_2 - \eta^{**} p_1^2 \right) + {t_{\alpha\delta}}^2 e^{-2i\alpha} p_1^3 \right) \right|, \tag{35}$$

where $\eta^* = \frac{4t_{\alpha\delta}e^{-i\alpha}}{6}$ and $\eta^{**} = \frac{3t_{\alpha\delta}e^{-i\alpha}}{8}$.
Furthermore, we discover that

$$\left.\begin{aligned} |p_3 - \eta^* p_1 p_2| &\leq 2 \max \left\{ 1, \left| \tfrac{4t_{\alpha\delta}e^{-i\alpha} - 3}{3} \right| \right\} = 2, \\ \left| p_2 - \eta^{**} p_1^2 \right| &\leq 2 \max \left\{ 1, \left| \tfrac{3t_{\alpha\delta}e^{-i\alpha} - 4}{4} \right| \right\} = 2. \end{aligned}\right\} \tag{36}$$

By implementing Lemma 1 and (36) into (35), as well as applying the triangle inequality, we achieve the desired bound. This completes the proof of Theorem 4.

4. Consequences and corollaries

Since $G(\alpha, \delta)$ generalizes R, $R(\delta)$, and $R(\alpha)$, several new consequences of Theorems 1-4 are highlighted out for specific choices of α and δ as follows:

Substituting $\alpha = 0$ and $\delta = 0$ in Theorems 1-4, we get the estimates bounds for the class R.

Corollary 1. *For any function $f(z)$ given by (1) for the class $G(0,0) \equiv R$, then*

(i) $|\gamma_1| \leq \frac{1}{2}$, $|\gamma_2| \leq \frac{1}{3}$, $|\gamma_3| \leq \frac{5}{12}$, $|\gamma_4| \leq \frac{23}{40}$

(ii) $|H_{2,2}(\gamma_f)| \leq \frac{301}{432}$

(iii) $|T_{2,2}(\gamma_f)| \leq \frac{1}{2}$

(iv) $|V_{2,2}(\gamma_f)| \leq \frac{3}{4}$

If we consider $\alpha = 0$ in Theorems 1-4, we obtain the estimates bounds for the class $R(\delta)$.

Corollary 2. *For any function $f(z)$ given by (1) for the class $G(0,\delta) \equiv R(\delta)$, then*

(i) $|\gamma_1| \leq \frac{1-\delta}{2}$, $|\gamma_2| \leq \frac{1-\delta}{3}$, $|\gamma_3| \leq \frac{1-\delta}{4} + \frac{(1-\delta)^3}{6}$, $|\gamma_4| \leq \frac{1-\delta}{5} + \frac{(1-\delta)^2}{4} + \frac{(1-\delta)^4}{8}$

(ii) $|H_{2,2}(\gamma_f)| \leq \frac{(1-\delta)^2}{2160} \left(\begin{array}{l} 36\,|5(1-\delta)+4| + 9(1-\delta)\,|5(1-\delta)+12| \\ +30(1-\delta)^3 + 80(1-\delta) + 135 \end{array} \right)$

(iii) $|T_{2,2}(\gamma_f)| \leq \frac{(1-\delta)^2}{144} \left(16 + 37(1-\delta)^2 + 4(1-\delta)^4 + 3\,|8(1-\delta) - 3| \right)$

(iv) $|V_{2,2}(\gamma_f)| \leq \frac{(1-\delta)\left(7 + 2(1-\delta)^2\right)}{12}$

Putting $\delta = 0$ in Theorems 1-4, we have the following results for the class $R(\alpha)$.

Corollary 3. *For any function $f(z)$ given by* (1) *for the class $G(\alpha, 0) \equiv R(\alpha)$, then*

(i) $|\gamma_1| \le \frac{\cos\alpha}{2}$, $|\gamma_2| \le \frac{\cos\alpha}{3}$, $|\gamma_3| \le \frac{\cos\alpha}{4} + \frac{\cos^3\alpha}{6}$, $|\gamma_4| \le \frac{\cos\alpha}{5} + \frac{\cos^2\alpha}{4} + \frac{\cos^4\alpha}{8}$

(ii) $|H_{2,2}(\gamma_f)| \le \frac{\cos^2\alpha}{2160} \left(\begin{array}{l} 36\left|5e^{-i\alpha}\cos\alpha + 4\right| + 9\cos\alpha\left|5e^{-i\alpha}\cos\alpha + 12\right| \\ +30\cos^3\alpha + 80\cos\alpha + 135 \end{array} \right)$

(iii) $|T_{2,2}(\gamma_f)| \le \frac{\cos^2\alpha}{144}\left(16 + 37\cos^2\alpha + 4\cos^4\alpha + 3\left|8e^{-i\alpha}\cos\alpha - 3\right|\right)$

(iv) $|V_{2,2}(\gamma_f)| \le \frac{\cos\alpha\left(7 + 2\cos^2\alpha\right)}{12}$

5. Conclusion

In this paper, we have obtained the estimates on logarithmic coefficients $|\gamma_n|$, $n = 1, 2, 3, 4$, thereby extending the properties of $G(\alpha, \delta)$, R, $R(\delta)$, and $R(\alpha)$. Recent research has sparked considerable interest in logarithmic coefficients and the Hankel, Toeplitz, and Vandermonde determinants. This has inspired us to define the Vandermonde determinant of logarithmic coefficients for functions $f(z) \in \mathcal{A}$. As a result of determining the logarithmic coefficients, we have established the upper bounds for three types of determinants: $|H_{2,2}(\gamma_f)|$, $|T_{2,2}(\gamma_f)|$, and $|V_{2,2}(\gamma_f)|$, where the logarithmic coefficients are considered as the entries, for functions from $G(\alpha, \delta)$, as well as R, $R(\delta)$, and $R(\alpha)$. The lemmas from the preliminary section have proven invaluable in establishing upper bounds for three types of determinants of logarithmic coefficients. The findings in this paper could inspire further research into determining upper bounds for Hankel, Toeplitz, and Vandermonde determinants with logarithmic coefficients as entries, particularly within other subclasses of univalent functions, while considering the inverse functions for $G(\alpha, \delta)$. Additionally, for new insights, one might refer to [41] for other coefficient-related problems in logarithmic functions such as Fekete Szegö inequality; however, consider subclasses of bi-univalent functions, which could expand upon, for example, the works of [17, 29].

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References

[1] N H A A Wahid, A Tumiran and T G Shaba. Hankel and toeplitz determinants of logarithmic coefficients of inverse functions for the subclass of starlike functions

with respect to symmetric conjugate points. *European Journal of Pure and Applied Mathematics*, 13(3):1818–1830, 2024.

[2] V Allu and V Arora. Second hankel determinant of logarithmic coefficients of certain analytic functions. *Rocky Mountain Journal of Mathematics*, 54(2):343–359, 2024.

[3] V V Andreev and P L Duren. Inequalities for logarithmic coefficients of univalent functions and their derivatives. *Indiana University Mathematics Journal*, 37(4):721–733, 1988.

[4] L De Branges. A proof of the bieberbach conjecture. *Acta Mathematica*, 154(1):137–152, 1985.

[5] S Bulut. Sharp bounds for the second hankel determinant of logarithmic coefficients for parabolic starlike and uniformly convex functions of order alpha. *Khayyam Journal of Mathematics*, 10(1):51–69, 2024.

[6] D G Cantor. Power series with integral coefficients. *Bulletin of the American Mathematical Society*, 69:362–366, 1963.

[7] M F Ali, D K Thomas, and A Vasudevarao. Toeplitz determinants whose elements are the coefficients of analytic and univalent functions. *Bulletin of the Australian Mathematical Society*, 97:253–264, 2018.

[8] S Cik Soh, D Mohamad, and H Dzubaidi. Coefficient estimate of the second hankel determinant of logarithmic coefficients for the subclass of close-to-convex function. *TWMS Journal Of Applied And Engineering Mathematics*, 24(2):563–570, 2024.

[9] P Dienes. *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable.* New York-Dover Publishing Company, Mineola, NY, USA, 1957.

[10] P L Duren. *Univalent Functions vol. 259.* Springer-Verlag., New York, Berlin, Heidelberg, Tokyo, 1983.

[11] I Efraimidis. A generalization of livingston's coefficient inequalities for functions with positive real part. *Journal of Mathematical Analysis and Applications*, 435(1):369–379, 2016.

[12] M M Elhosh. On the logarithmic coefficients of close-to-convex functions. *Journal of the Australian Mathematical Society*, 60(1):1–6, 1996.

[13] H Tang, Z Mujahid, N Khan, F Tchier, and M K Ghaffar Khan. Generalized bounded turning functions connected with gregory coefficients. *Axioms*, 13(6):359, 2024.

[14] D Girela. Logarithmic coefficients of univalent functions. *Annales Fennici Mathematici*, 25(2):337–350, 2000.

[15] S Giri and S S Kumar. Toeplitz determinants of logarithmic coefficients for starlike and convex functions. *arXiv preprint arXiv:2303.14712*, 2023.

[16] R M Goel and B S Mehrok. A subclass of univalent functions. *Journal of the Australian Mathematical Society*, 35(1):1–17, 1983.

[17] A Hussen and A. Zeyani. Coefficients and fekete–szegö functional estimations of bi-univalent subclasses based on gegenbauer polynomials. *Mathematics*, 11(13):2852, 2023.

[18] M Arif, M Raza, I Ullah, and P Zaprawa. Hankel determinants of order four for a set of functions with bounded turning of order α. *Lithuanian Mathematical Journal*, 62(2):135–145, 2022.

[19] M G Khan, W K Mashwani, J S Ro, and B Ahmad. Problems concerning sharp coefficient functionals of bounded turning functions. *AIMS Mathematics*, 8(11):27396–27413, 2023.

[20] I P Kayumov. On brennan's conjecture for a special class of functions. *Mathematical Notes*, 78:498–502, 2005.

[21] B Kowalczyk and A Lecko. The sharp bound of the third hankel determinant for functions of bounded turning. *Boletin de la Sociedad Matematica Mexicana*, 27:1–13, 2021.

[22] B Kowalczyk and A Lecko. Second hankel determinant of logarithmic coefficients of convex and starlike functions. *Bulletin of the Australian Mathematical Society*, 105(3):458–467, 2022.

[23] B Kowalczyk and A Lecko. Second hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha. *Bulletin of the Malaysian Mathematical Sciences Society*, 45(2), 2022.

[24] B Kowalczyk and A Lecko. The second hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 117(2):91, 2023.

[25] A Lecko and B Śmiarowska. The second hankel determinant for logarithmic coefficients of inverse functions of bounded turning of a given order. *Boletín de la Sociedad Matemática Mexicana*, 30(2):51, 2024.

[26] Y Li and X Ding. Vandermonde determinant and its applications. *Journal of Education and Culture Studies*, 7(4):16–24, 2023.

[27] O M Barukab, M Arif, M Abbas, and S A Khan. Sharp bounds of the coefficient results for the family of bounded turning functions associated with a petal-shaped domain. *Journal of Function Spaces*, 2021(1):5535629, 2021.

[28] P Sunthrayuth, I Aldawish, M Arif, M Abbas, and S El-Deeb. Estimation of the second-order hankel determinant of logarithmic coefficients for two subclasses of starlike functions. *Symmetry*, 14(10):2039, 2022.

[29] A Hussen, M S Madi, and A M Abominjil. Bounding coefficients for certain subclasses of bi-univalent functions related to lucas-balancing polynomials. *AIMS Mathematics*, 9(7):18034–18047, 2024.

[30] T H Macgregor. Functions whose derivative has a positive real part. *Transactions of the American Mathematical Society*, 104(3):532–537, 1962.

[31] I M Milin. *Univalent functions and orthonormal systems*, volume 49. American Mathematical Society, 1977.

[32] I M Milin. On a property of the logarithmic coefficients of univalent functions. *Metric Questions in the Theory of Functions*, pages 86–90, 1980.

[33] I M Milin. On one conjecture for the logarithhmic coefficients of univalent functions. *Zapiski Nauchnykh Seminarov POMI*, 125:135–143, 1983.

[34] D Mohamad. On a class of functions whose derivatives map the unit disc into a half plane. *Bulletin of The Malaysian Mathematical Sciences Society*, 23(2), 2000.

[35] D Mohamad, N H A A Wahid, and N N Hasni. Coefficient problems for star-like functions with respect to symmetric conjugate points connected to the sine function. *European Journal of Pure and Applied Mathematics*, 16(2):1167–1179, 2023.

[36] N H A A Wahid, D Mohamad, N M Kamarozzaman, and A A Shahminan. Toeplitz determinants for the class of functions with bounded turning. *European Journal of Pure and Applied Mathematics*, 15(4):1937–1947, 2022.

[37] K Noshiro. On the theory of schlicht functions. *Journal of Faculty of Science, Hokkaido Imperial University. Series I. Mathematics*, 2:129–155, 1934.

[38] C Pommerenke. On the coefficients and hankel determinants of univalent functions. *Journal of the London Mathematical Society*, 1(1):111–122, 1966.

[39] C Pommerenke. On the hankel determinants of univalent functions. *Mathematika*, 14(1):108–112, 1967.

[40] M G Khan, B Ahmad, J Sokol, Z Muhammad, W K Mashwani, R Chinram, and P Petchkaew. Coefficient problems in a class of functions with bounded turning associated with sine function. *European Journal of Pure and Applied Mathematics*, 14(1):53–64, 2021.

[41] A S Alshehry, R Shah, and A Bariq. The second hankel determinant of logarithmic coefficients for starlike and convex functions involving four-leaf-shaped domain. *Journal of Function Spaces*, 2022(1):2621811, 2022.

[42] O Roth. A sharp inequality for the logarithmic coefficients of univalent functions. *Proceedings of the American Mathematical Society*, 135(7):2051–2054, 2007.

[43] B Khan, I Aldawish, S Araci, and M G Khan. Third hankel determinant for the logarithmic coefficients of starlike functions associated with sine function. *Fractal and Fractional*, 6(5):261, 2022.

[44] S P Vijayalakshmi, S Bulut, and T V Sudharsan. Vandermonde determinant for a certain sakaguchi type function in limaçon domain. *Asian-European Journal of Mathematics*, 15(12):2250212, 2022.

[45] H Silverman and E M Silvia. On α-close-to-convex functions. *Publicationes Mathematicae Debrecen*, 49(3-4):305–316, 1996.

[46] V Allu, V Arora, and A Shaji. On the second hankel determinant of logarithmic coefficients for certain univalent functions. *Mediterranean Journal of Mathematics*, 20(2):81, 2023.

[47] S E Warschawski. On the higher derivatives at the boundary in conformal mapping. *Transactions of the American Mathematical Society*, 38:310–340, 1935.

[48] P Sunthrayuth, N Iqbal, M Naeem, Y Jawarneh, and S K Samura. The sharp upper bounds of the hankel determinant on logarithmic coefficients for certain analytic functions connected with eight-shaped domains. *Journal of Function Spaces*, 2022(1):2229960, 2022.

[49] K Ye and L H Lim. Every matrix is a product of toeplitz matrices. *Foundations of Computational Mathematics*, 16(3):577–598, 2016.